J. Korean Math. Soc. **37** (2000), No. 5, pp. 751–761

RADIAL SYMMETRY OF POSITIVE SOLUTIONS FOR SEMILINEAR ELLIPTIC EQUATIONS IN $\mathbb{R}^n$

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Abstract. Symmetry properties of positive solutions for semilinear elliptic problems in $\mathbb{R}^n$ are considered. We give a symmetry result for the problem in the general case, and then derive various results for certain classes of semilinear elliptic equations. We employ the moving plane method based on the maximum principle on unbounded domains to obtain the result on symmetry.

1. Introduction

In this paper, we study the radial symmetry of positive solutions for semilinear elliptic equations in $\mathbb{R}^n$. We consider the problem of the form

$$(1.1)\qquad \begin{cases} \Delta u + f(|x|, u) = 0 & \text{in } \mathbb{R}^n, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases}$$

where $n \geq 3$. We establish a symmetry result for the problem (1.1) in the general case, and then derive various results for certain classes of semilinear elliptic equations. Typical equations we shall be interested in are as follows:

$$\Delta u - u + u^p = 0, \quad p > 1;$$

$$\Delta u + \frac{1}{(1+|x|)^\ell} u^p = 0, \quad \ell \geq 0,\ p > 1;$$

$$\Delta u + \frac{\beta}{(1+|x|)^\mu} u^p - \frac{1}{(1+|x|)^\nu} u^q = 0, \quad \beta > 0,\ \mu, \nu > 2,\ p, q \geq 1.$$

Some of those results are already known, but our aim is to treat those results from a unified point of view.

Received August 18, 1999.

2000 Mathematics Subject Classification: 35J60.

Key words and phrases: radial symmetry, semilinear elliptic equation, moving plane method.

Our arguments are based on the moving plane method. The method was first developed by Serrin [13] in PDE theory, and later extended and generalized by Gidas, Ni, and Nirenberg [2, 3]. In this paper, we employ the moving plane method based on the maximum principle on unbounded domains to obtain the result on symmetry.

In Section 2, we state the main theorem, and give some corollaries of the theorem. In Section 3, we prove the theorem by using the method of moving planes.

2. Statement of the results

In (1.1), we assume that $f(r,u)$ is continuous and C^1 in $u \geq 0$, and that $f(r,u)$ is nonincreasing in $r > 0$ for each fixed $u \geq 0$. Our main result is the following:

THEOREM 1. *Let $u \in C^2(\mathbb{R}^n)$ be a positive solution of (1.1). Define U and Φ as*

(2.1)
$$U(r) = \sup\{u(x) : |x| \geq r\} \quad \text{and} \quad \Phi(r) = \sup\{f_u(r,s) : 0 \leq s \leq U(r)\},$$

respectively. Assume that there exists a positive function W on $|x| \geq R_0$ for some $R_0 > 0$ satisfying

$$\Delta W + \Phi(|x|)W \leq 0 \quad \text{in } |x| > R_0 \quad \text{and} \tag{2.2}$$

$$\lim_{|x|\to\infty} \frac{U(|x|)}{W(x)} = 0. \tag{2.3}$$

Then u must be radially symmetric about some point $x_0 \in R^n$ and $u_r < 0$ for $r = |x - x_0| > 0$.

We give some corollaries of the theorem. Some of the results presented here are already known, but our aim is to treat these results from a unified point of view.

COROLLARY 1. *Assume that*

$$f_u(r,u) \leq 0 \quad \text{for } r \geq r_0,\ 0 \leq u \leq u_0,$$

with some constants $r_0 \geq 0$ and $u_0 > 0$. Let u be a positive solution of (1.1). Then u must be radially symmetric about some point $x_0 \in \mathbb{R}^n$ and $u_r < 0$ for $r = |x - x_0| > 0$.

REMARK. This result has been obtained by [9]. Related results have been obtained by [3, 5].

Proof. We see that the function U defined by (2.1) satisfies $U(r) \to 0$ as $r \to \infty$. Take $R_0 \geq r_0$ so large that $U(r) \leq u_0$ for $r \geq R_0$. Define W as $W(x) \equiv 1$ on $|x| \geq R_0$. Then W satisfies (2.3). Since $\Phi(r) = \max\{f_u(r,u) : 0 \leq u \leq U(r)\} \leq 0$ for $r \geq R_0$, we have (2.2). Therefore, we can apply Theorem 1 to conclude the assertion.

