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# CONE VALUED LYAPUNOV TYPE STABILITY ANALYSIS OF NONLINEAR EQUATIONS

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ABSTRACT. We investigate various $\phi(t)$-stability of comparison differential equations and we obtain necessary and/or sufficient conditions for the asymptotic and uniform asymptotic stability of the differential equations $x' = f(t, x)$.

## 1. Preliminaries and Definitions

Lyapunov second methods are now well established subjects as the most powerful techniques of analysis for the stability and qualitative properties of nonlinear differential equations $x' = f(t, x)$, $x(t_0) = x_0 \in R^N$.

One of the original Lyapunov theorems is as follows :

LYAPUNOV THEOREM. *For $x' = f(t, x)$, assume that there exists a function $V : R_+ \times S_\rho \to R_+$ such that*
(i) *$V$ is $C^1$-function and positive definite,*
(ii) *$V$ is decresent,*
(iii) *$\frac{d}{dt}V(t, x) = V_t(t, x) + V_x \cdot f(t, x) \leq -a(\|x\|)$ for $t \geq 0$, $x \in S_\rho$, where $S_\rho = \{x \in R^N | \|x\| < \rho\}$ for $\rho > 0$, $a(r)$ is strictly increasing function with $a(0) = 0$.*
*Then the trivial solution $x(t) \equiv O$ is uniformly asymptotically stable.*

The advantage of the method is that it does not require the knowledge of solutions to analyse the stability of the equations. However in practical sense, how to find suitable Lyapunov functions $V$ for given

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equations are the most difficult questions. Hence weakening the conditions (i), (ii), and (iii), and enlarging the class of Lyapunov functions are basic trends in Lyapunov stability theory [2, 3, 4, 5, 6, 11].

In the unified comparison frameworks, Ladde [7] analysed the stability of comparison differential equations by using vector Lyapunov function methods.

Lakshmikantham and Leela [9] initiated the cone valued Lyapunov function methods to advoid the quasimonotonicity assumption of comparison equations. They obtained various useful differential inequalities with cone-valued Lyapunov functions. Akpan and Akinyele [1] extended and generalized the results of [7, 9] to the $\phi_0$-stabilities of the comparison differential equations by using the cone-valued Lyapunov functions.

Here we generalize, in some sense, the results of [1] to the $\phi(t)$-stabilities of comparison equations below.

Let $R^n$ denote the $n$-dimensional Euclidean space with any equivalent norm $\|\cdot\|$, and scalar product ( , ). $R_+ = [0, \infty)$. $C[R_+ \times R^n, R^n]$ denotes the space of continuous functions from $R_+ \times R^n$ into $R^n$.

DEFINITION 1.1 ([11]). A proper subset $K$ of $R^n$ is called a *cone* if (i) $\lambda K \subset K$, $\lambda \geq 0$; (ii) $K + K \subset K$; (iii) $K = \overline{K}$; (iv) $K^\circ \neq \emptyset$; (v) $K \cap (-K) = \{0\}$, where $\overline{K}$ and $K^\circ$ denote the closure and interior of $K$, respectively, and $\partial K$ denotes the boundary of $K$. The order relation on $R^n$ induced by the cone $K$ is defined as follows :

For $x, y \in R^n$, $x \leq_k y$ iff $y - x \in K$, and $x <_{k^\circ} y$ iff $y - x \in K^\circ$.

DEFINITION 1.2 ([11]). The set $K^* = \{\phi \in R^n : (\phi, x) \geq 0$, for all $x \in K\}$ is called the *adjoint cone* of $K$ if $K^*$ itself satisfies Definition 1.1.

Note that $x \in \partial K$ if and only if $(\phi, x) = 0$ for some $\phi \in {K_0}^*$, where $K_0 = K - \{0\}$.

Consider the differential equation

$$x' = f(t, x), \quad x(t_0) = x_0, t_0 \geq 0 \tag{1}$$

where $f \in C[R_+ \times R^N, R^N]$ and $f(t, 0) = 0$ for all $t \geq 0$. Let $S_\rho = \{x \in R^N : \|x\| < \rho\}$ , $\rho > 0$. Let $K \subset R^n$ be a cone in $R^n$, $n \leq N$. For $V \in C[R_+ \times S_\rho, K]$, at $(t, x) \in R_+ \times S_\rho$, let $D^+V(t, x) =$

$\limsup_{h\to 0^+}(\frac{1}{h})[V(t+h, x+hf(t,x)) - V(t,x)]$ be a Dini derivative of V along the solution curves of the equation (1).

