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ON STABILITY OF NONLINEAR NONAUTONOMOUS SYSTEMS BY LYAPUNOV'S DIRECT METHOD

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ABSTRACT. The paper deals with asymptotic stability of nonlinear nonautonomous systems by Lyapunov's direct method. The proposed Lyapunov-like function $V(t,x)$ needs not be continuous in t and Lipschitz in x in a Banach space. The class of systems considered is allowed to be nonautonomous and infinite-dimensional and we relax the boundedness, the Lipschitz assumption on the system and the definite decrescent condition on the Lyapunov function.

1. Introduction

Consider a nonlinear time-varying differential equation of the general form:

$$(1)\qquad \begin{cases} \dot{x}(t) = f(t, x(t)), & t \geq t_0 \in R, \\ x(t_0) = x_0, \end{cases}$$

where the states $x(t)$ take values in $X, f(t,x) : R \times X \to X$ is a given nonlinear function and $f(t,0) = 0$, for all $t \in R$. We shall assume that conditions are imposed on the system (1) such that existence of its solutions are guaranteed.

It is well-known that the Lyapunov direct or second method is one of the most useful and fruitful techniques in stability analysis of nonlinear differential equations and has gained increasing significance in the development of stability theory of dynamical systems [6, 9, 20, 21]. The classical direct method of stability is based essentially on the existence of a positive definite Lyapunov function with a negative derivative. There are a number of books and papers available

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expouding the extensions and generalizations of Lyapunov functions, see, e.g., [1, 3, 8, 11, 12, 18] and references therein. It is recognized that the Lyapunov-like functions serve as a main tool to reduce a given complicated system into a relatively simpler system and provide useful applications to control systems [4, 16, 17, 19]. However, the problem of Lyapunov-like functions and their characterization have remained under active investigation and finding Lyapunov-like functions for general nonlinear systems is usually a difficult task.

In [10], sufficient conditions for the asymptotic stability of system (1) were given with the boundedness assumption

$$\|f(t,x)\| \leq M, \quad \forall (t,x) \in R \times R^n, \tag{2}$$

and the Lyapunov function $V(t,x) : R \times D \to R$, is continuous in $(t,x) \in R \times D$ and Lipschitz in $x \in D$ satisfying

$$D_f^+ V(t,x) \leq -\gamma(\|x\|) < 0, \quad \forall (t,x) \in R \times D \setminus \{0\},$$

where $D \subseteq R^n$ is an open neighborhood of the origin, $\gamma(.) : R^+ \to R^-$ is a given non-decreasing continuous function and

$$D_f^+ V(t,x) = \lim_{h \to 0^+} \inf \frac{V(t+h, x+hf) - V(t,x)}{h}.$$

The weaker stability conditions were proposed and presented in [8] for system (1), where the Lyapunov function $V(t,x)$ satisfies the following conditions

$$\begin{aligned} &a(\|x\|) \leq V(t,x) \leq b(\|x\|), \\ &D_f^+ V(t,x) \leq -\gamma(V(t,x)), \end{aligned} \tag{3}$$

where $a(.), b(.), \gamma(.)$ are continuous strictly increasing functions.

Thanks to a result of [16], the assumption on Lyapunov-like functions in the asymptotic stability for a class of autonomous systems has been considerably relaxed by the existence of two continuous positive definite functions $V(x) : R^n \to R, \gamma(.) : R^n \to R^+ \setminus \{0\}$, where $V(x)$ is proper (i.e., $V(x) \to \infty$ as $\|x\| \to \infty$) satisfying condition

$$D_- V(x) \leq -\gamma(x), \tag{4}$$

where

$$D_{-}V(x) = \lim_{h \to 0^+} \inf \frac{V(x(t+h)) - V(t,x)}{h}.$$

Paper [2] proposed a nondifferentiable stepswise decreasing Lyapunov function $V(t,x)$ for system (1), where $X = R^n$, and the function $f(t,x)$ is assumed to be locally Lipschitzian.

Inspired by the results of [2, 8] we consider a Lyapunov-like function $V(t,x)$ for time-varying system (1), which needs not be continuous in t and Lipschitz in x in a Banach space. In this general setup, the class of systems considered is allowed to be nonautonomous and infinite-dimensional and we relax the boundedness, the Lipschitz assumption on the system and the definite decrescent condition on the Lyapunov function.