For simplicity we consider the equation of the form

$$\Delta u + \phi(|x|)f(u) = 0 \qquad \text{in } \mathbb{R}^n. \tag{2.4}$$

In equation (2.4), we assume that $\phi \in C[0,\infty)$ satisfies

$$\phi(r) \geq 0 \text{ for } r \geq 0 \quad \text{and} \quad \phi(r) \text{ is nonincreasing in } r > 0,$$

and that $f \in C^1[0,\infty)$ with $f(u) > 0$ for $u > 0$. □

COROLLARY 2. *Assume, furthermore, that*

$$\phi(r) = O(r^{\ell}) \qquad \text{as } r \to \infty \quad \text{for some } \ell \in [-2, 0] \quad \text{and}$$
$$|f'(u)| = O(u^{p-1}) \qquad \text{as } u \to 0 \quad \text{for some } p > \tfrac{n+\ell}{n-2}.$$

Let u be a positive solution of (2.4) *in* $\mathbb{R}^n$ *satisfying*

$$u(x) = \begin{cases} o(|x|^{-\frac{\ell+2}{p-1}}) & \text{as } |x| \to \infty \quad \text{if } -2 < \ell \leq 0 \\ o((\log|x|)^{-\frac{1}{p-1}}) & \text{as } |x| \to \infty \quad \text{if } \ell = -2. \end{cases}$$

Then u must be radially symmetric about some point $x_0 \in R^n$ and $u_r < 0$ for $r = |x - x_0| > 0$.

REMARK. Related results have been obtained by [3, 5, 7, 8]. Li[6] and Zou[14, 15] have obtained the symmetry results for positive solutions which have *slow decay* at infinity.

Proof. We see that the function U defined by (2.1) satisfies

$$U(r) = \begin{cases} o(r^{-\frac{\ell+2}{p-1}}) & \text{as } r \to \infty \quad \text{if } -2 < \ell \leq 0, \\ o((\log r)^{-\frac{1}{p-1}}) & \text{as } r \to \infty \quad \text{if } \ell = -2. \end{cases}$$

(i) *The case where* $-2<\ell\le 0$. Let $W(x)=|x|^{-\frac{\ell-2}{p-1}}$ for $|x|>0$. Then W satisfies (2.3) and

$$\Delta W+\frac{m(n-2-m)}{|x|^2}W=0,\quad |x|>0,$$

where $m=(\ell+2)/(n-2)$. We note that $n-2-m>0$ from $p>(n+\ell)/(n-2)$. We find that

$$\Phi(r)=\max\{\phi(r)f'(u):0\le u\le U(r)\}=o(|x|^{-2})\quad \text{as}\quad r\to\infty.$$

Then we have (2.2) for sufficiently large $R_0>0$. Therefore, we can apply Theorem 1 to conclude the assertion.

(ii) *The case where* $\ell=-2$. Let $W(x)=(\log|x|)^{-\frac{1}{p-1}}$ for $|x|>1$. Then W satisfies (2.3) and

$$\Delta W+\left(\frac{n-2}{p-1}-\frac{p}{(p-1)^2\log|x|}\right)\frac{1}{|x|^2\log|x|}W=0,\quad |x|>1.$$

We find that

$$\Phi(r)=o\left(|x|^{-2}(\log|x|)^{-1}\right)\quad \text{as}\quad r\to\infty.$$

Then we have (2.2) for sufficiently large $R_0>0$. Therefore, we can apply Theorem 1 to conclude the assertion. □

COROLLARY 3. *In equation* (2.4), *assume furthermore that* $\phi\not\equiv 0$ *satisfies*

$$\int_0^\infty r\phi(r)dr<\infty. \tag{2.5}$$

Let u *be a positive solution of* (2.4) *in* $\mathbb{R}^n$ *satisfying* $u(x)\to c$ *as* $|x|\to\infty$ *for some constant* $c\ge 0$. *Then* u *must be radially symmetric about the origin and* $u_r<0$ *for* $r=|x|>0$.

REMARK. Related results have been obtained by [7, 8, 11]. If ϕ is locally Hölder continuous, then every bounded positive solution u satisfies $u(x)\to c$ as $|x|\to\infty$ for some constant $c\ge 0$. (See [11].)