Consider a comparison differential equation

$$(2) \qquad u' = g(t,u), \quad u(t_0) = u_0, \ t_0 \geq 0$$

where $g \in C[R_+ \times K, R^n]$, $g(t,0) = 0$ for all $t \geq 0$ and $K$ is a cone in $R^n$.

Let $S(\rho) = \{u \in K : \|u\| < \rho\}$, $\rho > 0$. For $v \in C[R_+ \times S(\rho), K]$, at $(t,u) \in R_+ \times S(\rho)$, let $D^+v(t,u) = \limsup_{h\to 0^+}(\frac{1}{h})[v(t+h, u+hg(t,u)) - v(t,u)]$ be a Dini derivative of $v$ along the solution curves of the equation (2).

Definition 1.3 ([11]). A function $g : D \longrightarrow R^n$, $D \subset R^n$, is said to be *quasimonotone* nondecreasing relative to the cone $K$ when it satisfies that if $x, y \in D$ with $x \leq_K y$ and $(\phi_0, y - x) = 0$ for some $\phi_0 \in K_0^*$, then $(\phi_0, g(y) - g(x)) \geq 0$.

Definition 1.4 ([8, 10]). The trivial solution $x = 0$ of (1) is *equistable* if for each $\epsilon > 0$, $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_0, \varepsilon)$ such that the inequality $\|x_0\| < \delta$ implies $\|x(t, t_0, x_0)\| < \varepsilon$, for all $t \geq t_0$.

Other stability notions can be similarly defined [8, 10].

Now we give cone-valued $\phi(t)$-stability definitions of the trivial solution of (2).

Definition 1.5 ([12]). Let $\phi : [0, \infty) \longrightarrow K^*$ be a cone-valued function. The trivial solution $u = 0$ of (2) is

(a) *$\phi(t)$-equistable* if for each $\varepsilon > 0$, $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_0, \varepsilon)$ such that the inequality $(\phi(t_0), u_0) < \delta$ implies $(\phi(t), r(t)) < \varepsilon$, for all $t \geq t_0$ where $r(t)$ is a maximal solution of (2);

(b) *uniformly $\phi(t)$-stable* if the $\delta$ in (a) is independent of $t_0$ ;

(c) *quasi-equi asymptotically $\phi(t)$-stable* if, for each $\varepsilon > 0$, $t_0 \in R_+$, there exist positive numbers $\delta = \delta(t_0)$ and $T = T(t_0, \varepsilon)$ such that $(\phi(t_0), u_0) < \delta$ implies $(\phi(t), r(t)) < \varepsilon$ for all $t \geq t_0 + T$;

*Then the trivial solution $u = 0$ of (2) is equi-asymptotically $\phi(t)$-stable.*

*Proof.* By Theorem 2.1, the trivial solution of (2) is $\phi(t)$-equistable. By the formula (3), $(\phi(t), v(t, u(t)))$ is monotone decreasing in $t$ and hence the limit $v^* = \lim_{t\to\infty} (\phi(t), v(t, u(t)))$ exists. Suppose $v^* \neq 0$. Then $c(v^*) \neq 0$, $c \in \mathcal{K}$. Since $c(r)$ is monotone, $c[(\phi(t), v(t, u(t)))] \geq c(v^*)$, and so $D^+(\phi(t), v(t, u(t))) \leq -c[(\phi(t), v(t, u(t)))] \leq -c(v^*)$. Then

$$\int_{t_0}^{t} D^+(\phi(s), v(s, u(s)))ds \leq \int_{t_0}^{t} -c(v^*)ds.$$

Thus $(\phi(t), v(t, u(t))) \leq -c(v^*)(t - t_0) + (\phi(t_0), v(t_0, u_0))$. Accordingly, as $t \to \infty$, we have $(\phi(t), v(t, u(t))) \to -\infty$. This contradicts the condition $a[(\phi(t), r(t))] \leq (\phi(t), v(t, u(t)))$. It follows that $v^* = 0$. Thus $(\phi(t), v(t, u(t))) \to 0$ as $t \to \infty$ and so $(\phi(t), r(t)) \to 0$ as $t \to \infty$. Hence given $\varepsilon > 0$, and for each $t_0 \in R_+$, there exist $\delta = \delta(t_0)$ and $T = T(t_0, \varepsilon)$ such that for all $t \geq t_0 + T$, $(\phi(t_0), u_0) < \delta$ implies $(\phi(t), r(t)) < \varepsilon$. □

