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THE CONTROL OF THE BLOWING-UP TIME FOR THE SOLUTION OF THE SEMILINEAR PARABOLIC EQUATION WITH IMPULSIVE EFFECT

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Dedicated to Professor Norio Shimakura on the occasion of his sixtieth birthday.

Abstract. An impulsive semilinear parabolic equation subject to Robin boundary condition is considered. We prove that for certain classes of impulsive sources and continuous initial data, the solutions of the problem under consideration blow up in the desired time interval.

1. Introduction

The study of the impulsive partial differential equations started recently by the paper of Erbe, Freedman, Liu and Wu [5]. They considered an impulsive parabolic initial-boundary value problem (IBVP for short) discribing the growth of a population diffusing throughout its habitat. Since then a few works in this field were published. A study on the existence and uniqueness of the solutions to some more general impulsive partial differential equations of hyperbolic and parabolic type was done in the monograph [2]. A survey on the theory of impulsive partial differential equations and their applications can be found in [1]. Some important results have been published recently in [3, 4, 7, 8].

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We consider an impulsive IBVP under Robin boundary condition. Such problems arise from some discrete models of processes and phenomena which occur in discrete technologies, chemical reactor dynamics, combustion theory, thermal explosions, population dynamics etc. These and another applications can be found in the monographs of Whitham [10] and Pao [9] and the references therein. We are interested in studying the behaviour of the solution which is influenced by the reaction function and the impulsive source. In the smooth case (without impulses), it is well known that if the reaction function is bounded from above by a certain linear growth condition, then the solution of the problem under consideration converges to a steady-state solution. In the case when the reaction function is bounded from below by either linear or nonlinear growth condition then the solution may grow unboundedly as $t \to T^*$, where T^* is either a finite time or infinity (see cf. Pao [9] and Friedman and McLeod [6]). The blowing-up time T^* depends on the initial data. If the initial data is small enough, then T^* comes large.

In our case (with impulses), we fix the initial data which is not too small. We will investigate how to control the impulsive source to delay the blowing-up time T^* and to grow the solution unboundedly in the desired time interval.

The paper is organized as follows. In Section 2, we present some basic notations, definitions and results. In Section 3, we use these results to show that blowing-up phenomenon exists in the desired time interval provided that the reaction function, the initial data and also the impulsive functions satisfy certain conditions. In Section 4, we give one example which explains how to choose the impulsive sources.

2. Preliminaries

Let $\{t_i\}$ $(i = 0, 1, ..., m)$ and T be some nonnegative given numbers such that $0 = t_0 < t_1 < t_2 < ... < t_m < t_{m+1} = T < \infty$, and Ω be a smooth bounded domain in $\mathbb{R}^n$. We introduce the following notations:

$$Q_T \equiv (0,T) \times \Omega, \qquad \Gamma_T \equiv (0,T) \times \partial\Omega,$$

$$P_k \equiv \{(t_k, x) : x \in \Omega\}, \qquad P \equiv \cup_{k=1}^m P_k,$$

$$\Lambda_k \equiv \{(t_k, x) : x \in \partial\Omega\}, \qquad \Lambda \equiv \cup_{k=1}^m \Lambda_k.$$

We denote $u_t = \partial u/\partial t$, $u_{x_i} = \partial u/\partial x_i$, and $u_{x_i x_j} = \partial^2 u/\partial x_i \partial x_j$.

Consider the following impulsive parabolic IBVP:

$$(1) \quad \begin{cases} (a) \quad u_t - \Delta u = f(t,x,u) & \text{in } Q_T \setminus P, \\ (b) \quad Bu = 0 & \text{on } \Gamma_T \setminus \Lambda, \\ (c) \quad u(0,x) = u_0(x) & \text{on } \bar{\Omega}, \\ (d) \quad u(t_k, x) = g_k(u(t_k^-, x)) \ (1 \le k \le m) & \text{on } \bar{\Omega}, \end{cases}$$

where $u_0(x)$ is nonnegative and is in $C^2(\bar{\Omega})$. The "impulsive source" in (1d) is represented by the mapping $g_k : \mathbb{R} \to \mathbb{R}$ $(1 \le k \le m)$ which will be determined below. The boundary operator B is defined by $B \equiv \frac{\partial}{\partial \nu} + b(x)$, where b is a nonnegative function in $C^{1+\theta}(\partial\Omega)$ $(0 < \theta < 1)$ and not identically zero on $\partial\Omega$; $\partial/\partial\nu$ is the outward normal derivative on $\partial\Omega$.