Proof. Let $v(x)=u(x)-c$. Then v satisfies

$$\begin{cases}\Delta v+\phi(|x|)h(v)=0 \quad \text{in}\quad \mathbb{R}^n,\\ v(x)\to 0\quad \text{as}\quad |x|\to\infty,\end{cases} \tag{2.6}$$

where $h(v) = g(v + c)$. Since $-\Delta v = \phi h \geq 0$, we have $v > 0$ in $\mathbb{R}^n$ by the maximum principle. We apply Theorem 1 to the problem (2.6). We define U and Φ as

$$U(r) = \sup\{v(x) : |x| \geq r\} \quad \text{and} \quad \Phi(r) = \sup\{\phi(r)h'(s) : 0 \leq s \leq U(r)\},$$

respectively. Since $\Phi(r) \leq M\phi(r)$ for some constant $M > 0$ and (2.5) holds, we have

$$\int_0^\infty r\Phi(r)dr < \infty.$$

Then there exists a positive function W on $|x| \geq R_0$ for some $R_0 > 0$ satisfying

$$\Delta W + \Phi(|x|)W = 0 \quad \text{and} \quad \liminf_{|x|\to\infty} W(x) > 0.$$

(See, e.g., [11, Lemma B.1].) Then W satisfies (2.2) and (2.3). Therefore, Theorem 1 can be applied to conclude the assertion. $\square$

Finally, we consider the semilinear elliptic equations of the form

$$\Delta u + \frac{\beta}{(1+|x|)^\mu}u^p - \frac{1}{(1+|x|)^\nu}u^q = 0 \quad \text{in } \mathbb{R}^n, \tag{2.7}$$

where $\beta > 0$, μ, $\nu > 2$ and p, $q \geq 1$ are real constants. The problem of existence of positive solutions to (2.7) has been studied by [4, 10, 1], and the solutions structures have been investigated by Chern[1]. Combining the result [1, Theorem 1.4] and Theorem 1, we obtain the following:

COROLLARY 4. *Assume that $\nu \geq \mu > 2$ and $q > p \geq 1$. Let u be a positive solution of (2.7) satisfying $u(x) \leq (\beta\mu/\nu)^{1/(q-p)}$ in $\mathbb{R}^n$. Then u must be radially symmetric about the origin and $u_r < 0$ for $r = |x| > 0$. Furthermore, for the case $\nu = \mu > 2$, all bounded positive solutions of (2.7) are radially symmetric about the origin and $u_r < 0$ for $r = |x| > 0$.*

Proof. Let

$$f(r,u) = \frac{\beta}{(1+r)^\mu}u^p - \frac{1}{(1+r)^\nu}u^q.$$

Since

$$f_r(r,u) = -\frac{\beta\mu}{(1+r)^{\mu+1}}u^p + \frac{\nu}{(1+r)^{\nu+1}}u^q \leq \frac{1}{(1+r)^{\mu+1}}(-\beta\mu u^p + \nu u^q),$$

we have $f_r(r,u) \leq 0$ for $0 \leq u \leq (\beta\mu/\nu)^{1/(q-p)}$.

Let u be a positive solution of (2.7) satisfying $u \le (\beta\mu/\nu)^{1/(q-p)}$. By the similar arguments as in [1], we obtain $\lim_{|x|\to\infty} u(x) = c$ for some constant $c \ge 0$. Let $v(x) = u(x) - c$. Then v satisfies $v(x) \to 0$ as $|x| \to \infty$ and

$$\Delta v + g(|x|, v) = 0 \quad \text{in } \mathbb{R}^n,$$

where $g(r, v) = f(r, v + c)$. Therefore, we can apply Theorem 1 to conclude the assertion by the similar argument as in the proof of Corollary 3.

Let us consider the case where $\nu = \mu > 2$. Chern [1, Theorem 1.4] has shown that every bounded solution u satisfies

$$u(x) \to c \quad \text{as } |x| \to \infty \tag{2.8}$$

for some constant $c \ge 0$, and that u must be radially symmetric provided $u(x) \ge \beta^{1/(q-p)}$ in $\mathbb{R}^n$.

Let u be a positive solution of (2.7) satisfying (2.8). We claim that $u(x) \ge \beta^{1/(q-p)}$ in $\mathbb{R}^n$ if $c \ge \beta^{1/(q-p)}$, and $u(x) \le \beta^{1/(q-p)}$ in $\mathbb{R}^n$ if $c \le \beta^{1/(q-p)}$. We show the former case. The latter case can be shown similarly. Assume to the contrary that $u(x_0) < \beta^{1/(q-p)}$ for some $x_0 \in \mathbb{R}^n$. Define $\Omega = \{x \in \mathbb{R}^n : u(x) < \beta^{1/(q-p)}\}$. Since $f(r,u) \ge 0$ for $0 \le u \le \beta^{1/(q-p)}$, we have $-\Delta u \ge 0$ in Ω and $u = \beta^{1/(q-p)}$ on $\partial\Omega$. By the maximum principle, we obtain $u \ge \beta^{1/(q-p)}$ in Ω. This contradicts the assumption. Therefore $u(x) \ge \beta^{1/(q-p)}$ in $\mathbb{R}^n$.