THEOREM 2.4. *Let the hypothesis of Theorem 2.2 hold with*

$$D^+(\phi(t), v(t, u(t))) \leq -c[(\phi(t), r(t))]$$

*for each $t \geq t_0$ where $t_0 \in R_+$ and for some $c \in \mathcal{K}$.*
*Then the trivial solution $u = 0$ of (2) is uniformly asymptotically $\phi(t)$-stable.*

*Proof.* By Theorem 2.2, the trivial solution $u = 0$ of (2) is uniformly $\phi(t)$-stable. Let $\varepsilon > 0$ be arbitrarily given. Choose $\delta = \delta(\varepsilon)$ which is independent of $t_0$. Let $u(t)$ be a solution of (2) such that $(\phi(t_0), u_0) < \delta$. Let $v^* = \sup\{(\phi(t), v(t, u(t))) | (\phi(t_0), u_0) < \delta\}$. Set $T(\varepsilon) = v^*/c(\varepsilon)$. We claim that

$$(4) \qquad (\phi(t_0), u_0) < \delta \text{ implies } (\phi(t), r(t)) < \varepsilon, \quad t \geq t_0 + T(\varepsilon).$$

Suppose that (4) is not true. Then there would exist at least one $t \geq t_0 + T(\varepsilon)$ such that $(\phi(t_0), u_0) < \delta$ implies $(\phi(t), r(t)) \geq \varepsilon$. Since $c \in$

$\mathcal{K}$, from the condition $D^+(\phi(t), v(t, u(t))) \le -c[(\phi(t), r(t))]$, we have $D^+(\phi(t), v(t, u(t))) \le -c[(\phi(t), r(t))] \le -c(\varepsilon)$. Integrating, $\int_{t_0}^{t} D^+ (\phi(s), v(s, u(s)))\, ds \le \int_{t_0}^{t} -c(\varepsilon) ds$ implies that $(\phi(t), v(t, u(t))) \le (\phi(t_0), v(t_0, u_0)) - c(\varepsilon)(t - t_0)$ for all $t \ge t_0 + T(\varepsilon)$.
Then $\lim_{t\to\infty} (\phi(t), v(t, u(t))) = -\infty$ which is a contradiction. □

THEOREM 2.5. *Assume that*
(i) $V \in C[R_+ \times S_\rho, K]$, *$V(t, x)$ is locally Lipschitzian in $x$ relative to $K$ and for $(t, x) \in R_+ \times S_\rho$, $D^+V(t, x) \le_K g(t, V(t, x))$,*
(ii) *$g \in C[R_+ \times K, R^n]$ and $g(t, u)$ is quasimonotone in $u$ relative to $K$ for each $t \in R_+$,*
(iii) *there exist $a, b \in \mathcal{K}$ such that for some $\phi(t) \in {K_0}^*$, for each $x \in S_\rho$, $b(\|x\|) \le (\phi(t), V(t, x)) \le a(\|x\|)$, $t \ge t_0 \ge 0$.*
*Then the trivial solution $x = 0$ of (1) has the corresponding one of the stability properties if the trivial solution $u = 0$ of (2) has each one of the $\phi(t)$-stability properties in Definition 1.5.*

*Proof.* Suppose that the trivial solution $u = 0$ of (2) is $\phi(t)$-equistable. Let $0 < \varepsilon < \rho$ be arbitrarily given and $t_0 \in R_+$. Then there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that $(\phi(t_0), u_0) < \delta$ implies $(\phi(t), r(t)) < b(\varepsilon)$ for all $t \ge t_0$ where $r(t)$ be a maximal solution of (2) relative to $K$. For given $x_0 = x(t_0) \in S_\rho$, we can take $u_0 = u(t_0)$ in K such that $a(\|x(t_0)\|) = (\phi(t_0), u(t_0))$ and $V(t_0, x(t_0)) \le_K u_0$.

Note that if $x(t, t_0, x_0)$ is any solution of (1) such that $V(t_0, x(t_0)) \le_K u_0$, then by Lemma 1.6, $V(t, x(t)) \le_K r(t)$.