Let us define $C^{1,2}(Q_T, P)$ as the set of all functions $u : [0,T] \times \bar{\Omega} \to \mathbb{R}$ satisfying the conditions:

(i) $u(t,x)$ is continuously differentiable in $\bar{Q}_T \setminus (P \cup \Lambda)$.
(ii) $u_{x_i x_j}(t,x)$ $(i,j = 1,2,...,n)$ exist and are continuous in $Q_T \setminus P$.
(iii) There exist the finite limits

$$\lim_{(s,y)\to(t,x)} u(s,y) = u(t^-, x) \quad \text{for} \quad s < t,$$

$$\lim_{(s,y)\to(t,x)} u(s,y) = u(t^+, x) \quad \text{for} \quad s > t,$$

$$\text{and} \quad u(t,x) = u(t^+, x) \quad \text{for} \quad (t,x) \in \bar{Q}_T.$$

We give here the definition of the solution, the upper solution and the lower solution of (1).

DEFINITION 1. A function $u \in C^{1,2}(Q_T, P)$ is called a solution of the IBVP if it satisfies (1).

DEFINITION 2. A function $u \in C^{1,2}(Q_T, P)$ is called an upper solution of IBVP (1) if:

$$
(2)\quad
\begin{aligned}
&(a)\ \ u_t - \Delta u \geq f(t,x,u) && \text{for } (t,x) \in Q_T \setminus P,\\
&(b)\ \ Bu \geq 0 && \text{on } \Gamma_T \setminus \Lambda,\\
&(c)\ \ u(0,x) \geq u_0(x) && \text{on } \bar{\Omega},\\
&(d)\ \ u(t_k,x) \geq g_k(u(t_k^-,x))\ \ (1 \leq k \leq m) && \text{on } \bar{\Omega}.
\end{aligned}
$$

Lower solution is defined analogously by reversing the above inequalities.

Let $L_{loc}(\mathbb{R}_+)$ be the set of all locally Lipschitz continuous functions on $\mathbb{R}_+ = \{x \in \mathbb{R}^1 : \ x \geq 0\}$. We introduce the following assumptions:

H1. Let $f(t,x,u) \in L_{loc}(\mathbb{R}_+)$ and there exist positive constants γ, μ, σ_0, σ_1 such that for any $\eta \geq 0$ and $x \in \bar{\Omega}$,

$$(3)\qquad f(t,x,\eta) \geq \lambda_0 \eta + \sigma_0 t^{\mu-1}\eta^{\gamma+1}, \quad t \in [t_k, t_{k+1}) \ \ (0 \leq k \leq m),$$

and

$$(4)\qquad f(t,x,\eta) \leq \lambda_0 \eta + \sigma_1 t^{\mu-1}\eta^{\gamma+1}, \quad t \in [t_k, t_{k+1}) \ \ (0 \leq k \leq m),$$

where λ_0 stands for the principal eigenvalue of the eigenvalue problem

$$(5)\qquad -\Delta\Phi = \lambda\Phi \ \text{ in } \ \Omega, \quad B\Phi = 0 \ \text{ on } \ \partial\Omega.$$

It is well known that $\lambda_0 > 0$ and the corresponding eigenfunction $\Phi_0(x)$ is positive on $\bar{\Omega}$. In what follows we assume that $\Phi_0(x)$ is normalized by $\max\{\Phi_0(x) : \ x \in \bar{\Omega}\} = 1$.