The paper is organized as follows. In Section 2, we give main notations and definitions of Lyapunov-like functions needed later. Section 3 presents main theorems on asymptotic stability with proposed Lyapunov-like functions. The conclusion is drawn in Section 4.

2. Notations and definitions

We shall employ the following notations and definitions throughout:
X denotes an infinite-dimensional Banach space with the corresponding norm $\|.\|$,
B_ϵ denotes the open unit ball with radius ϵ,
R denotes the real line; R^+ denotes the set of non-negative real numbers,
Z^+ denotes the set of non-negative integers,
R^n denotes the $n-$dimensional Euclidean space.

DEFINITION 2.1. The zero solution of (1) is said to be stable if for every $\epsilon > 0, t_0 \in R$, there exists a number $\delta > 0$ (depending upon ϵ and t_0) such that for any solution $x(t)$ of (1) with $\|x_0\| < \delta$ implies $\|x(t)\| < \epsilon$, for all $t \geq t_0$.

DEFINITION 2.2. The zero solution of (1) is said to be asymptotically stable if it is stable and there is a number $\delta > 0$ such that any solution $x(t)$ with $\|x_0\| < \delta$ satisfies $\lim_{t\to\infty} \|x(t)\| = 0$.

In the above definitions, if the number $\delta > 0$ is independent on t_0, then the zero solution of the system is said to be uniformly (asymptotically) stable.

DEFINITION 2.3. Let H be a Hilbert space. A function $f(t,x)$: $R \times H \to H$ is dissipative if there is a number $L > 0$ such that for all $(t,x) \in R \times H$:

$$< x, f(t,x) > \leq L\|x\|^2, \tag{5}$$

where $< .,. >$ denotes the inner product in H.

Let $V(t,x) : R \times X \to R$ be some given function, $T > 0$, and $D \subseteq X$ be some open neighborhood of zero.

DEFINITION 2.4. A function $V(t,x) : R \times D \to X$ is a Lyapunov-like function for system (1) if it satisfies the following conditions:
(i) There exist a non-decreasing function $a(t) : R \to R^+$, and a non-increasing function $b(t) : R \to R^+$, and numbers $a > 0, b > 0$ such that for all $(t,x) \in R \times D$:

$$a(t)\|x\|^a \leq V(t,x) \leq b(t)\|x\|^b. \tag{6}$$

(ii) For every $\beta > 0$, there are a number $T > 0$ and functions $\gamma(.)$: $R^+ \to R^+, c(.) : R \to R^+$, strictly increasing, passing through zero and $\gamma(.)$ is integrable, such that, $\forall (t,x) \in R \times D \setminus \{0\}$,

$$\begin{aligned} \Delta_T V(t,x) &:= V(t+T, x(t+T)) - V(t,x) \\ &\leq -c(t) \int_t^{t+\beta} \gamma(\|x(s)\|) ds < 0, \end{aligned} \tag{7}$$

where $x(t)$ is a solution of (1) with $x(t) = x$. If the number T depending on β satisfies the condition $0 < T < \beta$, then we say that $V(t,x)$ is a strict Lyapunov-like function for system (1).

3. Stability results

In the following theorem, we relax the boundedness condition (2) on the system and the conditions (3), (4) on $V(t,x)$. We then provide a sufficient condition for asymptotic stability of system (1) with the proposed Lyapunov-like functions, where $V(t,x)$ needs not be continuous in t and Lipschitz in x.

THEOREM 3.1. *Assume that*

$$\|f(t,x)\| \leq M(t), \quad \forall (t,x) \in R \times X, \tag{8}$$

where $M(t) : R \to R^+$ is an integrable function satisfying the condition

$$\lim_{h\to 0} \int_t^{t+h} M(s)ds = 0, \quad \forall t \in R. \tag{9}$$

If the system (1) admits a strick Lyapunov-like function, then it is asymptotically stable.

Proof. a) The system is stable: We assume to the contrary that there are a number $\epsilon > 0$ and $t_0 \in R$, such that for every $\delta > 0$ and for some solution $x(t)$ of system (1) with $\|x_0\| < \delta$, there is a number $S > t_0$ such that $\|x(S)\| \geq \epsilon$. Let $\delta_1 > 0$ be chosen such that $B_{\delta_1} \subset D$. Let us take any $\delta_2 \in (0, \delta_1)$.