From (iii), we may assume that $V(t, 0) = 0$. Suppose $u_0 \in K^0$ and $(\phi(t_0), u_0) < \delta$. Since $V(t, x)$ is continuous in $x$, there exist $\bar{\delta}(u_0) > 0$ such that $V(t_0, x_0) \le_K u_0$ for any $\|x_0\| < \bar{\delta}$.
Now choose $\delta_1 > 0$ such that $a(\delta_1) \le \delta$ and $\delta_1 \le \bar{\delta}$. Then the inequalities $\|x(t_0)\| < \delta_1$ and $a(\|x(t_0)\|) < \delta$ hold simultaneously. Since $b(\|x(t)\|) \le (\phi(t), V(t, x(t))) \le (\phi(t), r(t)) < b(\varepsilon)$ for all $t \ge t_0$, $\|x(t; t_0, x_0)\| < \varepsilon$ whenever $\|x(t_0)\| < \delta_1$. Hence the trivial solution $x = 0$ of (1) is equistable.

In the above, choosing $\delta = \delta(\varepsilon)$ which is independent of $t_0$, the uniform stability follows from the same argument.

Suppose that the trivial solution $u = 0$ of (2) is quasi-equi asymptotically $\phi(t)$-stable. Then, following the same arguments for all $t \geq t_0 + T(\varepsilon)$, there exists a positive function $\delta = \delta(t_0, \varepsilon) < \varepsilon$ satisfying $\|x_0\| < \delta$ and $a(\|x_0\|) < \delta$ simultaneously. It follows that $\|x_0\| < \delta$ implies $\|x(t, t_0, x_0)\| < \varepsilon, t \geq t_0 + T(\varepsilon)$. If this is not true, then there exists a divergent sequence $\{t_k\}$, $t_k \geq t_0 + T$ such that $\|x(t_k, t_0, x_0)\| = \varepsilon$ for some $\|x_0\| < \delta$, $k = 1, 2, \cdots$. Using (iii) and Lemma 1.6 we have $b(\varepsilon) \leq (\phi(t_k), V(t_k, x(t_k, t_0, x_0)) \leq (\phi(t_k), r(t_k, t_0, u_0)) < b(\varepsilon)$ for some $u_0 \in K$ which is a contradiction. The other stability properties can be similarly proved. □

Now we investigate sufficient conditions for the existence of cone-valued Lyapunov functions.

THEOREM 2.6. *Assume that*
(i) $f \in C[R_+ \times S_\rho, R^n]$, $f(t, 0) = 0$, *and* $f(t, x)$ *satisfies a Lipschitz condition in* $x$ *such that* $\|f(t, x) - f(t, y)\| \leq L_1(t)\|x - y\|$, $(t, x), (t, y) \in R_+ \times S_\rho$ *with* $\theta > 0, t \geq 0$, $\int_t^{t+\theta} L(s)ds \leq N\theta$ *for some constant* $N > 0$.
(ii) *The solution* $x(t, 0, x_0)$ *of* (1) *satisfies that any* $x_0 \in S_\rho$,

$$(5) \qquad \beta_1(\|x_0\|) \leq \|x(t, 0, x_0)\| \leq \beta_2(\|x_0\|), \qquad \text{for some } \beta_1, \beta_2 \in \mathcal{K}.$$

(iii) $g \in C[R_+ \times K, R^n]$, $g(t, 0) = 0$, *and* $g(t, u)$ *satisfies a Lipschitz condition in* $u$ *such that* $\|g(t, u) - g(t, v)\| \leq L_2(t)\|u - v\|$, $(t, u), (t, v) \in R_+ \times K$.
(iv) *The solution* $u(t, 0, u_0)$ *of* (2) *satisfies that*

$$(6) \qquad \gamma_1[(\phi(t), u_0)] \leq (\phi(t), u(t, 0, u_0)) \leq \gamma_2[(\phi(t), u_0)], \ t \geq 0$$

*for some* $\phi(t) \in K_0^*$, *some* $\gamma_1$, $\gamma_2 \in \mathcal{K}$.

