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QUASILINEARIZATION FOR SECOND ORDER SINGULAR BOUNDARY VALUE PROBLEMS WITH SOLUTIONS IN WEIGHTED SPACES

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Abstract. In this paper, we develop the method of quasilinearization combined with the method of upper and lower solutions for singular second order boundary value problems in weighted spaces. The sequences constructed converge uniformly and monotonically to the unique solution of the second singular order boundary value problem. Further we prove the rate of convergence is quadratic.

1. Introduction

The study of second order singular mixed boundary value problem has received much attention due to its application [3]. Also in [3] the motivation to look for existence of solutions to singular boundary value problems in Weighted Banach Spaces is presented. In this paper, we develop the method of generalized quasilinearization [1, 4] to singular boundary value problems in weighted spaces using upper and lower solutions. The method yields two monotone sequences which are solutions of linear singular boundary value problems in weighted spaces. Further, the sequences converge quadratically to the unique solution of the nonlinear singular boundary value problem in weighted spaces. For this purpose, we have developed a comparison theorem [2] for second order singular boundary value problems. The results developed here paves way to study generalized quasilinearization method for singular problems in many different settings.

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2. Preliminaries

Consider the singular boundary value problem (SBVP) of 2nd order

$$(2.1)\qquad \begin{cases} -(t^n y')' = f(t,\, t^{n-1}y), \quad 0 < t < 1 \\ y(1) = y_1,\ \lim\limits_{t\to 0^+} t^{n-1}y(t) = y_0 \quad \text{with} \quad n > 2 \end{cases}$$

where $f : I \times R \to R$ is continuous, $I = [0,\, 1]$.

DEFINITION 2.1. By a solution of the SBVP (2.1) we mean a function $y \in C^2\,[(0,\, 1],\, R]$ with $t^{n-1}y \in C\,[[0,\, 1],\, R]$, and $t^n y' \in C\,[[0,\, 1],\, R]$ and $(t^{n-1}y)' \in L^1[0,\, 1]$ which satisfies the differential equation and the given boundary conditions.

Note: Whenever we write $y(t)$ is a solution of the SBVP (2.1) we mean that $t^{n-1}y(t)$ is a solution of SBVP (2.1). Similar notation is used for lower and upper solutions.

Throughout this paper, we work in the Banach space of functions

$E = \left\{y \in C^1\,[(0,1]\,,\, R]\,\left|x^{n-1}y\right| \in C\,[[0,\, 1],\, R] \text{ and } |x^n y'| \in C\,[[0,\, 1],\, R]\right\}$

with the norm

$$|y|_0 = \text{Max}\left\{\sup_{x\in[0,\,1]}\left|x^{n-1}y(x)\right|,\ \sup_{x\in[0,\,1]}|x^n y'(x)|\right\}.$$

We now state the existence theorem for (2.1) from [3] which we need in our work.

THEOREM 2.1. *Assume*
(i) *$f : [0,\, 1] \times R \to R$ is continuous.*
(ii) *There exist an upper solution β and a lower solution α of SBVP (2.1) with $t^{n-1}\,\alpha(t) \le t^{n-1}\,\beta(t)$ on $(0,\, 1)$, $\alpha(1) \le y_1 \le \beta(1)$ and*

$$\lim_{t\to 0^+} t^{n-1}\alpha(t) \le y_0 \le \lim_{t\to 0^+} t^{n-1}\beta(t).$$

Then the SBVP (2.1) has a solution $y \in E$ with $t^{n-1}\alpha(t) \le t^{n-1}y(t) \le t^{n-1}\beta(t)$ for $t \in [0,\, 1]$, where

$$(2.2)\qquad \begin{aligned} y(t) &= y_1 - y_0 + \tfrac{y_0}{t^{n-1}} - \tfrac{(1-t^{n-1})}{(n-1)t^{n-1}} \textstyle\int_0^t f\left(s,\, s^{n-1}y(s)\right) ds \\ &-\tfrac{1}{n-1}\textstyle\int_t^1 \tfrac{1-s^{n-1}}{s^{n-1}} f\left(s,\, s^{n-1}y(s)\right)\, ds. \end{aligned}$$

3. Main results

We first proceed to prove the comparison theorem which is an essential tool in our work.