In view of condition (9), we can find a number $\beta > 0$ such that

$$\int_t^{t+\beta} M(s)ds := h < \min\{\delta_1 - \delta_2, [\frac{a(t_0)\epsilon^a}{b(t_0)}]^{1/b}\}, \ \forall t \in R. \tag{10}$$

For this β, from the condition (7) it follows that there is a number $T \in (0, h)$ such that for all $(t,x) \in R \times B_{\delta_1}$:

$$V\big(t+T, x(t+T)\big) - V(t,x) \leq 0. \tag{11}$$

We now take a number $\delta > 0$ such that

$$\delta < \min\Big\{\delta_1 - h, [\frac{a(t_0)\delta_2^a}{b(t_0)}]^{1/b}, [\frac{a(t_0)\epsilon^a}{b(t_0)}]^{1/b} - h\Big\}.$$

For this δ there is, by the contrary assumption, a number $S > t_0$ such that $\|x(S)\| \geq \epsilon$. Consider the solution $x(t)$ with $\|x_0\| < \delta$. Since

$S > t_0$, there are an integer $k > 0$ and number $\tau \in [0, T)$ such that $S = t_0 + kT + \tau$. Since

$$\begin{aligned}\|x(t_0+T)\| &\leq \|x_0\| + \int_{t_0}^{t_0+T} M(s)\,ds \\ &< \delta + \int_{t_0}^{t_0+\beta} M(s)ds \\ &= \delta + h < \delta_1,\end{aligned}$$

which gives $x(t_0+T) \in B_{\delta_1}$. Applying (6) and (11), we have

$$\begin{aligned}a(t_0)\|x(t_0+T)\|^a &\leq V(t_0+T, x(t_0+T)) \\ &\leq V(t_0, x_0) \leq b(t_0)\|x_0\|^b < b(t_0)\delta^b < a(t_0)\delta_2^a\end{aligned}$$

and hence $\|x(t_0+T)\| < \delta_2$. On the other hand,

$$\begin{aligned}\|x(t_0+2T)\| &\leq \|x(t_0+T)\| + \int_{t_0+T}^{t_0+2T} M(s)\,ds \\ &\leq \delta_2 + \int_{t_0+T}^{t_0+T+\beta} M(s)ds < \delta_2 + h < \delta_1,\end{aligned}$$

which implies $x(t_0+2T) \in B_{\delta_1}$. Thus, applying (6) and (11) again the following estimate holds

$$\begin{aligned}a(t_0)\|x(t_0+2T)\|^a &\leq V\big(t_0+2T, x(t_0+2T)\big) \\ &\leq V\big(t_0+T, x(t_0+T)\big) \leq V(t_0, x_0) \leq a(t_0)\delta_2^a,\end{aligned}$$

and hence $\|x(t_0+2T)\| < \delta_2$. Repeating the same arguments, we obtain

$$\|x(t+kT)\| < \delta_2, \quad \forall k \in Z^+.$$

Therefore, for every $k \in Z^+$, we have

$$\begin{aligned}\|x(S)\| = \|x(t_0+kT+\tau)\| &\leq \|x(t_0+kT)\| + \int_{t_0+kT}^{t_0+kT+\tau} M(s)\,ds \\ &\leq \delta_2 + h < \delta_1\end{aligned}$$

which gives $x(S) \in B_{\delta_1}$. Applying (6) and (11), we have

$$\begin{aligned} a(t_0)\epsilon^a &\le a(S)\|x(S)\|^a \\ &\le V\big(S, x(S)\big) \\ &\le V\Big(t_0 + (k-1)T + \tau, x\big(t_0 + (k-1)T + \tau\big)\Big) \\ &\le \dots \le \\ &\le V\big(t_0 + \tau, x(t_0 + \tau)\big) \le b(t_0)\big\|x(t_0 + \tau)\big\|^b. \end{aligned}$$

Since

$$\begin{aligned} \|x(t_0 + \tau\| &\le \|x_0\| + \int_{t_0}^{t_0+\tau} M(s)\, ds \\ &\le \delta + \int_{t_0}^{t_0+\beta} M(s)\, ds \\ &\le \delta + h \\ &\le \Big[\frac{a(t_0)\epsilon^a}{b(t_0)}\Big]^{1/b}, \end{aligned}$$

we obtain $a(t_0)\epsilon^a < a(t_0)\epsilon^a$, which is a contradiction.