H2. The mapping $g_k : \mathbb{R} \to \mathbb{R}$ $(1 \leq k \leq m)$ possesses the following properties:

(i) $g_k(z)$ are nondecreasing smooth functions in $\mathbb{R}_+$;

(ii) for $k = 1, 2, ..., m$, there exist positive numbers E_k, e_k such that for $\eta \geq 0$,

$$(6)\qquad E_k \eta \geq g_k(\eta) \geq e_k \eta.$$

The constants E_k and e_k actually control the impulsive source g_k.

H3. There exist $\hat{\delta}_0 > 0$ and $\bar{\delta}_0 > 0$ such that

(7) $$\bar{\delta}_0\Phi_0(x) \geq u_0(x) \geq \hat{\delta}_0\Phi_0(x), \quad x \in \bar{\Omega}.$$

They satisfy

$$0 < \hat{\delta}_0 < \mu^{1/\gamma}\Psi_0^{-1}[\gamma\sigma_0(t_1^\mu - t_0^\mu)]^{-1/\gamma},$$

and

$$0 < \bar{\delta}_0 < \mu^{1/\gamma}[\gamma\sigma_1(t_1^\mu - t_0^\mu)]^{-1/\gamma},$$

where $\Psi_0 = \min_{x\in\bar{\Omega}} \Phi_0(x)$.

We first choose $\hat{\delta}_0$ and $\bar{\delta}_0$ suitablly and fix them. Let us introduce the following notations:

$$N_k \equiv 1 + \gamma\sigma_0\mu^{-1}\,(\hat{\delta}_k\,\Psi_0)^\gamma\,t_k^\mu \quad (0 \leq k \leq m),$$

$$\bar{N}_k \equiv 1 + \gamma\sigma_1\mu^{-1}\bar{\delta}_k^\gamma\,t_k^\mu \quad (0 \leq k \leq m),$$

$$M_k \equiv \gamma\sigma_0\mu^{-1}(\hat{\delta}_k\Psi_0)^\gamma \Big/ N_k \quad (0 \leq k \leq m),$$

$$\bar{M}_k \equiv \gamma\sigma_1\mu^{-1}\bar{\delta}_k^\gamma \Big/ \bar{N}_k \quad (0 \leq k \leq m).$$

We define $\hat{\delta}_k$ and $\bar{\delta}_k$ so that $\hat{\delta}_k \leq \bar{\delta}_k$ inductively.

$$\hat{\delta}_{k+1} = e_{k+1}\tfrac{\hat{\delta}_k}{(N_k(1-M_k t_{k+1}^\mu))^{1/\gamma}} \quad (0 \leq k \leq m-1),$$

$$\bar{\delta}_{k+1} = E_{k+1}\tfrac{\bar{\delta}_k}{(\bar{N}_k(1-\bar{M}_k t_{k+1}^\mu))^{1/\gamma}} \quad (0 \leq k \leq m-1).$$

H4. The constants $e_1, e_2, ..., e_{m-1}$ satisfy the inequalities

$$0 < e_k < \left\{\frac{\mu N_{k-1}(1 - M_{k-1}t_k^\mu)}{\gamma\sigma_0(t_{k+1}^\mu - t_k^\mu)}\right\}^{\frac{1}{\gamma}} (\hat{\delta}_{k-1}\Psi_0)^{-1} \quad (k = 1, 2, ..., m-1).$$

We choose $e_1, e_2, ..., e_{m-1}$ so that they satisfy **H4** and fix them. Then we have

$$0 < 1 - M_k t^\mu \quad \text{for} \quad t \in [t_k, t_{k+1}] \quad (k = 0, 1, 2, ..., m-1).$$

The constant e_m will be chosen in a different manner.