Let u be a positive solution of (2.7) satisfying (2.8). Assume that $c \ge \beta^{1/(q-p)}$. Then $u(x) \ge \beta^{1/(q-p)}$ in $\mathbb{R}^n$, and u must be radial symmetric by Chern's result. On the other hand, assume that $c \le \beta^{1/(q-p)}$. Then $u \le \beta^{1/(q-p)}$ in $\mathbb{R}^n$, and u must be radial symmetric by the first part of this corollary. This completes the proof. □

3. Proof of Theorem 1

To prove Theorem 1, we introduce a few notations. For $\lambda \in \mathbb{R}$, we define T_λ and Σ_λ as

$$T_\lambda = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_1 = \lambda\} \quad \text{and} \quad \Sigma_\lambda = \{x \in \mathbb{R}^n : x_1 < \lambda\}.$$

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, let x^λ be the reflection of x with respect to the hyperplane T_λ, i.e., $x^\lambda = (2\lambda - x_1, x_2, \dots, x_n)$. It is easy to see that, if $\lambda > 0$,

$$|x^\lambda| > |x| \quad \text{for } x \in \Sigma_\lambda. \tag{3.1}$$

We define $v_\lambda(x) = u(x) - u(x^\lambda)$ for $x \in \Sigma_\lambda$.

LEMMA 1. *Let $\lambda > 0$. Then v_λ satisfies*

$$\Delta v_\lambda + c_\lambda(x) v_\lambda \le 0 \quad \text{in } \Sigma_\lambda, \tag{3.2}$$

where

$$c_\lambda(x) = \int_0^1 f_u\Big(|x|, u(x^\lambda) + t\big(u(x) - u(x^\lambda)\big)\Big)dt. \tag{3.3}$$

Proof. Since $f(r,u)$ is nonincreasing in r and (3.1) holds, we have

$$\begin{aligned} 0 &= \Delta u(x) + f(|x|, u(x)) - \Delta u(x^\lambda) - f(|x^\lambda|, u(x^\lambda)) \\ &\ge \Delta\left(u(x) - u(x^\lambda)\right) + f(|x|, u(x)) - f(|x|, u(x^\lambda)) \\ &= \Delta v_\lambda(x) + c_\lambda(x) v_\lambda(x) \end{aligned}$$

for $x \in \Sigma_\lambda$, where $c_\lambda(x)$ is the function defined by (3.3). □

Define the set Λ as $\Lambda = \{\lambda \in (0,\infty) : v_\lambda(x) > 0 \text{ in } \Sigma_\lambda\}$. Put $B_0 = \{x \in \mathbb{R}^n : |x| < R_0\}$, where R_0 is the constant in the statement of Theorem 1.

LEMMA 2. *Let $\lambda > 0$. If $v_\lambda > 0$ on $\Sigma_\lambda \cap \overline{B_0}$ then $\lambda \in \Lambda$.*

Proof. From Lemma 1 and the assumption, we obtain

$$\Delta v_\lambda + c_\lambda(x) v_\lambda \le 0 \quad \text{in } \Sigma_\lambda \setminus \overline{B_0}, \quad v_\lambda \ge 0 \quad \text{on } \partial(\Sigma_\lambda \setminus \overline{B_0}).$$

Since $U(r)$ is nonincreasing, we have $0 \le u(x^\lambda) + t(u(x) - u(x^\lambda)) \le U(|x|)$ for $0 \le t \le 1$. Then, by (3.3), we have $c_\lambda(x) \le \Phi(|x|)$ in Σ_λ. From (2.2) and (2.3), the positive function W satisfies

$$\Delta W + c_\lambda(x) W \le 0 \quad \text{in} \quad \Sigma_\lambda \setminus \overline{B_0} \quad \text{and} \quad \frac{v_\lambda(x)}{W(x)} \le \frac{U(|x|)}{W(x)} \to 0$$

$$\text{as} \quad |x| \to \infty, \ x \in \Sigma_\lambda \setminus \overline{B_0}.$$