THEOREM 3.1. *Suppose that there exist $y, z \in E$ such that*

$$\begin{cases} -(t^n y')' \le f(t, t^{n-1}y) \\ -(t^n z')' \ge f(t, t^{n-1}z) \end{cases} \tag{3.1}$$

hold with

$$\begin{cases} \lim\limits_{t\to 0^+} t^{n-1}y(t) \le \lim\limits_{t\to 0^+} t^{n-1}z(t) \\ \qquad\qquad \text{and } y(1) \le z(1). \end{cases} \tag{3.2}$$

Further, assume that $f : [0,\, 1] \times R \to R$ is continuous and satisfies

$$f\left(t,\, t^{n-1}x\right) - f\left(t,\, t^{n-1}y\right) \le -Lt^{n-1}(x-y), \tag{3.3}$$

whenever, $x \ge y$ and $L > 0$.

Then $t^{n-1}y(t) \le t^{n-1}z(t)$ on $[0,\, 1]$.

Proof. Suppose the conclusion fails, i.e. $t^{n-1}y(t) > t^{n-1}z(t)$ at some $t \in (0,\, 1]$. Then by continuity, we must have a local maximum at $t_0 \in (0,\, 1)$. This implies that

$$\left[t^{n-1}(y-z)\right]' = 0 \text{ at } t = t_0 \tag{3.4}$$

and

$$\left[t^{n-1}(y-z)\right]'' \le 0 \text{ at } t = t_0\,. \tag{3.5}$$

Now considering the relation (3.5), and differentiating it twice and simplifying using the relation (3.4), we obtain

$$\begin{aligned} 0 &\ge \left[t^{n-1}(y-z)\right]'' \\ &= \frac{1}{t}\,(t^n y')' - \frac{1}{t}\,(t^n z')' \end{aligned}$$

which yields the inequality

$$-\frac{1}{t}\,(t^n y')' \ge -\frac{1}{t}\,(t^n z')' \quad \text{at } t = t_0 \in (0,\, 1].$$

Using the relation (3.1) in the above inequality at $t = t_0$, we get

$$f\left(t_0,\, t_0^{n-1}y\right) \ge -\left(t_0^n y'\right)' \ge -\left(t_0^n z'\right)' \ge f\left(t_0,\, t_0^{n-1}z\right)$$

which implies,

$$0 \leq f\left(t_0,\, t_0^{n-1}y\right) - f\left(t_0,\, t_0^{n-1}z\right) \leq -Lt_0^{n-1}(y-z) < 0$$

a contradiction. Thus the proof is complete. □

We next present the quasilinearization method which guarantees quadratic convergence of a sequence of functions to a solution g of the given SBVP.

THEOREM 3.2 *Assume that*
(i) $\alpha_0,\, \beta_0 \in E$ *are lower and upper solutions of the SBVP* (2.1) *such that* $t^{n-1}\alpha_0(t) \leq t^{n-1}\beta_0(t)$ *on* I *with*

$$\begin{cases} \lim\limits_{t\to 0^+} t^{n-1}\alpha_0(t) \leq y_0 \leq \lim\limits_{t\to 0^+} t^{n-1}\beta_0(t) \\ \quad \text{and} \quad \alpha_0(1) \leq y_1 \leq \beta_0(1); \end{cases} \tag{3.6}$$

(ii) $f(t,\, t^{n-1}y)$ *is convex* $(a) f_{uu}$ *exists, continuous and* $f_{uu}(t,u) > 0$ *for every* t, u *on* Ω, *where* Ω *is given by*

$$\Omega = \{(t,u); 0 < t \leq 1,\ \text{and}\ \alpha_0 \leq u \leq \beta_0\};$$

(iii) $f_u(t,\, t^{n-1}y) < 0$ *on* Ω.
Then there exists monotone sequences $\{t^{n-1}\alpha_k(t)\}$, $\{t^{n-1}\beta_k(t)\}$ *with*

$$\begin{aligned} t^{n-1}\alpha_0(t) &\leq t^{n-1}\alpha_1(t) \leq \cdots \leq t^{n-1}\alpha_n(t) \leq t^{n-1}\beta_n(t) \\ &\leq \cdots \leq t^{n-1}\beta_1(t) \leq t^{n-1}\beta_0(t), t \in [0,\, 1] \end{aligned}$$

which converges uniformly and quadratically to the unique solution $t^{n-1}\, y(t)$ *of the SBVP* (2.1).