*Then there exists a cone-valued function* $V$ *with the properties*
(a) $V \in C[R_+ \times S_\rho, K]$, $V(t, x)$ *is locally Lipschitzian in* $x$ *for a continuous function* $\beta(t) > 0$.
(b) $D^+V(t, x) \leq_K g(t, V(t, x))$.
(c) $b(\|x\|) \leq (\phi(t), V(t, x)) \leq a(\|x\|)$, *for some* $a, b \in \mathcal{K}$.

*Proof.* From (i) and (iii), the existence and uniqueness of solutions of (1) and (2) as well as their continuous dependence on the initial values are followed.

Let $x(t,0,x_0)$, $u(t,0,u_0)$ be the solutions of (1) and (2) passing through the points $(0,x_0)$ and $(0,u_0)$ satisfying (5) and (6), respectively.

Let us choose a function $G(r)$ such that $G(0) = 0$, $G'(0) = 0$, $G(r) > 0$, $G''(r) > 0$ for $r > 0$, and let $\alpha > 1$,

$$G(r) = \int_0^r du \int_o^u G''(v)dv \quad \text{and} \quad G(\frac{r}{\alpha}) = \int_0^{r/\alpha} du \int_0^u G''(v)dv,$$

we have, setting $u = w/\alpha$,

$$G(\frac{r}{\alpha}) = \frac{1}{\alpha}\int_0^r dw \int_0^{w/\alpha} G''(v)dv < \frac{1}{\alpha}\int_0^r dw \int_0^w G''(v)dv = \frac{1}{\alpha}G(r).$$

Let $w$ be any given point in $K_0$. Let $\sigma_w : S_\rho \longrightarrow K$ be a function with values in the cone $K \subset R^n$, defined by for $x \in S_\rho$,

$$\sigma_w(x) = \sup_{\delta \geq 0} G(\|x(t+\delta,t,x)\|)(\frac{1+\alpha\delta}{1+\delta})w. \tag{7}$$

For $\delta = 0$, we have from (7) that $G(\|x\|)w \leq_K \sigma_w(x)$, and $(\phi(t), G(\|x\|)w) \leq (\phi(t), \sigma_w(x))$ for any $\phi(t) \in K_0^*$ and each $t \geq 0$. Suppose that $\eta_1 = \inf\{(\phi(t),w) : t \geq 0\} > 0$. Let $\beta_3(r) = \eta_1 G(r), r > 0$. Then $\beta_3(\|x\|) \leq (\phi(t),\sigma_w(x))$, for each $\varepsilon > 0$, let $\delta = \beta_2^{-1}(\varepsilon)$. From the estimate (5), $\|x_0\| < \delta$ implies $\|x(t)\| < \beta_2(\|x_0\|) < \beta_2(\delta) = \beta_2(\beta_2^{-1}(\varepsilon)) = \varepsilon, t \geq 0$.

Hence the solution $x = 0$ of (1) is uniformly stable. Thus by Theorem 5.4.3 in [13], $\|x(t+\delta,t,x)\| < c(\|x\|)$, $c \in \mathcal{K}$. Therefore $G(\|x(t+\delta,t,x)\|) < G(c(\|x\|))$. Since $(1+\alpha\delta)/(1+\delta) < \alpha$, it follow that

$$\begin{aligned}(\phi(t),\sigma_w(x)) &\leq (\phi(t), \alpha G(c(\|x\|))w) \\ &\leq \eta_2 \alpha G(c(\|x\|)).\end{aligned}$$

Suppose that $\eta_2 = \sup\{(\phi(t),w) : t \geq 0\} < \infty$. Hence if $\beta_4(r) = \eta_2\alpha G(c(r))$, then

$$\beta_3(\|x\|) \leq (\phi(t),\sigma_w(x)) \leq \beta_4(\|x\|). \tag{8}$$

We now show that $\sigma_w(x)$ is locally Lipschitzian in $x$.

Define $V_0(t,x) = \sup_{\delta \geq 0} G(\|x(t+\delta,t,x)\|)((1+\alpha\delta)/(1+\delta))$. Then, for $\delta = 0$, we obtain $G(\|x\|) \leq V_0(t,x)$ and $V_0(t,x)$ is locally Lipschitzian in $x$, by Theorem 3.6.9 in [8],

i.e., $x, y \in S_\rho$, $|V_0(t,x) - V_0(t,y)| \leq l(t)\|x-y\|$ for each $t$.

$$\begin{aligned}\|\sigma_w(x) - \sigma_w(y)\| &= \|\sup_{\delta\geq 0} G(\|x(t+\delta,t,x)\|)(\frac{1+\alpha\delta}{1+\delta})w \\ &\quad - \sup_{\delta \geq 0} G(\|y(t+\delta,t,y)\|)(\frac{1+\alpha\delta}{1+\delta})w\| \\ &\leq \|w\||V_0(t,x) - V_0(t,y)| \leq \|w\|l(t)\|x-y\|.\end{aligned}$$

Then $\sigma_w(x)$ is also locally Lipschitzian in $x$.