b) The sytem (1) is asymptotically stable: We remain to show that there is a number $\delta > 0$, for every solution $x(t)$ of (1) with $\|x(t_0)\| < \delta$, for every $\epsilon > 0$, there exists a number $N > 0$ such that $\|x(t)\| < \epsilon$ for all $t > t_0 + N$. For this, from the stability of the system it follows that for $\delta_1 > 0$, where δ_1 is chosen so that $B_{\delta_1} \subset D$, we can find a number $\delta_2 > 0$ such that any solution $x(t)$ of the system with $\|x(t_0)\| < \delta_2$ implies

$$\|x(t)\| < \delta_1, \quad \forall t > t_0. \tag{12}$$

Consider any solution $x(t)$ of (1) with $\|x_0\| < \delta = \min\{\delta_1, \delta_2\}$. We then have $x(t) \in D$, for all $t > t_0$. Let $\epsilon > 0$ be an arbitrary given number, we define

$$\delta_3 = \Big[\frac{a(t_0)}{b(t_0)}\epsilon^a\Big]^{1/b}.$$

Let us take any $\delta_4 \in (0, \delta_3)$. Due to the condition (9), there is a number $\beta > 0$ such that

$$\int_t^{t+\beta} M(s)ds < \min\{\delta_4, \delta_3 - \delta_4\}, \quad \forall t \in R. \tag{13}$$

For this $\beta > 0$ there is, by condition (7), a number $T \in (0, \beta)$, and functions $c(.), \gamma(.)$ such that for all $(t, x) \in R \times D \setminus \{0\}$:

$$V\big(t+T, x(t+T)\big) - V(t, x) \leq -c(t) \int_t^{t+\beta} \gamma(\|x(s)\|)ds. \tag{14}$$

We shall show that there is an integer $K > 0$ such that

$$\|x(t_0 + KT)\| < \delta_4. \tag{15}$$

Indeed, if (15) is not satisfied, then $\|x(t_0 + kT)\| \geq \delta_4$ for all $k \in Z^{+}$. Taking (12) into account and applying (14), we have

(16)

$$\begin{aligned} &V\big(t_0 + (k+1)T, x(t_0 + (k+1)T)\big) \\ &\leq V\big(t_0 + kT, x(t_0 + kT)\big) - c(t_0 + kT) \int_{t_0+kT}^{t_0+kT+\beta} \gamma(\|x(s)\|)\, ds \\ &\leq V\big(t_0 + kT, x(t_0 + kT)\big) - c(t_0) \int_{\bar{t}_0}^{\bar{t}_0+\beta} \gamma(\|x(s)\|)\, ds, \end{aligned}$$

where $\bar{t}_0 := t_0 + kT$. On the other hand, for every $t \in [\bar{t}_0, \bar{t}_0 + \beta]$, we have

$$\begin{aligned} \|x(t)\| &\geq \|x(\bar{t}_0)\| - \int_{\bar{t}_0}^{t} |f(s, x(s))| ds \\ &\geq \delta_4 - \int_{\bar{t}_0}^{\bar{t}_0+\beta} M(s)ds. \end{aligned}$$

Taking (13) into account, we have

$$\|x(t)\| \geq \delta_4 - \int_{\bar{t}_0}^{\bar{t}_0+\beta} M(s)ds = \eta > 0,$$

and hence

$$\int_{\bar{t}_0}^{\bar{t}_0+\beta} \gamma(\|x(s)\|)ds \geq \beta\gamma(\eta) > 0.$$

Therefore, from (16), it follows that

$$V(t_0+(k+1)T, x(t_0+(k+1)T)) \leq V(t_0+kT, x(t_0+kT)) - c(t_0)\gamma(\eta)\beta.$$

Repeating the same arguments we obtain

$$V(t_0+(k+1)T, x(t_0+(k+1)T)) \leq V(t_0, x_0) - (k+1)c(t_0)\gamma(\eta)\beta.$$

Since the Lyapunov-like function $V(t,x)$ is non-negative, we have

$$0 \leq V(t_0, x_0) - (k+1)c(t_0)\gamma(\eta)\beta, \quad \forall k \in Z^+,$$

or equivalently

$$(k+1)c(t_0)\gamma(\eta)\beta \leq V(t_0, x_0) \leq b(t_0)\|x_0\|^b < b(t_0)\delta^b < +\infty$$

which is a contradiction when letting $k \to \infty$. Thus, the condition (15) is proved. The proof is completed as follows. For every $t \geq t_0 + KT$ there are numbers $k_0 > K, \tau \in [0,T)$ such that $t - t_0 = k_0T + \tau_0$. We have

$$\begin{aligned}\|x(t_0+KT+\tau_0)\| &\leq \|x(t_0+KT)\| + \int_{t+KT}^{t_0+KT+\tau} M(s)\,ds \\ &\leq \delta_4 + \int_{t_0+KT}^{t_0+KT+\beta} M(s)\,ds < \delta_3.\end{aligned}$$

