J. Korean Math. Soc. **37** (2000), No. 5, pp. 763–777

EXISTENCE OF GROUP INVARIANT SOLUTIONS OF A SEMILINEAR ELLIPTIC EQUATION

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Abstract. We investigate the existence of group invariant solutions of the Emden-Fowler equation, $-\Delta u = |x|^\sigma |u|^{p-1}u$ in B, $u = 0$ on ∂B and $u(gx) = u(x)$ in B for $g \in G$. Here B is the unit ball in $\mathbb{R}^n$, $n \geq 2$, $1 < p < (n+2)/(n-2)$, $\sigma \geq 0$ and G is a closed subgroup of the orthogonal group. A solution of the problem is called a G invariant solution. We prove that there exists a G invariant non-radial solution if and only if G is not transitive on the unit sphere.

1. Introduction

We consider the existence of group invariant solutions of the Emden-Fowler equation,

$$(1.1) \qquad -\Delta u = |x|^\sigma |u|^{p-1}u, \qquad x \in B,$$

$$(1.2) \qquad u = 0, \qquad x \in \partial B,$$

$$(1.3) \qquad u(gx) = u(x), \qquad x \in B, \quad g \in G,$$

where $B = \{x \in \mathbb{R}^n : |x| < 1\}$, $n \geq 2$, $1 < p < (n+2)/(n-2)$, $\sigma \geq 0$ and G is a closed subgroup of the orthogonal group $O(n)$. We call a solution of (1.1)-(1.3) a G *invariant solution*. In this paper, we extend the result of [8] with $\sigma = 0$ to the case $\sigma \geq 0$. It is known that (1.1)-(1.2) has infinitely many radially symmetric solutions. Any radially symmetric solution becomes a G invariant solution. We consider the converse problem: Must a G invariant solution be radially symmetric? Otherwise, does there exists a G invariant non-radial solution?

Received August 7, 1999.

2000 Mathematics Subject Classification: Primary 35J20; Secondary 35J60.

Key words and phrases: semilinear elliptic equation, Emden-Fowler equation, group invariant solution, non-radial solution, variational method.

Since G is a closed subgroup of the orthogonal group $O(n)$, G is a transformation group on the unit sphere S^{n-1},

$$S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}.$$

G is said to be *transitive* on S^{n-1} if for any two points $x, y \in S^{n-1}$ there exists a $g \in G$ such that $gx = y$. Our answer to the problem is as follows.

THEOREM 1. *The following two assertions are equivalent.*

(i) *G is not transitive on S^{n-1}.*

(ii) *There exists a sequence $\{u_k\}_{k=1}^{\infty}$ of solutions of* (1.1)-(1.3) *such that each u_k is G invariant, not radially symmetric and*

$$0 < \|u_1\|_{H_0^1(B)} < \|u_2\|_{H_0^1(B)} < \|u_3\|_{H_0^1(B)} < \cdots \nearrow \infty,$$

where $\|\cdot\|_{H_0^1(B)}$ denotes the L^2 Sobolev norm of the first order.

It is clear that when G is transitive on S^{n-1}, any G invariant solution has radial symmetry. Therefore the assertion (ii) implies (i). Theorem 1 asserts mainly that (i) implies (ii).

COROLLARY 1. *If $\dim G \leq n-2$, then the assertion* (ii) *of Theorem 1 holds.*

COROLLARY 2. *If G is a finite subgroup of $O(n)$, then the assertion* (ii) *of Theorem 1 remains valid.*

By Cartan's theorem [7, p115, Theorem 2.3], a closed subgroup G of $O(n)$ is also a Lie subgroup. Therefore $\dim G$ denotes the dimension of Lie group G. If $\dim G \leq n-2 < \dim S^{n-1}$, then G can not be transitive on S^{n-1}. Therefore Corollary 1 follows from Theorem 1. Corollary 2 is trivial because a finite group is not transitive.

We consider the question related to Theorem 1: What kind of G is transitive? This problem has already been solved by Montgomery, Samelson and Borel.