H5. The constants $E_1, E_2, \dots, E_{m-1}$ satisfy the inequalities

$$0 < E_k < \left\{ \frac{\mu \bar{N}_{k-1}(1 - \bar{M}_{k-1} t_k^{\mu})}{\gamma \sigma_1 (t_{k+1}^{\mu} - t_k^{\mu})} \right\}^{\frac{1}{\gamma}} \bar{\delta}_{k-1}^{-1} \quad (k = 1, 2, \dots, m-1).$$

We choose also $E_1, E_2, \dots, E_{m-1}$ so that they satisfy **H5** and fix them. Then we have

$$0 < 1 - \bar{M}_k t^{\mu} \quad \text{for} \quad t \in [t_k, t_{k+1}] \quad (k = 0, 1, 2, \dots, m-1).$$

The constants E_m will be chosen in a different manner.

Consider the problem (1) specifically without impulses, i.e., the equality ($1d$) being eliminated. Then the following result is known (Pao[9], Chapter 6, Theorem 3.1) concerning the IBVP $\big((1a), (1b), (1c)\big)$.

THEOREM 1 (Pao[9]). *Let $\bar{u}$, $\hat{u}$ be nonnegative functions in $[0, T^1) \times \bar{\Omega}$ and $[0, T^2) \times \bar{\Omega}$ such that $\bar{u} \geq \hat{u}$ and they are unbounded in $\bar{\Omega}$ as $t \to T^1$ and $t \to T^2$, respectively, where $T^1 \leq T^2$. If $f \in L_{loc}(\mathbb{R}_+)$ and for every $T < T^1$, $\bar{u}$ is an upper solution, and for $T < T^2$, $\hat{u}$ is a lower solution of the IBVP $\big((1a), (1b), (1c)\big)$ in Q_T then there exists $T^* \in [T^1, T^2]$ such that a unique solution u of the IBVP $\big((1a), (1b), (1c)\big)$ exists and satisfies the blowing-up property:*

$$\lim_{t \to T^*} [\max_{x \in \bar{\Omega}} u(t, x)] = +\infty. \tag{8}$$

3. Main result

THEOREM 2. *Assume that conditions H1–H5 hold and*

$$t_m < T_1 = \bar{M}_m^{-1/\mu} < T_2 = M_m^{-1/\mu} < T.$$

Then there exist $T^* \in [T_1, T_2]$ and a unique solution $u(t, x)$ of the IBVP (1) such that

$$\lim_{t \to T^*} [\max_{x \in \bar{\Omega}} u(t, x)] = +\infty. \tag{9}$$

Proof. The proof includes two parts. In the first part we construct a suitable function $\hat{u}(t, x)$ which is a lower solution for (1) possessing blowing-up property at the point T_2. In the second part we construct a suitable function $\bar{u}(t,x)$ which is an upper solution for (1) possessing blowing-up property at the point T_1.

Let us define the functions

$$p_k(t) = \hat{\delta}_k N_k^{-1/\gamma}(1 - M_k t^\mu)^{-1/\gamma} \text{ for } t \in [t_k, t_{k+1}) \ (k = 0, 1, 2, ..., m),$$

and

$$q_k(t) = \bar{\delta}_k \bar{N}_k^{-1/\gamma}(1 - \bar{M}_k t^\mu)^{-1/\gamma} \text{ for } t \in [t_k, t_{k+1}) \ (k = 0, 1, 2, ..., m).$$

All of $p_k(t)$ and all of $q_k(t)$ have only one singularity at $t = T_2$ and $t = T_1$ respectively. Define the function $\hat{u}(t,x)$ by

$$\hat{u}(t,x) \equiv p_k(t)\Phi_0(x) \text{ for } t \in [t_k, t_{k+1}) \text{ and } x \in \bar{\Omega}, \ (k = 0, 1, ..., m).$$

We will show that $\hat{u}(t,x)$ is a lower solution.

For $t \in [t_0, t_1)$, we have $\hat{u} \equiv p_0(t)\Phi_0(x)$ and

$$\hat{u}_t - \Delta\hat{u} = p_0'\Phi_0 + \lambda_0 p_0 \Phi_0 = (p_0' + \lambda_0 p_0)\Phi_0. \tag{10}$$

Here p' stands for dp/dt.