Hence the maximum principle (Lemma A in Appendix) implies that $v_\lambda > 0$ in $\Sigma_\lambda \setminus \overline{B_0}$. Then $v_\lambda(x) > 0$ in Σ_λ by the assumption, which implies that $\lambda \in \Lambda$. □

By Lemma 2, we obtain the following:

LEMMA 3. *Let $\lambda > 0$. If $\lambda \notin \Lambda$, then there exists $x_0 \in \Sigma_\lambda \cap \overline{B_0}$ such that $v_\lambda(x_0) \leq 0$.*

LEMMA 4. *Let $\lambda \in \Lambda$. Then $\partial u/\partial x_1 < 0$ on T_λ.*

Proof. By Lemma 1, we have (3.2) and $v_\lambda > 0$ in Σ_λ. Since $v_\lambda = 0$ on T_λ, we obtain $\partial v_\lambda/\partial x_1 < 0$ on T_λ by the Hopf boundary lemma. Therefore, we have

$$\frac{\partial u}{\partial x_1} = \frac{1}{2}\frac{\partial v_\lambda}{\partial x_1} < 0 \quad \text{on } T_\lambda. \qquad \square$$

Proof of Theorem 1. Since $u(x)$ is positive and $\lim_{|x|\to\infty} u(x) = 0$, there exists $R_1 > R_0$ such that

$$\max\{u(x) : |x| \geq R_1\} < \min\{u(x) : |x| \leq R_0\}, \tag{3.4}$$

where R_0 is the constant in the statement of Theorem 1. We decompose the proof of Theorem 1 into several steps.

Step 1. We have $[R_1, \infty) \subset \Lambda$.

Let $\lambda \geq R_1$. We note that $\overline{B_0} \subset \Sigma_\lambda$. From (3.4), we have $v_\lambda > 0$ in $\overline{B_0}$. Then, by Lemma 2, we have $\lambda \in \Lambda$, which implies that $[R_1, \infty) \subset \Lambda$.

Step 2. Let $\lambda_0 \in \Lambda$. Then there exists $\varepsilon > 0$ such that $(\lambda_0 - \varepsilon, \lambda_0] \subset \Lambda$.

Assume to the contrary that there exists an increasing sequence $\{\lambda_i\}$, $i = 1, 2, \ldots$, such that $\lambda_i \notin \Lambda$ and $\lambda_i \to \lambda_0$ as $i \to \infty$. By Lemma 3, we have a sequence $\{x_i\}$, $i = 1, 2, \ldots$, such that $x_i \in \Sigma_{\lambda_i} \cap \overline{B_0}$ and $v_{\lambda_i}(x_i) \leq 0$. A subsequence, which we call again $\{x_i\}$, converges to some point $x_0 \in \overline{\Sigma_{\lambda_0}} \cap \overline{B_0}$. Then $v_{\lambda_0}(x_0) \leq 0$. Since $v_{\lambda_0} > 0$ in Σ_{λ_0}, we must have $x_0 \in T_{\lambda_0}$. By the mean value theorem we observe that there exists a point y_i satisfying $(\partial u/\partial x_1)(y_i) \geq 0$ on the straight segment joining x_i to $x_i^{\lambda_i}$ for each $i = 1, 2, \ldots$. Since $y_i \to x_0$ as $i \to \infty$, we have $(\partial u/\partial x_1)(x_0) \geq 0$. On the other hand, since $x_0 \in T_{\lambda_0}$, we have $(\partial u/\partial x_1)u(x_0) < 0$ by Lemma 4. This is a contradiction and Step 2 is established.

Step 3. We have either

$$u(x) \equiv u(x^{\lambda_1}) \quad \text{for some } \lambda_1 > 0 \quad \text{and} \quad \frac{\partial u}{\partial x_1} < 0 \quad \text{on } T_\lambda \text{ for } \lambda > \lambda_1 \tag{3.5}$$

or

$$(3.6) \qquad u(x) \geq u(x^0) \quad \text{in } \Sigma_0 \quad \text{and} \quad \frac{\partial u}{\partial x_1} < 0 \quad \text{on } T_\lambda \text{ for } \lambda > 0.$$

Let $\lambda_1 = \inf\{\lambda > 0 : (\lambda, \infty) \subset \Lambda\}$. We distinguish the following two cases: (i) $\lambda_1 > 0$; (ii) $\lambda_1 = 0$.