Therefore

$$\begin{aligned}a(t_0)\|x(t)\|^a &\leq a(t)\|x(t)\|^a \leq V(t,x(t)) \\ &\leq V(t_0+KT+\tau, x(t_0+KT+\tau_0)) \\ &\leq b(t_0)\|x(t_0+KT+\tau)\|^b \leq b(t_0)\delta_3^b = a(t_0)\epsilon^a,\end{aligned}$$

which gives $\|x(t)\| < \epsilon$. The theorem is proved. □

REMARK 3.1. Note that since the functions $a(t), b(t)$ need not be continuous, the Lyapunov function $V(t, x)$ is not continuous in t and it is, in general, not decreasing in t, we can not apply the techniques used in [8]. Furthermore, if the functions $a(t), b(t)$ are independent on t, i.e., are constant, then Theorem 3.1 holds for uniform asymptotic stability.

Theorem below asserts that if the boundedness of $f(t, x)$ is replaced by the dissipativity assumption (5) in Hilbert space, then we can derive asymptotic stability conditions for system (1) with a weaker Lyapunov-like function than the strict Lyapunov-like function.

THEOREM 3.2. *Assume that $X = H$ is a Hilbert space and the function $f(t, x)$ is dissipative in H. If the system* (1) *admits a Lyapunov-like function then it is asymptotically stable.*

Proof. Let $f(t, x)$ be a dissipative function satisfying condition (5) with the constant $L > 0$. We first note that for every solution $x(t)$ of (1) with $x(t_0) = x_0$ the following property holds:

$$\|x(t)\| \leq \|x_0\| e^{L(t-t_0)}, \quad \forall t \geq t_0. \tag{17}$$

Indeed, since

$$\frac{d}{dt} < x(t), x(t) > = \frac{d}{dt}\|x(t)\|^2 = 2 < x(t), f(t, x(t)) >,$$

and by integrating both sides of the above relation, we have

$$\|x(t)\|^2 = \|x_0\|^2 + 2\int_{t_0}^{t} < x(s), f(s, x(s)) > ds.$$

Then

$$\|x(t)\|^2 \leq \|x_0\|^2 + 2L\int_{t_0}^{t} \|x(s)\|^2 ds.$$

Applying the Gronwall's inequality, we obtain

$$\|x(t)\|^2 \leq \|x_0\|^2 e^{2L(t-t_0)},$$

which gives (17).

a) The system is stable. The proof is similar to that of Theorem 3.1, part a), where the positive numbers δ_2, δ are defined by

$$\delta_2 < \delta_1 e^{-LT},\ \delta < \min\Big\{\delta_1 e^{-KL}, \Big[\frac{a(t_0)\delta_2^a}{b(t_0)}\Big]^{1/b}, \Big[\frac{a(t_0)\epsilon_0^a}{b(t_0}\Big]^{1/b} e^{-KL}\Big\}.$$

b) The system is asymptotically stable. From the stability of the system it follows that there is a number $\delta_2 > 0$ such that for every solution $x(t)$ of (1) with $\|x_0\| < \delta_2$ implies $\|x(t)\| < \delta_1$, where $\delta_1 > 0$ is a number chosen so that $B_{\delta_1} \subset D$. Consider any solution $x(t)$ of (1) with $\|x_0\| < \delta = \min\{\delta_1, \delta_2\}$. We then have $x(t) \in B_{\delta_1}$, for all $t > t_0$. Let $\epsilon > 0$ be a given number. We take

$$\delta_3 = \Big[\frac{a(t_0)}{b(t_0)}\epsilon^a\Big]^{1/b},\ \delta_4 < \delta_3 e^{-LT}.$$

Since $V(t,x)$ is a Lyapunov-like function, take any number $\delta > 0$ such that

$$\delta < \frac{\delta_4^2}{2L\delta_1^2}.$$

We first prove that there is an integer $K > 0$ such that

$$\|x(t_0 + KT)\| < \delta_4. \tag{18}$$

Indeed, if (18) is not satisfied, we have $\|x(t_0 + kT)\| \geq \delta_4$, for all $k \in Z^+$. In view of (14) we have for all $k \in Z^+$:

(19)

$$\begin{aligned} &V\Big(t_0 + (k+1)T, x(t_0 + (k+1)T)\Big) \\ &\leq V\Big(t_0 + kT, x(t_0 + kT)\Big) - c(t_0 + kT)\int_{t_0+kT}^{t_0+kT+\beta} \gamma(\|x(s)\|)\,ds \\ &\leq V\Big(t_0 + kT, x(t_0 + kT)\Big) - c(t_0)\int_{t_0+kT}^{t_0+kT+\beta} \gamma(\|x(s)\|)\,ds. \end{aligned}$$

We estimate the value $\|x(t)\|$ on the interval $I_\beta := [t_0 + kT, t_0 + kT + \beta]$. On the other hand, since for all $t \in I_\beta$:

$$\begin{aligned} \|x(t)\|^2 &\geq \|x(t_0 + kT)\|^2 - \Big|2\int_{t_0+kT}^{t_0+kT+\beta} < x(s), f\big(s, x(s)\big) > ds\Big| \\ &\geq \|x(t_0 + kT)\|^2 - 2L\int_{t_0+kT}^{t_0+kT+\beta} \|x(s)\|^2\,ds, \end{aligned}$$

and since $\|x(t)\| < \delta_1$ for all $t > t_0$, we have

$$\|x(t)\|^2 \geq \delta_4^2 - 2Lh\delta_1^2 := \eta^2 > 0,$$

which gives

$$\int_{t_0+kT}^{t_0+kT+\beta} \gamma(\|x(s)\|)ds \geq \beta\gamma(\eta). \tag{20}$$

Therefore, combining (19) and (20), we obtain

$$\begin{aligned} &V\big(t_0 + (k+1)T, x(t_0 + (k+1)T)\big) \\ &\leq V\big(t_0 + kT, x(t_0 + kT)\big) - c(t_0)\beta\gamma(\eta) \\ &\leq V\big(t_0 + kT, x(t_0 + kT)\big) - M, \\ &\leq \dots \leq \\ &\leq V(t_0, x_0) - (k+1)M, \end{aligned}$$

where $M := c(t_0)\beta\gamma(\eta)$ which gives

$$(k+1)M \leq V(t_0, x_0) \leq b(t_0)\delta^b < +\infty.$$

The last inequality leads to a contradiction when letting $k \to \infty$. Thus, (18) is proved.

For every $t > t_0 + KT$, there are numbers $k_0 > K, \tau \in [0, T)$, such that $t - t_0 = k_0T + \tau$. We have

$$\begin{aligned} a(t_0)\|x(t)\|^a &\leq V\big(t_0 + k_0T + \tau, x(t_0 + k_0T + \tau)\big) \\ &\leq V(t_0 + KT + \tau, x(t_0 + KT + \tau)) \\ &\leq b(t_0)\|x(t_0 + KT + \tau)\|^b. \end{aligned}$$

On the other hand, in view of (17) we have

$$\|x(t_0 + KT + \tau_0)\| \leq \|x(t_0 + KT)\|e^{K\tau_0} \leq \delta_4 e^{LT} < \delta_3,$$

and hence

$$a(t_0)\|x(t)\|^a \leq b(t_0)\delta_3^b = a(t_0)\epsilon^a,$$

which gives $\|x(t)\| < \epsilon$. The proof is complete. □

REMARK 3.2. The dissipative condition (2) can be replaced by the Holder-type condition

$$\exists L > 0,\ \exists \alpha \in (0,1] : \|f(t,x)\| \le L\|x\|^{\alpha},\ \forall (t,x) \in R \times H.$$

In this case, the generalized Gronwall inequality [8, 22] is applied to obtain the estimate (17) of the form

$$\|x(t)\| \le \begin{cases} \|x_0\| e^{L(t-t_0)}, & \text{if } \alpha = 1, \\ \|x_0\| [1 + (1-\alpha)L(t-t_0)]^{\frac{1}{2-\alpha}}, & \text{if } \alpha \in (0,1), \end{cases}$$

and the positive numbers ϵ_1, δ_4, and h in the proof of Theorem 3.2 are defined respectively by

$$\epsilon_1 < \min\{\epsilon\eta^{-1}, \delta_1\eta^{-1}\}, \quad \delta_4 < \delta_3\eta^{-1}, \quad h < \frac{\delta_4}{L\delta_1^{\alpha}},$$

where

$$\eta := \begin{cases} e^{LT}, & \text{if } \alpha = 1, \\ \left[1 + (1-\alpha)LT\right]^{\frac{1}{2-\alpha}}, & \text{if } \alpha \in (0,1). \end{cases}$$