Proof. Integrating the assumption (ii) twice, we get

$$f\left(t,\, t^{n-1}x\right) \geq f(t,\, t^{n-1}y) + f_u(t,\, t^{n-1}y)(x-y)t^{n-1}, x \geq y. \tag{3.7}$$

Consider the singular BVP (SBVP)

$$\begin{cases} -\left(t^n\alpha_1'\right)' = F\left(t,\, t^{n-1}\alpha_1;\, \alpha_0\right) \\ \text{with}\ \ \alpha_1(1) = y_1 \text{ and } \lim\limits_{t\to 0^+} t^{n-1}\alpha_1(t) = y_0, \end{cases} \tag{3.8}$$

where $F\left(t,\, t^{n-1}\alpha_1,\, \alpha_0\right) = f\left(t,\, t^{n-1}\alpha_0\right) + f_u\left(t,\, t^{n-1}\alpha_0\right)\left(\alpha_1 - \alpha_0\right) t^{n-1}$.

Since α_0 and β_0 are lower and upper solutions of SBVP (2.1) with

$$t^{n-1}\alpha_0(t) \leq t^{n-1}\beta_0(t),$$

using the relation (3.7), we get

$$\begin{aligned} -\left(t^n\alpha_0'\right)' &\le f\left(t,\, t^{n-1}\alpha_0\right) \\ &= F\left(t,\, t^{n-1}\alpha_0;\, \alpha_0\right) \end{aligned}$$

and

$$\begin{aligned} -\left(t^n\beta_0'\right)' &\ge f\left(t,\, t^{n-1}\beta_0\right) \\ &\ge f\left(t,\, t^{n-1}\alpha_0\right) + f_u\left(t,\, t^{n-1}\alpha_0\right)\left(\beta_0 - \alpha_0\right) t^{n-1} \\ &= F\left(t,\, t^{n-1}\beta_0;\, \alpha_0\right). \end{aligned}$$

The above two inequalities, along with relation (3.6) imply that α_0 and β_0 are lower and upper solutions for the SBVP (3.8). Furthermore, α_0 and β_0 satisfy the hypothesis of Theorem 2.1, hence we can conclude that there exists a function say $\alpha_1(t) \in E$ such that $\alpha_1(t)$ is a solution of the SBVP (3.8) with

$$t^{n-1}\alpha_0(t) \le t^{n-1}\alpha_1(t) \le t^{n-1}\beta_0(t), \qquad t \in [0,\, 1].$$

We now claim that this solution is unique. The proof is as follows. If possible, suppose that $\overline{\alpha_1}\,(t)$ is another solution of the SBVP (3.8). Then

$$(3.9) \qquad \begin{cases} -\left(t^n\overline{\alpha_1}'\,(t)\right)' = F\left(t,\, t^{n-1}\overline{\alpha_1};\, \alpha_0\right) \\ \text{with } \overline{\alpha_1}(1) = y_1 \text{ and } \lim_{t\to 0^+} t^{n-1}\overline{\alpha_1}(t) = y_0. \end{cases}$$

Now writing (3.8) and (3.9) as inequalities

$$(3.10) \qquad \begin{cases} -\left(t^n\overline{\alpha_1}'\right)' \ge F\left(t,\, t^{n-1}\overline{\alpha_1};\, \alpha_0\right) \\ \text{and} \\ -\left(t^n\alpha_1'\right)' \le F\left(t,\, t^{n-1}\alpha_1;\, \alpha_0\right) \end{cases}$$

along with boundary conditions and observing that the function F satisfies the condition (3.3), we apply the comparison theorem and obtain the relation $t^{n-1}\overline{\alpha_1}(t) \ge t^{n-1}\alpha_1(t)$, $t \in [0,\, 1]$.