THEOREM A ([9], [3]). *Let $n \geq 2$ and G be a connected closed subgroup of $SO(n)$. Then the following are equivalent.*

(i) *G is transitive on S^{n-1}.*

(ii) *G is $O(n)$-conjugate to one of the following subgroups : $SO(n)$; $SU(m)$, $U(m)$ $(n = 2m)$; $Sp(m)$, $Sp(m)Sp(1)$, $Sp(m)U(1)$ $(n = 4m)$; $Spin(9)$ $(n = 16)$; $Spin(7)$ $(n = 8)$; G_2 $(n = 7)$.*

If G is not necessarily connected, Theorem A implies the next theorem.

THEOREM B. *Let $n \geq 2$ and G be a closed subgroup of $O(n)$. Then the following are equivalent.*

(i) *G is transitive on S^{n-1}.*

(ii) *The connected component of G which has the unit matrix is $O(n)$-conjugate to one of the Lie groups listed in* (ii) *of Theorem A.*

2. G invariant eigenvalues

In this section, we investigate the upper estimate for the G invariant eigenvalues of the problem,

$$\begin{cases} -\Delta u = \lambda |x|^\sigma u, & x \in B, \\ u = 0, & x \in \partial B, \\ u(gx) = u(x), & x \in B, \quad g \in G. \end{cases} \tag{2.1}$$

This problem has countably infinitely many eigenvalues $\{\lambda_k\}$,

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \nearrow .$$

Here the eigenvalue is repeated a number of times equal to its multiplicity. We define

$$G(x) = \{gx : g \in G\} \qquad \text{for } x \in S^{n-1}, \tag{2.2}$$

$$m = \max\{\dim G(x) : x \in S^{n-1}\}. \tag{2.3}$$

For each $x \in S^{n-1}$, the orbit $G(x)$ is a closed submanifold of S^{n-1}, and so the number m is well-defined. If G is not transitive, then $0 \leq m \leq n-2$. The main assertion in this section is as follows.

PROPOSITION 2.1. *Let G be not transitive on S^{n-1}. Then $0 \leq m \leq n-2$ and there exists a constant $C > 0$ such that*

$$\lambda_k \leq C k^{2/(n-m)} \textit{ for } k \in \mathbb{N}.$$

To prove this proposition, we introduce norms, function spaces and operators.

DEFINITION 2.2. We define

$$\|u\|_{q,\sigma} = \left(\int_B |u|^q |x|^\sigma dx \right)^{1/q},$$

$$\|u\|_q = \|u\|_{q,0} = \left(\int_B |u|^q dx \right)^{1/q}.$$

In case of $q = 2$, the inner product is defined by

$$(u, v)_{2,\sigma} = \int_B uv|x|^\sigma dx,$$

$$(u, v)_2 = (u, v)_{2,0} = \int_B uv dx.$$

We set

$L^q_\sigma(B) = \{u : \|u\|_{q,\sigma} < \infty\}$,
$L^q_\sigma(B, G) = \{u \in L^q_\sigma(B) : u(gx) = u(x) \ (x \in B, g \in G)\}$,
$H^1_0(B, G) = \{u \in H^1_0(B) : u(gx) = u(x) \ (x \in B, g \in G)\}$,
$Au = -|x|^{-\sigma} \Delta u$,
$A_G u = -|x|^{-\sigma} \Delta u$,
$D(A) = \{u \in L^2_\sigma(B) \cap H^1_0(B) : Au \in L^2_\sigma(B)\}$,
$D(A_G) = D(A) \cap L^2_\sigma(B, G)$.

LEMMA 2.3. *The operator A_G is self-adjoint in $L^2_\sigma(B, G)$ and has a compact resolvent.*

We will give the proof of the lemma in Appendix. Lemma 2.3 means that the spectrum of A_G consists only of discrete eigenvalues and the system of eigenfunctions is complete in $L^2_\sigma(B, G)$. The eigenvalue λ_k is characterized by the minimax value of the Rayleigh quotient,

$$(A_G u, u)_{2,\sigma} / (u, u)_{2,\sigma} = -\int_B |x|^{-\sigma} \Delta u u |x|^\sigma dx \Big/ \int_B u^2 |x|^\sigma dx$$
$$= \int_B |\nabla u|^2 dx \Big/ \int_B u^2 |x|^\sigma dx \equiv R(u).$$

