J. Korean Math. Soc. **37** (2000), No. 5, pp. 735–749

# MULTIPLICITY RESULT FOR PERIODIC SOLUTIONS OF SEMILINEAR DISSIPATIVE HYPERBOLIC EQUATIONS WITH COERCIVE GROWTH NONLINEARITY

WAN SE KIM

ABSTRACT. The multiplicity of periodic solutions of semilinear dissipative hyperbolic equations is treated

## 1. Introduction

Let $R$ be the set of all reals and $\Omega \subseteq R^n$, $n \geq 1$, be a bounded domain with smooth boundary $\partial\Omega$ which is assumed to be of class $C^2$.

Let $Q = (0, 2\pi) \times \Omega$ and $L^2(Q)$ be the space of measurable and Lebesgue square integrable real-valued functions on $Q$ with usual inner product $< \cdot, \cdot >$ and corresponding norm $\| \cdot \|_2$.

By $H_0^1(\Omega)$ we mean the completion of $C_0^1(\Omega)$ with respect to the norm $\| \cdot \|_1$ defined by

$$\|\phi\|_1^2 = \int_\Omega \sum_{|\alpha| \leq 1} |D^\alpha \phi(x)|^2 dx.$$

$H^2(\Omega)$ stands for the usual Sovolev space ; i.e., the completion of $C^2(\bar{\Omega})$ with respect to the norm $\| \cdot \|_2$ defined by

$$\|\phi\|_2^2 = \int_\Omega \sum_{|\alpha| \leq 2} |D^\alpha \phi(x)|^2 dx.$$

Let $g : R \to R$ be a continuous function. Moreover, we assume that there exist constants $a_0$ and $b_0$ such that

$(H_1)$ $$|g(u)| \leq a_0|u| + b_0 \quad \text{for all } u \in R.$$

---

Received October 27, 1999.
2000 Mathematics Subject Classification: 35S20, 35L70.
Key words and phrases: multiplicity, semilinear, dissipative, hyperbolic equation.
Supported by Hanyang University Research Grant 1999.

The purpose of this work is to investigate the multiplicity for periodic solutions of the semilinear dissipative hyperbolic equations

$$(E) \qquad \beta\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} - \triangle_x u - \lambda_1 u + g(u) = h(t,x) \ \text{ in } \ Q,$$

$$(B_1) \qquad u(t,x) = 0 \ \text{ on } \ (0,2\pi)\times\partial\Omega,$$

$$(B_2) \qquad u(0,x) = u(2\pi,x) \ \text{ on } \ \Omega$$

where $\lambda_1$ and $\lambda_2$ denotes the first and second eigenvalues of $-\Delta$ with zero Dirichlet boundary data and $\phi_1$ is the positive normalized eigenfunction corresponding to $\lambda_1$ and $h \in L^2(Q)$.

The purpose of this paper is to give a multiplicity result for semilinear dissipative hyperbilic equations. Orginally, the linear dissipative hyperbolic equations are derived from physical principle(see [4]). The existence and asymptotic theory of dissipative hyperbolic equations have been developed by several authors for initial value problems, boundary value problems, or mixed problems. For information on dissipative hyperbolic equations, we refer to [24]. On the existence of doubly-periodic solutions of semilinear dissipative hyperbolic equations have been done by Mawhin [22], Fucik and Mawhin [7]. Mawhin treat the existence of double-periodic solutions for semilinear dissipative hyperbolic equations of several types of $g(u)$ with at most linear growth in connection with the set $\Sigma = \{k^2 - j^2 | k, j \ \text{ integers}\}$. Fucik and Mawhin consider also the existence double-periodic solutions of seminear dissipative hyperbolic equations with nonlinear term of the form $g(u) = \mu u^+ - \nu u^- - \phi(u)$, where $\phi$ is a continuous and bounded function, and $\mu, \nu$ are real numbers related to the set $\Sigma$. In [9, 15], the existence of solutions for Dirichlet-periodic problem for semilinear dissipative hyperbolic equations at resonsnce, in [13, 14], the existence of Dirichlet-periodic solutions for semilinear dissipative hyperbolic problems with superlinear growth, in [16], the existence of double-periodic solutions for semilinear dissipative hyperbolic equations with non-decreasing type of non-linear term, in [19, 20], the multiple existence of double-periodic and Dirichlet-periodic problem, respectably, for semilinear dissipative hyperbolic equations and, in [17], the asymptotic behavior of Dirichlet-initial problem of semi-linear dissipative hyperbolic equations are disscused. Our result is related to the results in [19, 20] which are so called the Ambrosetti-prodi type multiplicity result which has been initiated by Ambrosetti-Prodi [1] in the study of a Dirichlet problem to elliptic equations and developed in various directions by several authors to ordinary and partial differential equations. For more information on this problem for semilinear elliptic, parabolic and ordinary equations, we refer to [3, 5, 6, 10, 11, 12, 18, and 21] and their references.