Now reversing the above inequalities in (3.10) and applying the comparison theorem again, we get

$$t^{n-1}\overline{\alpha_1}(t) \le t^{n-1}\alpha_1(t), \;\; t \in [0,\, 1].$$

The above inequalities together imply the uniqueness of the solution $t^{n-1}\alpha_1(t)$ of the SBVP (3.8). We proceed next to show that there exists a unique solution $\beta_1(t)$ for the SBVP

$$(3.11)\qquad \begin{cases} -\left(t^n\beta_1'\right)' = G\left(t,\, t^{n-1}\beta_1;\, \alpha_0,\, \beta_0\right) \\ \text{with } \lim_{t\to 0^+} t^{n-1}\beta_1(t) = y_0,\ \beta_1(1) = y_1 \end{cases}$$

such that $t^{n-1}\alpha_0(t) \leq t^{n-1}\beta_1(t) \leq t^{n-1}\beta_0(t)$; where $G\left(t,\, t^{n-1}\beta_1;\ \alpha_0,\, \beta_0\right) = f\left(t,\, \beta_0\right) + f_u\left(t,\, t^{n-1}\alpha_0\right)\left(\beta_1 - \beta_0\right)t^{n-1}$.

The fact that α_0, β_0 are lower and upper solutions of SBVP (2.1), together with the relation (3.7) imply that

$$\begin{aligned} -\left(t^n\alpha_0'\right)' &\leq f\left(t,\, t^{n-1}\alpha_0\right) \\ &\leq f\left(t,\, t^{n-1}\beta_0\right) - f_u\left(t,\, t^{n-1}\alpha_0\right)\left(\beta_0 - \alpha_0\right)t^{n-1} \\ &= f\left(t,\, t^{n-1}\beta_0\right) + f_u\left(t,\, t^{n-1}\alpha_0\right)\left(\alpha_0 - \beta_0\right)t^{n-1} \\ &= G\left(t,\, t^{n-1}\alpha_0;\, \alpha_0,\, \beta_0\right) \end{aligned}$$

and

$$\begin{aligned} \left(-t^n\beta_0'\right)' &\geq f\left(t,\, t^{n-1}\beta_0\right) \\ &= G\left(t,\, t^{n-1}\beta_0;\ \alpha_0,\, \beta_0\right). \end{aligned}$$

The above inequalities, along with relation (3.6) yield that α_0, β_0 are also lower and upper solutions for the SBVP (3.11). Now using Theorem 2.1 we get the existence of a solution of SBVP (3.11). The proof of uniqueness is similar to the uniqueness proof of the solution of the SBVP (3.8). Thus we have $\beta_1 \in E$ such that $\beta_1(t)$ is the unique solution of SBVP (3.11).

The proof of $t^{n-1}\alpha_1(t) \leq t^{n-1}\beta_1(t)$ is as follows.

We know that $t^{n-1}\alpha_0 \leq t^{n-1}\beta_0$. Using it in relation (3.7), we get

$$\begin{aligned} -\left(t^n\alpha_1'\right)' &= F\left(t,\, t^{n-1}\alpha_1;\, \alpha_0\right) \\ &= f\left(t,\, t^{n-1}\alpha_0\right) + f_u\left(t,\, t^{n-1}\alpha_0\right)\left(\alpha_1 - \alpha_0\right)t^{n-1} \\ &\leq f(t,\, t^{n-1}\beta_0) - f_u(t,\, t^{n-1}\alpha_0)(\beta_0 - \alpha_0)t^{n-1} \\ &\quad + f_u(t,\, t^{n-1}\alpha_0)(\alpha_1 - \alpha_0)t^{n-1} \\ &= f(t,\, t^{n-1}\beta_0) + f_u(t,\, t^{n-1}\alpha_0)(\alpha_1 - \beta_0)t^{n-1} \\ &= G(t,\, t^{n-1}\alpha_1;\, \alpha_0,\, \beta_0). \end{aligned}$$

Since G satisfies the relation (3.3), the above inequality together with the SBVP (3.11) and the boundary conditions satisfy the hypothesis of the comparison theorem. Hence, we have $t^{n-1}\alpha_1(t) \leq t^{n-1}\beta_1(t)$, $t \in [0,\, 1]$.