It is well-known that (see [5, p. 405], [11, p. 76])

$$\lambda_1 = \inf\{R(u) : u \in H^1_0(B, G) \setminus \{0\}\},$$
$$\lambda_k = \sup\{d(v_1, \cdots, v_{k-1}) : v_1, \cdots, v_{k-1} \in H^1_0(B, G)\} \quad \text{for } k \geq 2,$$

where

$$d(v_1, \cdots, v_{k-1}) = \inf \big\{ R(u) : u \in H_0^1(B, G) \setminus \{0\}, \\ (u, v_i)_{2,\sigma} = 0 \text{ for } 1 \le i \le k-1 \big\}.$$

Proof of Proposition 2.1. We set

$$D = \{x \in \mathbb{R}^n : \ 1/2 < |x| < 1\}.$$

Let μ_k denote the k-th eigenvalue of the problem (2.1) with D instead of B. From $D \subset B$, it follows that $\lambda_k \le \mu_k$. The Rayleigh quotient associated with μ_k is

$$R(u) = \frac{\displaystyle\int_D |\nabla u|^2 dx}{\displaystyle\int_D u^2 |x|^\sigma dx} \le 2^\sigma \frac{\displaystyle\int_D |\nabla u|^2 dx}{\displaystyle\int_D u^2 dx}.$$

The minimax value of the last quotient is the k-th eigenvalue ν_k of the problem,

$$\begin{cases} -\Delta u = \nu u, & x \in D, \\ u = 0, & x \in \partial D, \\ u(gx) = u(x), & x \in D, \quad g \in G. \end{cases}$$

Therefore $\lambda_k \le \mu_k \le 2^\sigma \nu_k$. By [8, Proposition 3.3], there exists a constant $C > 0$ such that $\nu_k \le C k^{2/(n-m)}$ for $k \in \mathbb{N}$. This completes the proof. □

3. G invariant critical values

Our argument to prove Theorem 1 is based on the variational method of the functional $I(u)$,

$$I(u) = \int_B \Big(\frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} |x|^\sigma |u|^{p+1} \Big) dx, \quad u \in H_0^1(B, G).$$

The solution of (1.1)-(1.3) is considered as a critical point of $I(\cdot)$ because $I'(u)$ is calculated as

$$I'(u)v = \int_B \Big(\nabla u \nabla v - |x|^\sigma |u|^{p-1} u v \Big) dx \qquad \text{for} \quad v \in H_0^1(B, G).$$

If $I'(u)v = 0$ for all $v \in H_0^1(B, G)$, then this identity remains valid for any $v \in H_0^1(B)$. See [8] for the proof. Therefore u satisfies (1.1)-(1.3) in the distribution sense. The elliptic regularity theory proves that u is of class $C^2(\overline{B})$. Consequently, to prove Theorem 1, we have only to find a non-radial critical point in $H_0^1(B,G)$.

In this section, we use the symmetric mountain pass method by Ambrosetti-Rabinowitz [1] to construct G invariant critical values.

DEFINITION 3.1. Let K be the closed subset of $H_0^1(B,G)$ such that $0 \notin K$ and $-u \in K$ for $u \in K$. We define the *genus* $\gamma(K)$ of K by the smallest integer k such that there exists an odd continuous mapping from K to $\mathbb{R}^k \setminus \{0\}$. When there does not exist a finite such k, we set $\gamma(K) = \infty$.

In the next proposition, we obtain G invariant critical values.

PROPOSITION 3.2. *There exists a sequence $\{\alpha_k\}$ of real numbers which satisfies the following conditions.*

(i) *Each α_k is a G invariant critical value of $I(\cdot)$.*

(ii) $0 < \alpha_1 \le \alpha_2 \le \cdots \nearrow \infty$.

(iii) *If $\alpha_k = \alpha_{k+1} = \cdots = \alpha_{k+j} \equiv \alpha$, then it holds that $\gamma(K_\alpha) \ge j+1$, where*

$$K_\alpha = \{u \in H_0^1(B,G) : I'(u) = 0 \text{ and } I(u) = \alpha\}.$$

(iv) *There is a constant $C > 0$ such that*

$$\alpha_k \le C k^{\frac{2(p-1)}{(n-m)(p-1)}} \quad \text{for } k \in \mathbb{N},$$

where m is defined by (2.3).