In our result, we will trerat a multiplicity result for Dirichlet-periodic solutions of semilinear dissipative hyperbolic equations in $n$-dimensional space. we assume the coercive growth on $g$ with restriction on the left-hand and our proof based on Mawhin's continuation theorem in [8].

## 2. Preliminary results

Let us define the linear operator

$$L : \mathrm{Dom}L \subseteq L^2(Q) \to L^2(Q)$$

by

$$\mathrm{Dom}L = \{u \in L^2((0,2\pi), H^2(\Omega) \cap H_0^1(\Omega)) | \frac{\partial u}{\partial t} \in L^2(Q), \frac{\partial^2 u}{\partial t^2} \in L^2(Q), \\ u(0,x) = u(2\pi, x), x \in \Omega\}$$

and

$$Lu = \beta \frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} - \Delta u - \lambda_1 u.$$

Using Fourier series and Parseval inequality, we get easily

$$< Lu, \frac{\partial u}{\partial t} >= \beta \| \frac{\partial u}{\partial t} \|_{L^2}^2 \quad \text{for all} \quad u \in \mathrm{Dom}L.$$

Hence $\ker L = \ker(\Delta + \lambda_1 I) = \ker L^*$ since $\Delta + \lambda_1 I$ is self-adjoint and $\ker(\Delta + \lambda_1 I)$ is one space dimension generated by the eigenfunction $\phi_1$. Therefore $L$ is a closed, densely defined linear operator and

$\mathrm{Im}(L) = [\ker L]^{\perp}$; i.e., $L^2(Q) = \ker L \bigoplus \mathrm{Im}L$. Let's consider a continous projection $P_1 : L^2(Q) \to L^2(Q)$ such that $\mathrm{Im}P_1 = \ker L$. Then $L^2(Q) = \ker L \bigoplus \ker P_1$. We consider another continuous projection $P_2 : L^2(Q) \to L^2(Q)$ defined by

$$(P_2 h)(t,x) = \frac{1}{2\pi} \iint_Q h(t,x)\phi(x)dtdx\phi(x).$$

Then we have $L^2(Q) = \mathrm{Im}P_1 \bigoplus \mathrm{Im}L$, $\ker P_2 = \mathrm{Im}L$, and $L^2(Q)/\mathrm{Im}L$ is isomorphism to $\mathrm{Im}P_2$.

Since $\dim[L^2(Q)/\mathrm{Im}L] = \dim[\mathrm{Im}P_2] = \dim[\ker L] = 1$, we have an isomorphism $J : \mathrm{Im}P_2 \to \ker L$.

By the closed graph theorm, the generalized right inverse of $L$ defined by

$$K = [L|_{\mathrm{Dom}L \cap \mathrm{Im}L}]^{-1} : \mathrm{Im}L \to \mathrm{Im}L$$

is continuous. If we equip the space $\mathrm{Dom}L$ with the norm

$$\|u\|_{\mathrm{Dom}L} = \iint_Q [u^2 + (\frac{\partial u}{\partial t})^2 + (\frac{\partial^2 u}{\partial t^2})^2 + \sum_{|\beta| \le 2} (D_x^{\beta} u)^2] dtdx.$$

Then there exist a constant $c > 0$ independently of $h \in \mathrm{Im}L$, $u = Kh$ such that

$$\|Kh\|_{\mathrm{Dom}L} \le c\|h\|_{L^2}.$$

Therefore $K : \mathrm{Im}L \to \mathrm{Im}L$ is continuous and by the compact imbedding of $\mathrm{Dom}L$ in $L^2(Q)$, we have that $K : \mathrm{Im}L \to \mathrm{Im}L$ is compact.

LEMMA 2.1. *$L$ is closed, densely defined linear operator such that $\ker L = [\mathrm{Im}L]^{\perp}$ and such that the right inverse $K : \mathrm{Im}L \to \mathrm{Im}L$ is completely continuous.*

*Proof.* See [2, 23]. □

## 3. Multiplicity result

Let us consider the following

$$(E_h^{\mu}) \qquad \beta\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} - \triangle_x u - \lambda_1 u + \mu g(u) = \mu h(t,x) \text{ in } Q,$$

$$(B_1) \qquad u(t,x) = 0 \text{ on } (0,2\pi) \times \partial\Omega,$$

$$(B_2) \qquad u(0,x) = u(2\pi, x) \text{ on } \Omega$$

where $\mu \in [0,1]$.