Working in a similar fashion as above, we now proceed to show that for any $k > 1$

$$t^{n-1}\alpha_k(t) \leq t^{n-1}\alpha_{k+1}(t) \leq t^{n-1}\beta_{k+1}(t) \leq t^{n-1}\beta_k(t),$$

where $\alpha_k(t)$, $\beta_k(t)$ are known.

Consider the SBVP

$$\begin{cases} -(t^n\alpha_{k+1}')' = F(t,\, t^{n-1}\alpha_{k+1},\, \alpha_k) \\ \text{with } \lim\limits_{t\to 0^+} t^{n-1}\alpha_{k+1}(t) = y_0 \text{ and } \alpha_{k+1}(1) = y_1 \end{cases} \tag{3.12}$$

where

$$F(t,\, t^{n-1}\alpha_{k+1},\, \alpha_k) \equiv f(t,\, t^{n-1}\alpha_k) + f_u(t,\, t^{n-1}\alpha_k)(\alpha_{k+1} - \alpha_k)t^{n-1}.$$

Using the fact that $t^{n-1}\alpha_{k-1} \leq t^{n-1}\alpha_k$ in the relation (3.7) and substituting in the equality below, we get

$$\begin{aligned} -(t^n\alpha_k')' &= F(t,\, t^{n-1}\alpha_k,\, \alpha_{k-1}) \\ &= f(t,\, t^{n-1}\alpha_{k-1}) + f_u(t,\, t^{n-1}\alpha_{k-1})(\alpha_k - \alpha_{k-1})t^{n-1} \\ &\leq f(t,\, t^{n-1}\alpha_k) \\ &= F(t,\, t^{n-1}\alpha_k,\, \alpha_k). \end{aligned}$$

Since $t^{n-1}\alpha_0 \leq t^{n-1}\alpha_k$ and f_u is an increasing function in u, we have $f_u(t,\, t^{n-1}\alpha_0) \leq f_u\,(t,\, t^{n-1}\alpha_k)$. Using this in relation (3.7) and as $t^{n-1}\alpha_k \leq t^{n-1}\beta_k \leq t^{n-1}\beta_{k-1}$, the equality below reduces to

$$\begin{aligned} -(t^n\beta_k')' &= G(t,\, t^{n-1}\beta_k;\, \alpha_0,\, \beta_{k-1}) \\ &= f(t,\, t^{n-1}\beta_{k-1}) + f_u(t,\, t^{n-1}\alpha_0)(\beta_k - \beta_{k-1})t^{n-1} \\ &\geq f(t,\, t^{n-1}\alpha_k) + f_u(t,\, t^{n-1}\alpha_k)(\beta_{k-1} - \alpha_k)t^{n-1} \\ &\quad - f_u(t,\, t^{n-1}\alpha_0)(\beta_{k-1} - \beta_k)t^{n-1} \\ &\geq f(t,\, t^{n-1}\alpha_k) + f_u(t,\, t^{n-1}\alpha_k)(\beta_k - \alpha_k)t^{n-1} \\ &= F(t,\, t^{n-1}\beta_k;\, \alpha_k). \end{aligned}$$

Thus we get the inequalities

$$-(t^n\alpha_k')' \leq F(t,\, t^{n-1}\alpha_k;\, \alpha_k)$$

and

$$-(t^n\beta_k')' \geq F(t,\, t^{n-1}\beta_k;\, \alpha_k)$$

along with the boundary conditions.

Now discussing as earlier, we obtain that there is a unique solution $\alpha_{k+1}(t) \in E$ for the SBVP (3.12) such that

$$t^{n-1}\alpha_k(t) \leq t^{n-1}\alpha_{k+1}(t) \leq t^{n-1}\beta_k(t), \ t \in [0, 1]. \tag{3.13}$$

Next consider the SBVP

$$\begin{cases} -(t^n\beta'_{k+1})' = G(t, t^{n-1}\beta_{k+1}; \alpha_0, \beta_k) \\ \text{with } \lim_{t\to 0^+} \text{lim} t^{n-1}\beta_{k+1}(t) = y_0 \text{ and } \beta_{k+1}(t) = y_1. \end{cases} \tag{3.14}$$