We can easily obtain that, from the definitions of p_0 and Ψ_0,

$$p_0' = \sigma_0 \, p_0^{\gamma+1} \, \Psi_0^\gamma \, t^{\mu-1} \le \sigma_0 \, p_0^{\gamma+1} \, \Phi_0^\gamma \, t^{\mu-1}.$$

Hence, making use of **H1**, we get

$$\hat{u}_t - \Delta\hat{u} \le \lambda_0 p_0 \Phi_0 + \sigma_0 \, p_0^{\gamma+1} \, \Phi_0^\gamma \, t^{\mu-1} \le f(t, x, \hat{u}).$$

We also have $\hat{u}(0,x) = \hat{\delta}_0\Phi_0(x) \le u_0(x)$, and $B\hat{u} = p_0(t)B\Phi_0 = 0$. It is easy to see that at the point $t_0^* = M_0^{-1/\mu}$, the function $p_0(t)$ tends to infinity. On the other hand, we also have that $t_0^* > t_1$ from **H4**. Since

$$\hat{u}(t_1^-, x) = p_0(t_1^-)\Phi_0(x) = \frac{\hat{\delta}_0}{(1 - M_0 t_1^\mu)^{1/\gamma}}\Phi_0(x)$$

and $g_1(\eta)$ is a nondecreasing function in η, we have

$$\begin{aligned} g_1(\hat{u}(t_1^-, x)) &\ge g_1\left(\frac{\hat{\delta}_0}{(1 - M_0 t_1^\mu)^{1/\gamma}}\Phi_0(x)\right) \\ &\ge e_1 \frac{\hat{\delta}_0}{(1 - M_0 t_1^\mu)^{1/\gamma}}\Phi_0(x) = \hat{\delta}_1\Phi_0(x) = \hat{u}(t_1, x). \end{aligned}$$

For $t \in [t_1, t_2)$ we have $\hat{u}(t,x) \equiv p_1(t)\Phi_0(x)$ and $\hat{u}_t - \Delta\hat{u} = (p_1' + \lambda_0 p_1)\Phi_0$. Making use of that

$$p_1' = \sigma_0 \, p_1^{\gamma+1} \, \Psi_0^\gamma \, t^{\mu-1} \le \sigma_0 \, p_1^{\gamma+1} \, \Phi_0^\gamma \, t^{\mu-1},$$

we obtain that $\hat{u}_t - \Delta\hat{u} \leq f(t,x,\hat{u})$. Moreover, we can easily see that $B\hat{u} = p_1 B\Phi_0 = 0$. Hence $\hat{u}(t,x)$ has the same properties for $t \in [t_k, t_{k+1})$ $(k = 1, 2, ..., m-1)$. It is easy to show that at the point $t_k^* = M_k^{-1/\mu}$, the function $p_k(t)$ tends to infinity. On the other hand, we also have that $t_k^* > t_{k+1}$ $(k = 1, 2, ..., m-1)$ from **H4**. Therefore, $\hat{u}(t,x)$ is a lower solutin of (1) and blows up at $t = M_m^{-1/\mu}$.

Let us define the function $\bar{u}(t,x)$ by

$$\bar{u}(t,x) \equiv q_k(t)\Phi_0(x) \quad \text{for} \quad (t,x) \in [t_k, t_{k+1}) \times \bar{\Omega} \quad (k = 0, 1, ..., m).$$

By the similar arguments as in the above, we can prove that $\bar{u}(t,x)$ is an upper solution of (1) and also blows up at $t = \bar{M}_m^{-1/\mu}$.

Therefore we can apply Theorem 1 and we obtain desired result. □

We should notice that the constants e_m and E_m are defined by the values of T_2 and T_1 respectively.

Remark. From the direct calculation of this theorem with the Neumann boundary condition for (1), that is the case when $b \equiv 0$, the principal eigenvalues λ_0 become zero and $\Phi_0(x) = 1$. In this case the solution of (1) blows up at some finite time T_1^* for any initial function $u_0(x) \geq 0$.