Let $L : \mathrm{Dom}L \subseteq L^2(Q) \to L^2(Q)$ be defined as before. If we define a substitution operator $N_h^\mu : L^2(Q) \to L^2(Q)$ by

$$(N_h^\mu)(t,x) = \mu g(u) - \mu h(t,x)$$

for $u \in L^2(Q)$ and $(t,x) \in Q$, then $N_h^\mu$ maps continuously into itself and take bounded sets into bounded set. Let $G$ be any open bounded subset of $L^2(Q)$, then $P_2 N_h^\mu : \bar{G} \to L^2(Q)$ is bounded and $K(I - P_2) : \bar{G} \to L^2(Q)$ is compact and continuous. Thus $N_h^\mu$ is L-compact on $\bar{G}$.

The coincidence degree $D_L(L + N_h^\mu, G)$ is well defined and constant in $\mu$ if $Lu + N_h^\mu u \neq 0$ for $\mu \in [0,1]$ and $u \in \mathrm{Dom}L \cap \partial G$. It is easy to check that $(u, \mu)$ is a weak solution of $(E_h^\mu)$ if and only if $u \in \mathrm{Dom}L$ and

$$(3.1_h^\mu) \qquad Lu + N_h^\mu u = 0.$$

Here, we assume the following

$$(H_2) \qquad \lim_{|u| \to \infty} \inf g(u) = +\infty,$$

$$(H_3) \qquad \lim_{u \to -\infty} \sup \left|\frac{g(u)}{u}\right| < \lambda_2 - \lambda_1.$$

From $(H_2)$ and $(H_3)$, we may assume that

$$m = \inf_{u \in R} g(u) > 0$$

and there exist $a \in (0, \lambda_2 - \lambda_1)$ and $b \geq 0$ such that

$$|g(u)| \leq a|u| + b \text{ for all } u \leq 0.$$

For $h \in L^2(Q)$, we write $\wedge h = \iint_Q h(t,x)\phi(x)\,dt\,dx$.

LEMMA 3.1. *If $(H_1)$, $(H_2)$ and $(H_3)$ are satisfied, then, for each $h^* \in L^2(Q)$, there exists $M(h^*) > 0$ independently of $\mu$ such that*

$$\|\tilde{u}\|_{L^2} \leq M$$

*holds for each possible weak solution $u = \alpha\phi + \tilde{u}$, with $\alpha \in R$ and $\tilde{u} \in \mathrm{Im}L$, of $(E_h^\mu)$ with $\mu \in [0.1]$, and with $\wedge h \leq \wedge h^*$ and $\|h\|_{L^2} \leq \|h^*\|_{L^2}$.*

*Proof.* Suppose there exists $h \in L^2(Q)$ with $\wedge h \leq \wedge h^*$ and $\|h\|_{L^2} \leq \|h^*\|_{L^2}$ and the corresponding sequence of solutions $\{(u_n, \mu_n)\}$, with $\mu \in [0,1]$, of $(3.1_h^{\mu_n})$ such that

$$\lim_{n\to\infty} \|\tilde{u}_n\|_{L^2} = \infty,$$

then clearly

$$\lim_{n\to\infty} \|u_n\|_{L^2} = \infty.$$

For each $n \geq 1$, we put $u_n(t,x) = \alpha_n\phi(x) + \tilde{u}_n(t,x)$.

First, we are going to prove that

$$\lim_{n\to\infty} \frac{|\alpha_n|}{\|\tilde{u}_n\|_{L^2}} = c < \infty.$$

If it is not the case, we may assume that, by extracting subsequence if it is necessary,

$$\lim_{n\to\infty} \frac{\|\tilde{u}_n\|_{L^2}}{|\alpha_n|} = 0.$$

Therefore, we may have a subsequence, say again, $\{\tilde{u}_n\}$ such that we have easily

$$\lim_{n\to\infty} |u_n(t,x)| = \infty \text{ a.e. on } Q.$$

By taking the inner product with $\phi$ on both sides of $(3.1_h^\mu)$, we have

$$\iint_Q g(u_n(t,x))\phi(x)dtdx = \iint_Q h\phi(x)dtdx \leq \wedge h^*.$$

On the other hand, by $(H_2)$ and Fatou's lemma, we have

$$\lim_{n\to\infty} \iint_Q g(u_n(t,x))\phi(x)dtdx = \infty$$

which leads to a contradiction. First, we assume that $0 < c < \infty$, then there exist $n_0 \in N$ such that

$$(c/2)\|\tilde{u}_n\|_{L^2} \leq |\alpha_n| \leq (3c/2)\|\tilde{u}_n\|_{L^2} \text{ for all } n \geq n_0.$$