REMARK 3.3. Theorem 3.2 remains true even if we replace the conditions (6), (7) by the following conditions:

$$a(t,\tau) \le V(t,x) \le b(t,\tau)$$

$$\Delta_T V(t,x) \le -\int_t^{t+\beta} \gamma(s, \|x(s)\|)ds < 0, \tag{21}$$

where $a(t,\tau), b(t,\tau) : R \times R^+ \to R^+ \backslash \{0\}$ are continuous strictly increasing function in τ, $a(t,\tau)$ is non-decreasing in t, $b(t,\tau)$ is non-increasing in t, and $\gamma(t,h) : R \times R^+ \to R^+$ is a strictly increasing in $h \in R^+$ function satisfying

$$\lim_{h \to 0^+} \inf \int_t^{t+h} \gamma(s,h)ds > 0, \quad \forall t \in R,\ h > 0.$$

EXAMPLE 3.1. To illustrate the stability result, we consider the asymptotic stability of a semilinear system in Hilbert space of the form

$$\dot{x}(t) = \begin{cases} Ax(t) + g(t, x(t)), & t \geq 0, \\ x(t_0) = x_0, x(t) \in H. \end{cases} \tag{22}$$

Let us assume that the linear operator $A : H \to H$ is stable. Then, by a result of [5], there is a positive definite symmetric linear operator $Q : H \to H$ such that

$$2 < QAx, x > \leq -\|x\|^2, \quad \forall x \in H.$$

Assume that the nonlinear function $g(t, x)$ satisfies the following growth condition

$$\|g(t, x)\| \leq K(t)\|x\|, \quad \forall (t, x) \in R^+ \times H, \tag{23}$$

where $K(t) : R^+ \to R^+$ is a bounded and integable function. Consider the Lyapunov function $V(t, x) =< Qx, x >$. The derivative along the trajectories $x(t)$ of system (22) is given by

$$\frac{d}{dt}V(t, x) = -\|x(t)\|^2 + < Qx(t), g(t, x(t)) > .$$

Thus, we can not apply the classical stability Lyapunov theorem since the derivative of $V(t, x)$ may take positive and negative values. We will show that if

$$\lim_{h \to 0^+} \inf \int_t^{t+h} [1 - \|Q\|K(s)]ds > 0, \quad \forall t \in R, \tag{24}$$

then the system is asymptotically stable. Indeed, let us consider any solution $x(t)$ of (22) with $x(t_0) = x_0$. By the assumption (24), there exist a sequence of positives numbers $\{t_n\}$ going to zero and a number $N > 0$ such that

$$\int_t^{t+t_n} [1 - \|Q\|K(s)]ds \geq a > 0, \quad \forall n > N. \tag{25}$$

Let $\beta \in (0, t_n), n \geq N$ be an arbitrary number. We will show that there are a positive number $T > 0$, and a function $\gamma(t, h)$ satisfying the condition (21) and hence, by Theorem 3.2 and Remark 3.3, the system is asymptotically stable. For this, let x_0 be an arbitrary initial state such that $\|x_0\| = \eta > 0$. Since the solution $x(t)$ is continuous, we can find a number $\delta > 0$ such that $\|x(t_0+t)\| \geq \eta > 0$ for all $t \in [t_0, t_0+\delta]$. Let us take a number $n > N$ such that $t_n < \delta$ satisfying (25) and let $T > t_n > \beta$. From (23) it follows that for every $t_0 > 0$, the following relation holds:

$$\begin{aligned}
\Delta_T V(t_0, x) &= V\big(t_0 + T, x(t_0 + T)\big) - V(t_0, x_0) \\
&= \int_{t_0}^{t_0+T} \frac{d}{dt} V(s, x(s))\, ds \\
&= \int_{t_0}^{t_0+T} \big[- \|x(s)\|^2 + < Qx(s), g(s, x(s)) > \big]\, ds \qquad (26) \\
&\leq - \int_{t_0}^{t_0+\beta} [1 - \|Q\| K(s)] \|x(s)\|^2\, ds.
\end{aligned}$$

On the other hand, setting $\gamma(t,h) = [1 - \|Q\|K(t)]h^2$ and using (25) we have

$$\begin{aligned}
&\int_{t_0}^{t_0+t_n} \gamma(s,h)\, ds \\
&= \int_{t_0}^{t_0+t_n} [1 - \|Q\|K(s)]h^2\, ds \\
&\geq h^2 \int_{t_0}^{t_0+t_n} [1 - \|Q\|]K(s)\, ds \\
&= h^2 a > 0.
\end{aligned}$$

Therefore

$$\lim_{h \to 0^+} \inf \int_t^{t+h} \gamma(s,h) ds > 0, \quad t \in R, h > 0,$$

which together with (26) proves the assertion.