Proof. This proposition except for (iv) is due to Ambrosetti and Rabinowitz [1]. The proof of (iv) is proved in the same way as [8] by using Proposition 2.1. However, for the reader's convenience, we give the proof. Let λ_k be the k-th eigenvalue of (2.1) and ϕ_k the eigenfunction. We set

$$E_k = \text{span}\{\phi_1, \cdots, \phi_k\}.$$

Recall the definition of $I(u)$,

$$I(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1,\sigma}^{p+1}, \tag{3.1}$$

where the norm $\|\cdot\|_{q,\sigma}$ and $\|\cdot\|_q$ are defined by Definition 2.2. Since each E_k is a finite dimensional linear space, the $H_0^1(B)$ norm is equivalent to the $L_\sigma^{p+1}(B)$ norm, and so we have a sequence $\{R_k\}$ such that

$$I(u) \le 0 \quad \text{for } \|u\|_{H_0^1} \ge R_k \text{ and } u \in E_k. \tag{3.2}$$

We may assume $R_1 < R_2 < \cdots \nearrow \infty$ without loss of generality. Set

$$\begin{aligned} \Gamma_k &= \{h \in C(D_k, H_0^1(B,G)) : h \text{ is odd on } D_k \text{ and } h(u) = u \text{ on } \partial D_k\}, \\ D_k &= \{u \in E_k : \|u\|_{H_0^1} \leq R_k\}, \\ \partial D_k &= \{u \in E_k : \|u\|_{H_0^1} = R_k\}. \end{aligned}$$

We define

$$\alpha_k = \inf_{h\in\Gamma_k} \max_{u\in D_k} I(h(u)). \tag{3.3}$$

The assertions (i)-(iii) are proved in the same method as [10, pp55-60]. To prove (iv), we take $h =$ identity $\in \Gamma_k$ and obtain

$$\alpha_k \leq \max_{u\in D_k} I(u) \leq \sup_{E_k} I(u). \tag{3.4}$$

It follows from the definition of E_k that $\|\nabla u\|_2^2 \leq \lambda_k \|u\|_{2,\sigma}^2$ for $u \in E_k$. Since $p > 1$, there is a $C > 0$ such that $\|u\|_{2,\sigma} \leq C\|u\|_{p+1,\sigma}$. Therefore there is a constant c_0 such that $c_0\lambda_k^{-(p+1)/2}\|\nabla u\|_2^{p+1} \leq \|u\|_{p+1,\sigma}^{p+1}$ for $u \in E_k$. Hence we have

$$\alpha_k \leq \sup_{E_k}\{\frac{1}{2}\|\nabla u\|_2^2 - \frac{c_0}{p+1}\lambda_k^{-(p+1)/2}\|\nabla u\|_2^{p+1}\} \leq C_1\lambda_k^{(p+1)/(p-1)}.$$

Here C_1 is independent of k. This inequality with Proposition 2.1 completes the proof. □

4. Radially symmetric critical values

In this section, we consider the radially symmetric solutions of (1.1)–(1.2). We set $u = u(r), r = |x|$ and reduce (1.1)–(1.2) to

$$u'' + \frac{n-1}{r}u' + r^\sigma |u|^{p-1}u = 0, \qquad 0 < r < 1, \tag{4.1}$$

$$u'(0) = 0, \qquad u(1) = 0. \tag{4.2}$$

PROPOSITION 4.1. *For each integer $k \geq 1$ there exists a unique solution u of (4.1)-(4.2) such that $u(0) > 0$ and $u(r)$ has exactly k zeros in $[0,1]$. Denote it by u_k. Then the set of all solutions of (4.1)-(4.2) consists of u_k, $-u_k$ for all $k \in \mathbb{N}$ and the zero solution. Set $\beta_k = I(u_k) = I(-u_k)$. Then it holds that :*

(i) $0 < \beta_1 < \beta_2 < \cdots \nearrow \infty$.
(ii) $I(u_k) = \beta_k$ *and* $I'(u_k) = 0$.