For given $\epsilon > 0$, we may choose $\delta > 0$ such that

$$\iint_A |\phi|^2 dtdx < \epsilon\|\phi\|_{L^2}^2$$

for any measurable set $A \subset \bar{Q}$ with $|A| \leq \delta$.

Let $0 < \gamma < \|\phi\|_\infty$ and $\Omega_0 = \{x \in \Omega : \phi(x) \geq \gamma\}$. Choose $M_0 > 0$ such that

$$\delta M_0 - |m| \iint_Q \phi dtdx > \iint_Q h^* \phi(x) dtdx.$$

Then, since $\lim_{u\to\infty} g(u) = \infty$, we have that

$$m_0 = \sup\{|u| : \gamma g(u) < M_0\} < \infty.$$

We put

$$Q_n = \{(t,x) \in [0,2\pi] \times \Omega_0 : |u_n(t,x)| \geq m_0\}.$$

Then we have $|Q_n| \leq \delta$. In fact, if $|Q_n| > \delta$, then from the definition of $m_0$ we have

$$\begin{aligned}
&\iint_Q g(u_n(t,x))\phi(x)dtdx \\
&= \iint_{Q_n} g(u_n)\phi(x)dtdx + \iint_{Q\setminus Q_n} g(u_n)\phi(x)dtdx \\
&> \delta M_0 - m \iint_Q \phi(x)dtdx \\
&> \iint_Q h^* \phi(x)dtdx
\end{aligned}$$

and this leads to a contradiction. Therefore, we have

$$\iint_{Q\setminus Q_n} |\alpha_n \phi|^2 \geq (1-\epsilon) \iint_Q |\alpha_n \phi|^2.$$

On the other hand,

$$\begin{aligned}0 &= \iint_Q \alpha_n \phi \tilde{u}_n \\ &= \iint_{Q\setminus Q_n} \alpha_n \phi \tilde{u}_n + \iint_{Q_n} \alpha_n \phi \tilde{u}_n \\ &\leq (1/2) \iint_{Q\setminus Q_n} (|\alpha_n \phi + \tilde{u}_n|^2 - |\alpha_n \phi|^2 - |\tilde{u}_n|^2) + \iint_{Q_n} |\alpha_n \phi||\tilde{u}_n|.\end{aligned}$$

From the definition of $m_0$ and the above facts, we have, for all $n \geq n_0$,

$$\begin{aligned}0 &\leq (1/2)m_0^2 - (1/2)(1-\epsilon)(c/2)\|\tilde{u}_n\|_{L^2}^2 + \epsilon(3c/2)\|\tilde{u}_n\|_{L^2}^2 \\ &= (1/2)m_0^2 - (c/4)(1+5\epsilon c)\|\tilde{u}_n\|_{L^2}^2.\end{aligned}$$

Therefore, $\{\|\tilde{u}_n\|_{L^2}\}$ is bounded which leads to a contradiction.

Next, we assume $c = 0$, then $\lim_{n\to\infty} \frac{\|\tilde{u}_n\|}{\|u_n\|_{L^2}} = 1$.
Multiplying $(3.1_h^\mu)$ by $\frac{\partial u}{\partial t}$ and integrate over $Q$, we find from the periodicity of $u$ that

$$\|\frac{\partial u}{\partial t}\|_{L^2} \leq \frac{1}{|\beta|}\|h\|_{L^2}.$$

Again, taking the inner prodnct with $u_n$ on both sides of $(3.1_h^\mu)$, we have

$$(\lambda_2 - \lambda_1)\|\tilde{u}_n\|_{L^2}^2 - \|\frac{\partial u_n}{\partial t}\|_{L^2}^2 + < g(u_n), u_n > \leq \|h\|_{L^2}\|\tilde{u}_n\|_{L^2}$$

and hence

$$\lim_{n\to\infty} \sup(\lambda_2 - \lambda_1 - a)\|\tilde{u}_n\|_{L^2}^2 \leq [\max\{m, b\}|Q|^{1/2} + \frac{1}{|\beta|^2}\|h^*\|_{L^2}^2 + \|h^*\|_{L^2}].$$