Define a cone-valued function $V(t,x)$ by, $t \geq 0, x \in S_\rho$,

$$V(t,x) = u(t,0,\sigma_w(x(t,0,x))), \tag{9}$$

where $u(t,0,u_0)$ are solutions of (2) passing through $(0,u_0)$. By hypotheses (i) and (iii) and the choice of $\sigma_w(x)$, $V(t,x)$ is continuous in $t$ and $x$.

From conditions (i), (iii), and Corollary 2.7.1 in [8], we obtain, $x, y \in S_\rho$,

$$\begin{aligned}\|V(t,x) - V(t,y)\| &= \|u(t,0,\sigma_w(x(t,0,x))) - u(t,0,\sigma_w(x(t,0,y)))\| \\ &\leq \|\sigma_w(x(t,0,x)) - \sigma_w(x(t,0,y))\| \exp \int_0^t L_2(s)ds \\ &\leq l(t)\|w\|\|x(t,0,x) - x(t,0,y)\| \exp \int_0^t L_2(s)ds \\ &\leq l(t)\|w\|\|x-y\| \exp \int_0^t L_1(s)ds \exp \int_0^t L_2(s)ds \\ &= \beta(t)\|x-y\|,\end{aligned}$$

where $\beta(t) = l(t)\|w\| \exp[\int_0^t (L_1(s) + L_2(s))ds] > 0$, which implies that $V$ satisfies a local Lipschitz condition.

Next, for $h > 0$ sufficiently small,

$$\begin{aligned}
&V(t+h, x+hf(t,x)) - V(t,x) \\
&\leq_K \beta(t)\|x+hf(t,x) - x(t+h,t,x)\|e(t,x,h) \\
&+ V(t+h, x(t+h,t,x)) - V(t,x), \quad \text{where } \limsup_{t\to\infty} \frac{1}{h} e(t,x,h) = 0.
\end{aligned}$$

Divide both sides by $h$ and take limsup as $h \to 0^+$ to obtain

$$\begin{aligned}
&D^+V(t,x) \\
&\leq_K \limsup_{h\to 0^+} [\frac{1}{h} V(t+h, x(t+h,t,x)) - V(t,x)] \\
&= \limsup_{h\to 0^+} \frac{1}{h}[u(t+h, 0, \sigma_w(x(t+h,0,x))) - u(t,0,\sigma_w(x(t,0,x)))] \\
&= u'(t,0,\sigma_w(x(t,0,x))) = g(t, V(t,x)).
\end{aligned}$$

Now from (5), we can take $\beta_1, \beta_2 \in \mathcal{K}$ which satisfy (5) and (10) simultaneously,

$$\beta_2^{-1}(\|x\|) \leq \|x(t,0,x)\| \leq \beta_1^{-1}(\|x\|). \tag{10}$$

Since $(\phi(t), V(t,x)) = (\phi(t), u(t,0,\sigma_w(x(t,0,x)))$ from (6), (8), and (10) we have

$$\begin{aligned}
(\phi(t), V(t,x) &\geq \gamma_1((\phi(t), \sigma_w(x(t,0,x)))) \\
&\geq \gamma_1(\beta_3(\|x(t,0,x)\|)) \\
&\geq \gamma_1(\beta_3(\beta_2^{-1}[\|x\|])) \equiv b(\|x\|), \quad b \in \mathcal{K}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
(\phi(t), V(t,x) &\leq \gamma_2((\phi(t), \sigma_w(x(t,0,x)))) \\
&\leq \gamma_2(\beta_4(\|x\|)) \\
&\leq \gamma_2(\beta_4(\beta_1^{-1}(\|x\|))) \equiv a(\|x\|), \quad a \in \mathcal{K}.
\end{aligned}$$

This completes the proof of Theorem 2.6. □

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