(iii) *There exists a constant* $c > 0$ *such that*

$$ck^{2(p+1)/(p-1)} \le \beta_k \quad \text{for } k \in \mathbb{N}.$$

Proof. Instead of (4.1)-(4.2), we consider the initial value problem,

$$w'' + \frac{n-1}{r}w' + r^{\sigma}|w|^{p-1}w = 0, \quad 0 < r < \infty, \tag{4.3}$$

$$w'(0) = 0, \quad w(0) = 1. \tag{4.4}$$

This problem has a unique global solution $w(r)$ and it is oscillatory, i.e., it has an unbounded sequence of zeros. To check this, we consider the energy function,

$$E(r) = \frac{1}{2}w'(r)^2 r^{-\sigma} + \frac{1}{p+1}|w(r)|^{p+1}.$$

Then

$$E'(r) = -(n + \frac{\sigma}{2} - 1)w'(r)^2 r^{-\sigma-1} \le 0,$$

which means that $E(r)$ is bounded above. This shows the global existence of $w(r)$.

Let $n \ge 3$. To show that w is oscillatory, we take a change of variable, $t = r^{n-2}$, $v(t) = tw(r)$, which reduces (4.3)-(4.4) to

$$v'' + a(t)|v|^{p-1}v = 0, \quad t > 0,$$

$$v'(0) = 1, \qquad v(0) = 0,$$

where $a(t) = (n-2)^{-2}t^{(\sigma+2)/(n-2)-p-1}$. Since $p < (n+2)/(n-2)$ and $\sigma \ge 0$, $t^{(p+3)/2}a(t)$ is nondecreasing. By [4, Corollary 10], the solution $v(t)$ with a zero at $t = 0$ is oscillatory. Hence $w(r)$ is also oscillatory.

When $n = 2$, we set $t = \log r$, $v(t) = w(r)$, which reduces (4.3) to

$$v'' + a(t)|v|^{p-1}v = 0, \qquad a(t) = e^{(\sigma+2)t}. \tag{4.5}$$

Since $\int_0^{\infty} ta(t)dt = \infty$, all solutions of (4.5) are oscillatory by [2].

We denote the zeros of the solution w of (4.3)-(4.4) by $\{t_k\}$, $0 < t_1 < t_2 < t_3 < \cdots \nearrow \infty$. We set

$$u_k(r) = t_k^{(\sigma+2)/(p-1)}w(t_k r). \tag{4.6}$$

It is a solution of (4.1)-(4.2) such that $u(0) > 0$ and it has exactly k zeros in $[0, 1]$. Such a solution is unique because $\lambda^{(\sigma+2)/(p-1)}w(\lambda r)$ $(\lambda > 0)$ represents any solution of (4.1) with $u'(0) = 0$ and $u(0) > 0$. It is clear that the set of all solutions of (4.1)-(4.2) consists of u_k, $-u_k$, for $k \in \mathbb{N}$ and the zero solution.

We show the assertion (i) of Proposition 4.1. Multiplying (4.1) by $u(r)r^{n-1}$ and integrating by parts, we get

$$\int_0^1 u'(r)^2 r^{n-1} dr = \int_0^1 |u|^{p+1} r^{\sigma+n-1} dr.$$

Therefore we have

$$I(u) = \frac{(p-1)\omega_n}{2(p+1)} \int_0^1 |u|^{p+1} r^{\sigma+n-1} dr, \tag{4.7}$$

for any solution u of (4.1)-(4.2). Here ω_n is the surface area of the unit sphere. This identity with (4.6) gives

$$\beta_k = I(u_k) = \frac{(p-1)\omega_n}{2(p+1)} t_k^{(\sigma+2)(p+1)/(p-1)-n-\sigma} \int_0^{t_k} |w(s)|^{p+1} s^{\sigma+n-1} ds,$$

which proves the assertion (i) because of $(\sigma+2)(p+1)/(p-1)-n-\sigma > 0$.

The assertion (ii) is trivial. We show (iii). Let $n \geq 3$. We take the change of variable,

$$t = r^{1/\alpha}, \quad v(t) = t^{\beta} u(r), \tag{4.8}$$

where $2\beta = (n-2)\alpha + 1$ and $\alpha > 1$ will be determined later. Then equation (4.1) is reduced to

$$v'' + \alpha^2 t^{\gamma} |v|^{p-1} v - \beta(\beta-1) t^{-2} v = 0, \qquad 0 < t < 1, \tag{4.9}$$

where $\gamma = (\sigma+2)\alpha - (p-1)\beta - 2$. We need the next lemma.