An important result concerning the existence and uniqueness of bounded solutions of (1) in Q_T puts in an appearance as a consequence of the previous Theorem 2, i.e., one can prove the following statement in a similar way as before:

Corollary. *Assume that the requirements of Theorem 2 hold and let $T < T_1 = \bar{M}_m^{-1/\mu}$. Then there exists a unique solution of* (1) *which is bounded in $\bar{Q}_T$.*

4. Example

We give one simple example and consider how to control the impulsive source really. Let $m = 3$, $t_0 = 0$, $t_1 = 1$, $t_2 = 2$, $t_3 = 3$, and $t_4 = 50 = T$. We can caluculate the principal eigenvalue λ_0 in a usual manner, but for convenience now we assume that $\lambda_0 = 1$. The exact form of the corresponding eigenfunction $\Phi_0(x)$ is genarally unknown. So, we also assume that the minimum Ψ_0 of $\Phi_0(x)$ on $\bar{\Omega}$ is equal to 0.1. Moreover we put $\mu = 2$, $\gamma = 2$, $\sigma_0 = 1$, and $\sigma_1 = 1.5$. Then the reaction function f satisfies

$$\eta + t\eta^3 < f(t,x,\eta) < \eta + 1.5t\eta^3.$$

We assume that $\hat{\delta}_0 = 0.398$ and $\bar{\delta}_0 = 0.407$ so that **H3** holds. Then we have

$$p_0(t) = \frac{0.398248}{\sqrt{1 - 0.00158602t^2}}, \quad \text{and} \quad q_0(t) = \frac{0.408248}{\sqrt{1 - 0.25t^2}}.$$

The blowing-up time of $p_0(t)$ $(q_0(t))$ is $M_0^{-1/\mu} = 25.11$ $(\bar{M}_0^{-1/\mu} = 2.01)$, respectively.

We choose the constants e_1 and E_1 so that they satisfy **H4** and **H5**. Namely, we put $e_1 = 0.9$ and $E_1 = 1.05$. In this case, we have $\hat{\delta}_1 = 0.358$ and $\bar{\delta}_1 = 0.429$ and we obtain

$$p_1(t) = \frac{0.358477}{\sqrt{1 - 0.00128506t^2}}, \quad \text{and} \quad q_1(t) = \frac{0.379771}{\sqrt{1 - 0.21634t^2}}.$$

Moreover, we choose the constants e_2 and E_2 so that they also satisfy **H4** and **H5**. Namely, $e_2 = 0.9$ and $E_2 = 0.91$. Then we have $\hat{\delta}_2 = 0.323$ and $\bar{\delta}_2 = 0.346$ and we obtain

$$p_2(t) = \frac{0.322787}{\sqrt{1 - 0.00104192t^2}}, \quad \text{and} \quad q_2(t) = \frac{0.264168}{\sqrt{1 - 0.104677t^2}}.$$

Finally, we make the solution to grow unboundedly as $t \to T^*$, where $T^* \in [3.4, 36]$. We put $e_3 = 0.9$ and $E_3 = 1.7$. Then we have $T_1 = \bar{M}_3^{-1/\mu} = 3.50351$ and $T_2 = M_3^{-1/\mu} = 34.3917$. In this case, we have $\hat{\delta}_3 = 0.29188$ and $\bar{\delta}_3 = 0.451206$ and we also obtain

$$p_3(t) = \frac{0.290768}{\sqrt{1 - 0.000845459t^2}}, \quad \text{and} \quad q_3(t) = \frac{0.233051}{\sqrt{1 - 0.081469t^2}}.$$

If we define the impulsive source g_k $(k = 1, 2, 3)$ so that they satisfy **H2** for the above constants, then there exists $T^* \in [T_1, T_2] \subset [3.4, 36]$ such that the unique solution of (1) exists in $[0, T^*)$ and grows unboundedly as $t \to T^*$.

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