(i) *The case where* $\lambda_1 > 0$. Let $v_{\lambda_1}(x) = u(x) - u(x^{\lambda_1})$. From the continuity of u, we have $v_{\lambda_1}(x) \geq 0$ in Σ_{λ_1}. It follows from Lemma 1 that $\Delta v_{\lambda_1} + c_{\lambda_1}(x) v_{\lambda_1} \leq 0$ in Σ_{λ_1}. Hence, by the strong maximum principle, we have that either $v_{\lambda_1} > 0$ in Σ_{λ_1} or $v_{\lambda_1} \equiv 0$ in Σ_{λ_1}. Assume that $v_{\lambda_1} > 0$ in Σ_{λ_1}. Then $\lambda_1 \in \Lambda$. From Step 2, there exists $\varepsilon > 0$ such that $(\lambda_1 - \varepsilon, \lambda_1] \subset \Lambda$. This contradicts the definition of λ_1. Therefore, $v_{\lambda_1} \equiv 0$ in Σ_{λ_1}. Since $(\lambda_1, \infty) \subset \Lambda$, we have $\partial u / \partial x_1 < 0$ on T_λ for $\lambda > \lambda_1$ by Lemma 4. Thus we obtain (3.5).

(ii) *The case where* $\lambda_1 = 0$. From the continuity of u, we have $u(x) \geq u(x^0)$ in Σ_0. By Lemma 4 we have $\partial u / \partial x_1 < 0$ on T_λ for $\lambda > 0$. Thus, (3.6) holds.

If (3.6) occurs in Step 3, we can repeat the previous Steps 1-3 for the negative x_1-direction to conclude that either u is symmetric in the x_1-direction about some plane $x_1 = \lambda_1 < 0$ or

$$(3.7) \qquad u(x) \leq u(x^0) \quad \text{in } \Sigma_0.$$

If (3.7) occurs, then $u(x) \equiv u(x^0)$ in Σ_0. Therefore, u must be symmetric in x_1-direction about some plane, and strictly decreasing away from the plane. Since the equation (1.1) is invariant under rotation, we may take any direction as the x_1-direction and conclude that u is symmetric in every direction about some plane. Therefore, u is radially symmetric about some point $x_0 \in \mathbb{R}^n$ and $u_r < 0$ for $r = |x - x_0|$. □

Appendix

Let Ω be an unbounded domain in $\mathbb{R}^n$, and let L denote a uniformly elliptic differential operator of the form

$$Lu \equiv a^{ij}(x)\partial_{ij}u + b^i(x)\partial_i u + c(x)u,$$

where a^{ij}, b^i, $c \in L^\infty(\Omega)$. (We use the notation $\partial_i = \partial / \partial x_i$ and $\partial_{ij} = \partial^2 / \partial x_i \partial x_j$, together with the summation convention for repeated indices.)

Lemma A. *Suppose that $u \not\equiv 0$ satisfies $Lu \le 0$ in Ω and $u \ge 0$ on $\partial\Omega$. Suppose, furthermore, that there exists a function w such that $w > 0$ on $\Omega \cup \partial\Omega$ and $Lw \le 0$ in Ω. If*

$$\frac{u(x)}{w(x)} \to 0 \quad \text{as } |x| \to \infty,\ x \in \Omega, \tag{A.1}$$

then $u > 0$ in Ω.

Remark. If Ω is bounded, we do not require the condition (A.1). See [12, Chap. 2, Theorem 10].

Proof. First we show that $u \ge 0$ in Ω. Assume to the contrary that $u(x_0) < 0$ for some $x_0 \in \Omega$. Choose $\delta > 0$ so that

$$u(x_0) + \delta w(x_0) = 0. \tag{A.2}$$

From (A.1), there exists $R > |x_0|$ satisfying $u(x) + \delta w(x) \ge 0$ on $\{|x| = R\} \cap \Omega$. Define $B_R = \{x \in \mathbb{R}^n : |x| < R\}$. Then $u + \delta w$ satisfies $L(u+\delta w) \le 0$ on $\Omega \cap B_R$ and $u + \delta w \ge 0$ on $\partial(\Omega \cap B_R)$. By [12, Chap.2, Theorem 10], $(u + \delta w)/w$ cannot attain a nonpositive minimum at an interior point of $\Omega \cap B_R$ unless it is a constant. This contradicts (A.2). Therefore, $u \ge 0$ in Ω. By the strong maximum principle, we conclude that $u > 0$ in Ω. □

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