Lemma 4.2 ([6, p. 346, Corollary 5.2]). *Let $q(t)$ be a continuous function on $[a,b]$. Let $v(t) \not\equiv 0$ be a solution of the equation,*

$$v'' + q(t)v = 0, \quad t \in [a,b].$$

Assume that $v(t)$ has exactly k zeros in $(a,b]$. Then we have

$$k < \frac{1}{2}\left((b-a)\int_a^b q^+(t)dt\right)^{1/2} + 1,$$

where $q^+(t) \equiv \max\{q(t), 0\}$.

We set $q(t) = \alpha^2 t^{\gamma} |v(t)|^{p-1} - \beta(\beta-1)t^{-2}$. Since $\alpha > 1$ will be chosen very large, we see $\beta > 1$, and so we have $q^+(t) \leq \alpha^2 t^{\gamma} |v(t)|^{p-1}$. Let v be a solution of (4.9) corresponding to u_k. Then we choose $\varepsilon > 0$ so small that v has k zeros in $[\varepsilon, 1]$ and $q(t)$ is continuous on $[\varepsilon, 1]$. Applying Lemma 4.2 on $[\varepsilon, 1]$, we get

$$\begin{aligned} k &< \frac{1}{2}\left((1-\varepsilon)\int_\varepsilon^1 q^+(t)dt\right)^{1/2} + 1 \\ &\leq \frac{\alpha}{2}\left(\int_0^1 |v(t)|^{p-1}t^\gamma dt\right)^{1/2} + 1, \\ &\leq \frac{\sqrt{\alpha}}{2}\left(\int_0^1 |u(r)|^{p-1}r^{\sigma+1-1/\alpha}dr\right)^{1/2} + 1, \end{aligned}$$

where we have used relation (4.8) in the last inequality. We set

$$\mu = \sigma + 1 - 1/\alpha - (\sigma + n - 1)(p-1)/(p+1)$$

and rewrite the last integral into

$$\int_0^1 (|u|^{p+1}r^{\sigma+n-1})^{(p-1)/(p+1)}r^\mu dr.$$

By Hölder's inequality with exponents $(p+1)/(p-1)$ and $(p+1)/2$, this integral is estimated from above by

$$\left(\int_0^1 |u|^{p+1}r^{\sigma+n-1}dr\right)^{(p-1)/(p+1)} \left(\int_0^1 r^{\mu(p+1)/2}dr\right)^{2/(p+1)}.$$

We choose $\alpha > 1$ so large that $\mu(p+1)/2 > -1$ and the last integral is finite. Therefore we have

$$k \leq C\left(\int_0^1 |u|^{p+1}r^{\sigma+n-1}dr\right)^{(p-1)/2(p+1)} + 1,$$

or equivalently

$$k - 1 \leq C\beta_k^{(p-1)/2(p+1)},$$

where C is independent of k. This shows the assertion (iii) except for $k=1$.

We show a priori lower bound of any solution of (1.1)-(1.2). Multiplying (1.1) by u and integrating by parts, we get

$$\int_B |\nabla u|^2 dx = \int_B |u|^{p+1}|x|^\sigma dx \leq \int_B |u|^{p+1}dx \leq C(\int_B |\nabla u|^2 dx)^{(p+1)/2},$$

where we have used Sobolev's inequality. This gives a constant $C > 0$ such that

$$C \le \int_B |\nabla u|^2 dx = \int_B |u|^{p+1}|x|^\sigma dx$$

for any solution $u \not\equiv 0$ of (1.1)-(1.2). This proves (iii) for $k = 1$.

Let $n = 2$. The change of variable $t = 1/(1 - \log r)$, $v(t) = tu(r)$, transforms (4.1) to

$$v'' + t^{-p-3} e^{(\sigma+2)(t-1)/t} |v|^{p-1} v = 0, \qquad 0 < t < 1.$$

We set

$$q(t) = t^{-p-3} e^{(\sigma+2)(t-1)/t} |v(t)|^{p-1}$$

and apply Lemma 4.2 to obtain

$$\begin{aligned} k \ &< \frac{1}{2}\left(\int_0^1 q^+(t)dt\right)^{1/2} + 1 \\ &= C\left(\int_0^1 |u(r)|^{p-1} r^{\sigma+1} |\log(r/e)|^2 dr\right)^{1/2} + 1 \\ &\le C\left(\int_0^1 |u|^{p+1} r^{\sigma+1} dr\right)^{(p-1)/2(p+1)} \left(\int_0^1 r^{\sigma+1} |\log(r/e)|^{p+1} dr\right)^{1/(p+1)} + 1, \end{aligned}$$