Again using the relation (3.7), f_u is increasing in u, and the inequalities $t^{n-1}\alpha_0 \leq t^{n-1}\alpha_{k-1} \leq t^{n-1}\alpha_k \leq t^{n-1}\beta_k \leq t^{n-1}\beta_{k-1} \leq t^{n-1}\beta_0$ in the SBVP (3.14), we deduce as follows

$$\begin{aligned} -(t^n\beta'_k)' &= G(t, t^{n-1}\beta_k; \alpha_0, \beta_{k-1}) \\ &= f(t, t^{n-1}\beta_{k-1}) + f_u(t, t^{n-1}\alpha_0)(\beta_k - \beta_{k-1})t^{n-1} \\ &\geq f(t, t^{n-1}\beta_k) + f_u(t, t^{n-1}\beta_k)(\beta_{k-1} - \beta_k)t^{n-1} \\ &\quad -f_u(t, t^{n-1}\alpha_0)(\beta_{k-1} - \beta_k)t^{n-1} \\ &\geq f(t, t^{n-1}\beta_k) \\ &= G(t, t^{n-1}\beta_k; \alpha_0, \beta_k). \end{aligned}$$

Now considering

$$\begin{aligned} -(t^n\alpha'_k)' &= F(t, t^{n-1}\alpha_k, \alpha_{k-1}) \\ &= f(t, t^{n-1}\alpha_{k-1}) + f_u(t, t^{n-1}\alpha_{k-1})(\alpha_k - \alpha_{k-1})t^{n-1} \\ &\leq f(t, t^{n-1}\beta_k) + f_u(t, t^{n-1}\alpha_{k-1}) \times \\ &\quad \left[-(\beta_k - \alpha_{k-1})t^{n-1} + (\alpha_k - \alpha_{k-1})t^{n-1}\right] \\ &= f(t, t^{n-1}\beta_k) - f_u(t, t^{n-1}\alpha_{k-1})(\beta_k - \alpha_k)t^{n-1} \\ &\leq f(t, t^{n-1}\beta_k) - f_u(t, t^{n-1}\alpha_0)(\beta_k - \alpha_k)t^{n-1} \\ &= f(t, t^{n-1}\beta_k) + f_u(t, t^{n-1}\alpha_0)(\alpha_k - \beta_k)t^{n-1} \\ &= G(t, t^{n-1}\alpha_k; \alpha_0, \beta_k). \end{aligned}$$

Thus we get the inequalities

$$-(t^n\beta'_k)' \geq G(t, t^{n-1}\beta_k; \alpha_0, \beta_k)$$

$$-(t^n\alpha'_k)' \leq G(t, t^{n-1}\alpha_k; \alpha_0, \beta_k)$$

together with the boundary conditions.

Now using the existence Theorem 2.1, we conclude that $\beta_{k+1}(t)$ is the unique solution of (3.4) such that $t^{n-1}\alpha_k(t) \leq t^{n-1}\beta_{k+1}(t) \leq t^{n-1}\beta_k(t)$.

Now consider the SBVPs (3.13) and (3.14). Working in a similar fashion as in the proof of $t^{n-1}\alpha_1(t) \le t^{n-1}\beta_1(t)$ we can show that $t^{n-1}\alpha_{k+1}(t) \le t^{n-1}\beta_{k+1}(t)$.

Thus we have proved

$$
\begin{aligned}
t^{n-1}\alpha_0(t) \ &\le t^{n-1}\alpha_1(t) \ \le \cdots \le t^{n-1}\alpha_{k+1}(t) \le t^{n-1}\beta_{k+1}(t) \\
&\le \cdots \le t^{n-1}\beta_1(t) \le t^{n-1}\beta_0(t), \quad t \in [0,\, 1].
\end{aligned} \tag{3.15}
$$

These monotone sequences $\{t^{n-1}\alpha_{k+1}(t)\}$ and $\{t^{n-1}\beta_{k+1}(t)\}$ are both bounded and hence they converge to $t^{n-1}\rho(t)$ and $t^{n-1}\gamma(t)$ point-wise respectively.

Next we show that $\{t^{n-1}\alpha_{k+1}(t)\}$ converges uniformly to $t^{n-1}\rho(t)$. The proof for the convergence of the other sequence is similar.