4. Conclusions

In this paper, asymptotic stability of nonlinear time-varying differential equations by Lyapunov direct method have been investigated. Discontinuous Lyapunov-like function are proposed for sufficient stability conditions. Several stability results obtained earlier can be derived. The stability results obtained in the paper can be considered as a further development of Lyapunov function characterization in stability analysis of nonlinear dynamical systems.

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References

[1] Aeyels D., *Asymptotic stability of autonomous systems by Lyapunov's direct method*, Syst. Contr. Lett. **25** (1995), 273–280.

[2] Aeyels D. and Peuteman J., *A new asymptotic stability criterion for nonlinear time-variant differential equations*, IEEE, Aut. Contr. **43** (1998), 968–971.

[3] Clarke F., Ledyev Y., and Stern R., *Asymptotic stability and smooth Lyapunov function*, J. of Diff. Equats. **149** (1998), 69–114.

[4] Coron J. M., *Global asymptotic stabilization for controllable systems without drift*, Math. Contr. Signal Systems **5** (1992), 295–312.

[5] Datko R., *Extending a theorem of Lyapunov to Hilbert space*, J. Math. Anal. Appl. **32** (1970), 610–616.

[6] Hahn W., *Theory and Applications of Lyapunov's Direct Methods*, Prentice Hall, NJ. 1963.

[7] Lakshmikantham V. and Leela S., *A technique in stability theory for delay differential equations*, Nonlinear Analysis, Theory Method & Appl. **3** (1979), 317–323.

[8] Lakshmikantham V., S. Leela S., and Martynyuk M., *Stability Analysis of Nonlinear Systems*, Marcel Dekker, New York, 1989.

[9] LaSalle J. P. and Lefschetz S., *Stability by Lyapunov's Direct Method with Applications*, Academic Press, New York, 1961.

[10] Marachkov M., *On a theorem on stability*, Bull. Soc. Phys. Math. Kazan **12** (1940), 171–174.

[11] Martynyuk A. and Podilchuk V., *On asymptotic practical stability of systems*, Appl. Mech. **19** (1983), 89–94.

[12] Narendra K. S. and Annaswamy A. M., *Persistent exitation in adaptive systems*, Int. J. of Control **45** (1987), 127–160.

[13] Phat V. N., *Weak asymptotic stabilizability of discrete-time systems given by set-valued operators*, J. Math. Anal. Appl. **202** (1996), 363–378.

[14] ———, *On the stability of time-varying differential equations*, Optimization, (to appear), 1999.
[15] Phat V. N. and Park J. Y., *Asymptotic stability of nonlinear perturbed systems with multiple delays*, Proc. Int. Conf. on Math. Anal. Appl. Chinju, Korea, **1** (1998), 345–356.
[16] Sontag E. D., *A Lyapunov-like characterization of asymptotic controllability*, SIAM J. Contr. Optim. **21** (1983), 462–471.
[17] Sontag E. D. and Sussmann H., *Nonsmooth control Lyapunov functions*, Proc. IEEE Conf. on DC, New Orleans, 1995, pp. 61–81.
[18] Sun Y., Hsieh J., and Hsieh Y., *Exponential stability criterion for uncertain retarded systems*, J. Math. Anal. Appl. **201** (1996), 430–446.
[19] Sun Y., Lieu C., and Hsieh J., *Global exponential stabilization for a class of uncertain nonlinear systems with control constraints*, IEEE Aut. Contr. **43** (1998), 674–677.
[20] Vidiyasaga M., *Nonlinear System Analysis*, Prentice Hall, 1978.
[21] Zubov V. I., *Methods of Lyapunov and Their Applications*, Gronningen, 1964.
[22] Yang En-Hao, *Perturbation of nonlinear systems of ordinary differential equations*, J. Math. Anal. Apll. **103** (1984), 1–15.

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