by Hölder's inequality. Therefore we obtain the assertion (iii) for $n = 2$. □

5. Proof of Theorem 1

Our idea to prove Theorem 1 is to make use of the difference between the distribution of G invariant critical values $\{\alpha_k\}$ and that of the radially symmetric critical values $\{\beta_k\}$.

Proof of Theorem 1. If G is transitive, any G invariant solution becomes a radially symmetric solution. That is, the assertion (ii) implies (i).

We prove the converse by contradiction. Suppose that the condition (i) holds but (ii) does not. Then there exists an integer $k_0 \ge 1$ such that

$$\{\alpha_k : k \ge k_0\} \subset \{\beta_k : k \ge 1\}.$$

If $\alpha_k = \alpha_{k+1}$ with some $k \geq k_0$, then (iii) of Proposition 3.2 gives $\gamma(K) \geq 2$, where

$$K = \{u \in H_0^1(B,G) : I'(u) = 0 \text{ and } I(u) = \alpha_k\}.$$

Since $\alpha_k = \beta_j$ with a certain $j \geq 1$, we see $K = \{u_j, -u_j\}$. Here u_j is the radially symmetric solution with exactly j zeros. Since the genus of a finite set is one by definition, it holds that $\gamma(K) = 1$. This contradiction shows that $\{\alpha_k\}$ is strictly increasing for $k \geq k_0$. Therefore we have a sequence $\{\mu(k)\}$ of positive integers such that $\alpha_{k+k_0} = \beta_{\mu(k)}$ for $k \geq 1$. Since $\{\alpha_k\}$ and $\{\beta_k\}$ are strictly increasing, so is $\{\mu(k)\}$, and it holds that $k \leq \mu(k)$ for $k \geq 1$. Hence (iv) of Proposition 3.2 and (iii) of Proposition 4.1 show

$$C_1 k^{2(p+1)/(p-1)} \leq \beta_k \leq \alpha_{k+k_0} \leq C_2(k+k_0)^{2(p+1)/((n-m)(p-1))}.$$

This yields a contradiction by letting $k \to \infty$ because of $0 \leq m \leq n-2$. The proof is complete. □

Appendix

Lemma 2.3 can be proved in the same way as [8] with the help of the next lemma.

LEMMA A.1. *Let the operator A be defined in Definition 2.2. Then A is a self-adjoint operator in $L^2_\sigma(B)$ and has a compact resolvent.*

Proof. For $u \in D(A)$, we have the identity

$$(Au, u)_{2,\sigma} = -\int_B \Delta u \cdot u dx = \|\nabla u\|_2^2.$$

Poincaré's inequality shows

$$\text{(A.1)} \qquad \|u\|_{2,\sigma}^2 = \int_B u^2 |x|^\sigma dx \leq \int_B u^2 dx \leq C\|\nabla u\|_2^2 \quad \text{for } u \in H_0^1(B).$$

Here C is a positive constant independent of u. These two inequalities imply

$$\text{(A.2)} \qquad \|\nabla u\|_2 \leq C\|Au\|_{2,\sigma}.$$

Let $Au_k = v_k$ and $\{v_k\}$ be bounded in $L^2_\sigma(B)$. Then inequality (A.2) means that $\{u_k\}$ is bounded in $H_0^1(B)$. Therefore $\{u_k\}$ has a convergent subsequence in $L^2(B)$, and so in $L^2_\sigma(B)$. Thus A has a compact resolvent.