Thus $\{\|\tilde{u}_n\|_{L^2}\}$ is bounded which leads to another contradiction. □

LEMMA 3.2. *If $(H_1)$, $(H_2)$ and $(H_3)$ are satisfied, then, for each $h^* \in L^2(Q)$, there exists $r = r(h^*) > 0$ independently of $\mu$ such that*

$$|\bar{u}| \leq r$$

*holds for each possible weak solution $u = \bar{u} + \tilde{u}$, with $\bar{u} = \alpha\phi(x)$, $\alpha \in R$ and $\tilde{u} \in \mathrm{Im}L$, of $(3.1_h^\mu)$ where $\mu \in [0,1]$, and with $\wedge h \leq \wedge h^*$ and $\|h\|_{L^2} \leq \|h^*\|_{L^2}$.*

*Proof.* Suppose there exists $h \in L^2(Q)$ with $\wedge h \leq \wedge h^*$ and $\|h\|_{L^2} \leq \|h^*\|_{L^2}$, and the corresponding sequence of weak solutions $\{(u_n, \mu_n)\}$ of $(3.1_h^{\mu_n})$ with $\{|\bar{u}_n|\}$ is unbounded. Then $(u_n, \mu_n)$ is a solution of $(3.1_h^{\mu_n})$ where $u_n = \bar{u}_n + \tilde{u}_n$ with $\bar{u}_n = \alpha_n\phi(x)$ and $\tilde{u}_n \in \mathrm{Im}L$. We may choose a subsequence, say again $\{\bar{u}_n\}$ with $\bar{u}_n = \alpha_n\phi(x)$ such that $|\alpha_n| \to +\infty$ as $n \to +\infty$. Now, let $\tilde{M} > M$ which is given ln Lemma 3.1. Let

$$Q_0 = \{(t,x) \in Q | |\tilde{u}(t,x)| \geq \frac{1+\tilde{M}}{|Q|}\}.$$

Then

$$\begin{aligned}\tilde{M}^2 &\geq \iint_Q |\tilde{u}(t,x)|^2 dtdx \\ &\geq \iint_{Q_0} |\tilde{u}(t,x)|^2 dtdx \\ &\geq |Q_0| [\frac{1+\tilde{M}}{|Q|}]^2.\end{aligned}$$

Therefore $|Q_0| \leq [\frac{\tilde{M}}{1+\tilde{M}}]^2 |Q|$ and hence $|Q \setminus Q_0| = |\{(t,x) \in Q | |\tilde{u}(t,x)| \leq \frac{1+\tilde{M}}{|Q|}\}| \geq [1 - \frac{\tilde{M}}{1+\tilde{M}}]^2 |Q| > 0$.

Let $W = (0.2\pi) \times \Omega_0$. Then we have $|\alpha_n\phi(x)| \to \infty$ for each $x \in \Omega_0$ as $n \to \infty$. Hence, by Fatou's lemma and $(H_2)$, we have

$$\begin{aligned}&\liminf_{n\to\infty} \iint_Q g(\alpha_n\phi(x) + \tilde{u}(t,x))\phi(x)\, dtdx \\ &= \liminf_{n\to\infty} \iint_Q g(\alpha_n\phi(x) + \tilde{u}(t,x))\phi(x)\, dtdx \\ &\geq \iint_{W\cap(Q\setminus Q_0)} \liminf_{n\to\infty} g(\alpha_n\phi(x) + \tilde{u}(t,x))\phi(x)\, dtdx \\ &= \infty.\end{aligned}$$

Hence, there exists $r_0(h^*) > 0$ such that, for $|\alpha_n| > r_0$, we have

(3.1)

$$\iint_Q g(\alpha_n\phi(x) + \tilde{u}_n(t,x))\phi(x)\, dtdx > \iint_Q h^*\phi(x)\, dtdx.$$