Using the SBVP (3.12), the relation (3.15) and the integral equation (2.2) it is easy to observe that $\left\{\ t^{n-1}\ \alpha_{k+1}\ (t)\ \right\}$, $\left\{\ (t^{n-1}\alpha_{k+1}(t))'\ \right\}$, $\left\{t^n\alpha'_{k+1}(t)\right\}$, and $\left\{\left(t^n\alpha'_{k+1}(t)\right)'\right\}$ are uniformly bounded sequences. Hence by Ascoli's theorem, we have that the sequence $\{t^{n-1}\alpha_{k+1}(t)\}$, $\left\{t^n\alpha'_{k+1}(t)\right\}$ have uniform convergent subsequences. Since $\{t^{n-1}\alpha_{k+1}(t)\}$ is a monotone sequence in E, it follows that the sequence is uniformly convergent in E. Thus $\{t^{n-1}\alpha_{k+1}(t)\}$ converges uniformly to a solution $t^{n-1}\rho(t)$ of the SBVP (2.1).

We now claim that this convergence is quadratic in nature. For proof, set

$$
p_{k+1}(t) = \rho(t) - \alpha_{k+1}(t).
$$

Then

$$
\begin{aligned}
& -\left(t^n p'_{k+1}(t)\right)' \\
= \ & -\left(t^n\rho'(t)\right)' + \left(t^n\alpha'_{k+1}(t)\right)' \\
= \ & f\left(t,\, t^{n-1}\rho\right) - f\left(t,\, t^{n-1}\alpha_k\right) - f_u\left(t,\, t^{n-1}\alpha_k\right)\left(\alpha_{k+1} - \alpha_k\right) t^{n-1} \\
= \ & f_u(t,\, t^{n-1}\xi)(\rho - \alpha_k)t^{n-1} - f_u\left(t,\, t^{n-1}\alpha_k\right)\left(\alpha_{k+1} - \alpha_1\right) t^{n-1}, \\
\le \ & f_u(t,\, t^{n-1}\rho)(\rho - \alpha_k)t^{n-1} - f_u(t,\, t^{n-1}\alpha_k)(\rho - \alpha_k)t^{n-1} \\
& + f_u(t,\, t^{n-1}\alpha_k)(\rho - \alpha_{k+1})t^{n-1} \\
= \ & f_{uu}(t,\, t^{n-1}\xi)\left[(\rho - \alpha_k)\left(t^{n-1}\right)\right]^2 + f_u(t,\, t^{n-1}\alpha_k)\left(\rho - \alpha_{k+1}\right) t^{n-1} \\
< \ & -M p_{k+1}(t) t^{n-1} + N\left[p_k(t)t^{n-1}\right]^2 \\
\le \ & -M p_{k+1}(t) t^{n-1} + N\left|p_k\right|_0^2,
\end{aligned}
$$

where $t^{n-1}\xi \in (t^{n-1}\alpha_k,\, t^{n-1}\rho]$. Thus we have the inequality

$$-\left(t^n p'_{k+1}\right)' \leq -M p_{k+1}(t) t^{n-1} + N \left|p_k\right|_0^2 . \tag{3.16}$$

Clearly given M, N and $|p_k|_0^2$, there exist $z \in R^+$ such that

$$0 = -M z t^{n-1} + N \left|p_k\right|_0^2 . \tag{3.17}$$

Now using the comparison theorem [2] for (3.16) and (3.17), we get

$$p_{k+1}(t) t^{n-1} \leq z t^{n-1},\ t \in [0,\, 1].$$

(i.e.)

$$\left(\rho(t) - \alpha_{k+1}(t)\right) t^{n-1} \leq \frac{N}{M} \left|p_k\right|_0^2$$

which yields

$$\left|\rho - \alpha_{k+1}\right|_0 \leq \frac{N}{M} \left|p_k\right|_0^2 .$$

This proves the quadratic convergence of the sequence $\{t^{n-1}\alpha_{k+1}(t)\}$. Similarly, we can prove the quadratic convergence of the sequence $\{\, t^{n-1} \beta_{k+1}(t) \,\}$ to the solution of the SBVP (2.1). Further, it is noted that these sequences converge to the unique solution of (2.1) from hypothesis (iii) and the comparison Theorem 3.1. Thus the proof of the theorem is complete. □

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