We show that $D(A)$ is dense in $L^2_\sigma(B)$. To this end, we define

$$X = \{u \in C_0^\infty(B) : u(x) \equiv \text{constant near } x = 0\}.$$

Then $X \subset D(A) \subset L^2_\sigma(B)$. Let $u \in L^2_\sigma(B)$ and set

$$u_\varepsilon = \begin{cases} u & (\varepsilon < |x| < 1-\varepsilon) \\ 0 & (\text{otherwise}). \end{cases}$$

Then u_ε converges to u in $L^2_\sigma(B)$ as $\varepsilon \to 0$. We use a mollifier $J_\delta * u_\varepsilon \in X$, which is a convolution of J_δ and u_ε. Here $J_\delta(x) = \delta^{-n} J(x/\delta)$ with $0 < \delta < \varepsilon$ and J satisfies

$$J \in C_0^\infty(\mathbb{R}^n), \quad \text{supp } J \subset B, \quad J \geq 0, \quad \int_{\mathbb{R}^n} J(x)dx = 1.$$

Since $J_\delta * u_\varepsilon$ converges to u_ε in $L^2_\sigma(B)$ as $\delta \to 0$, the space X, and $D(A)$ also, is dense in $L^2_\sigma(B)$. Therefore the adjoint operator A^* is well-defined.

We show that A is self-adjoint. For $u, v \in D(A)$, the relation

$$(Au, v)_{2,\sigma} = -\int_B \Delta u v dx = (u, Av)_{2,\sigma},$$

implies that A is a symmetric operator, i.e., $A \subset A^*$. Therefore we have only to prove $D(A^*) \subset D(A)$. Let $v \in D(A^*)$. Then there exists a $w \in L^2_\sigma(B)$ such that

$$(Au, v)_{2,\sigma} = (u, w)_{2,\sigma} \qquad \text{for } u \in D(A),$$

or equivalently

$$\text{(A.3)} \qquad -\int_B \Delta u v dx = \int_B u w |x|^\sigma dx \qquad \text{for } u \in D(A).$$

Let $B_\delta = \{x \in \mathbb{R}^n : \delta < |x| < 1\}$ for $0 < \delta < 1$. Relation (A.3) for any $u \in C_0^\infty(B_\delta) \subset D(A)$ shows

$$-\Delta v = |x|^\sigma w \qquad \text{a.e. on } B_\delta.$$

Since $0 < \delta < 1$ is arbitrarily, this relation is valid for a.e. on B, and $Av = -|x|^{-\sigma}\Delta v = w \in L^2_\sigma$.

We show that $v \in H_0^1(B)$. We set $u_\varepsilon = (1 + \varepsilon A)^{-1} v$ for $\varepsilon > 0$. This is well-defined because $-1/\varepsilon$ is in the resolvent set of A by (A.1) and (A.2). It holds that

$$\text{(A.4)} \qquad u_\varepsilon \in D(A), \quad \|u_\varepsilon\|_{2,\sigma} \leq \|v\|_{2,\sigma},$$

and u_ε converges to v in $L^2_\sigma(B)$ as $\varepsilon \to 0+$. Substituting $u = u_\varepsilon$ into (A.3), we get

$$\text{(A.5)} \qquad -\int_B \Delta u_\varepsilon v dx = \int_B u_\varepsilon w |x|^\sigma dx.$$

The definition of u_ε gives

$$u_\varepsilon - \varepsilon |x|^{-\sigma} \Delta u_\varepsilon = v.$$

We multiply both sides by $-\Delta u_\varepsilon$ to get

$$-\Delta u_\varepsilon u_\varepsilon \leq -\Delta u_\varepsilon v.$$

Since $u_\varepsilon \in D(A) \subset H^2(B) \cap H^1_0(B)$, the integration by parts yields

$$\|u_\varepsilon\|^2_{H^1_0(B)} \equiv \int_B |\nabla u_\varepsilon|^2 dx \leq -\int_B \Delta u_\varepsilon v dx,$$

which together with (A.5) and (A.4) shows

$$\|u_\varepsilon\|^2_{H^1_0(B)} \leq \left(\int_B u_\varepsilon^2 |x|^\sigma dx\right)^{1/2} \left(\int_B w^2 |x|^\sigma dx\right)^{1/2} \leq \|v\|_{2,\sigma} \|w\|_{2,\sigma}.$$

Therefore there exists a sequence $\{\varepsilon_k\}$ convergent to 0 such that $\{u_{\varepsilon_k}\}$ has a weak limit in $H^1_0(B)$. Since u_ε converges to v in $L^2_\sigma(B)$, v is in $H^1_0(B)$. Consequently, $v \in D(A)$. This completes the proof. □

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