On the other hand, by taking the inner product with $\phi(x)$ on the both sides of $(3.1_h^{\mu_n})$, we have

$$\iint_Q g(\alpha_n\phi(x)+\tilde{u}_n(t,x))\phi(x)dtdx = \iint_Q h\phi(x)dtdx \le \wedge h^*$$

which is impossible. The proof is complete. □

LEMMA 3.3. *If $(H_1)$, $(H_2)$ and $(H_3)$ are satisfied, then, for each $h^* \in L^2(Q)$, we can find an open bounded set $G(h^*)$ in $L^2(Q)$ such that, for each open bounded set $G$ in $L^2(Q)$ such that $G \supseteq G(h^*)$, we have*

$$D_L(L+N_h^1, G) = 0 \quad \text{for all } \ h \in L^2(Q)$$

*with $\wedge h \le \wedge h^*$ and $\|h\|_{L^2} \le \|h^*\|_{L^2}$.*

*Proof.* By similar fashion as we did in the proof of Lemma 3.2 to get (3.1), there exists $\bar{r}(h^*) > 0$ such that, for $|\alpha| > \bar{r}$, we have

$$\iint_Q g(\alpha\phi(x))\phi(x)dtdx > \iint_Q h^*\phi(x)dtdx.$$

Let

$$G(h^*) = \{u \in L^2(Q) | -\tilde{r}\phi(x) < \alpha\phi(x) < \tilde{r}\phi(x) \text{ for } x \in \Omega, \|\tilde{u}\|_{L^2} < \tilde{M}\}$$

where $u = \alpha\phi(x) + \tilde{u}$ with $\tilde{r}(h^*) > \max\{r(h^*), r_0(h^*), \bar{r}(h^*)\}$ and $\tilde{M} > M$ which are given in Lemma 3.1 and Lemma 3.2. If $(3.1_{\bar{h}}^{\mu})$ has a solution $u$ for some $\bar{h} \in L^2(Q)$ such that $\wedge\bar{h} < 2\pi m \int_\Omega \phi(x)dx$ and $\mu \in [0,1]$, then by taking the inner product with $\phi$ on the both sides of the equation $(3.1_{\bar{h}}^{\mu})$, we have

$$2\pi m \int_\Omega \phi(x)dx \le \iint_Q g(u(t,x))\phi(x)dtdx = \iint_Q \bar{h}\phi(x)dtdx.$$

Thus $(3.1_{\bar{h}}^{\mu})$ has no solution for $\bar{h} \in L^2(Q)$ such that $\wedge\bar{h} < 2\pi m$ $\int_\Omega \phi(x)\,dx$.

Hence, for each open bounded set $G \supseteq G(h^*)$, we have

$$D_L(L + N^1_{\bar{h}}, G) = 0 \quad \text{for} \quad \bar{h} \in L^2(Q)$$

such that $\wedge \bar{h} < 2\pi m \int_\Omega \phi(x)dx$. Choose $\bar{h} \in L^2(Q)$ with $\wedge \bar{h} \leq 2\pi\ m \int_\Omega \phi(x)dx$ and $\|\bar{h}\|_{L^2} \leq \|h^*\|_{L^2}$, and define

$$F : (D(L) \cap G) \times [0,1] \to L^2(Q) \quad \text{by}$$

$$F(u, \lambda) = Lu + N_{(1-\lambda)\bar{h}+\lambda h}(u) \quad \text{for} \quad h \in L^2(Q)$$

with $\wedge h \leq \wedge h^*$ and $\|h\|_{L^2} \leq \|h^*\|_{L^2}$. Then by Lemma 3.1 and Lemma 3.2, we have

$$0 \notin F(D(L) \cap \partial G) \times [0,1] \quad \text{for} \quad h \in L^2(Q)$$

with $\wedge h \leq \wedge h^*$ and $\|h\|_{L^2} \leq \|h^*\|_{L^2}$. By the homotopy invariance of degree, we have, for all $h \in L^2(Q)$ with $\wedge h \leq \wedge h^*$ and $\|h\|_{L^2} \leq \|h^*\|_{L^2}$,

$$\begin{aligned} D_L(L + N^1_h, G) &= D_L(F(\cdot, 1), G) \\ &= D_L(F(\cdot, 0), G) \\ &= D_L(L + N^1_{\bar{h}}, G) \\ &= 0 \end{aligned}$$

and the proof is completed. □

THEOREM. *Assume $(H_1)$, $(H_2)$ and $(H_3)$. Then there there exist a constant $\alpha_0$ such that the boundary value problem $(E), (B_1)$, and $(B_2)$ has at least two solutions for $h$ such that*

$$\iint_Q g(\alpha_0\phi(x) + \tilde{u}(t,x))\phi(x)dtdx < \iint_Q h\phi(x)dtdx \tag{3.2}$$

*for every $\tilde{u} \in L^2(\Omega)$ having mean value zero on $\Omega$, satisfying the conditions $(B_1)$, and $(B_2)$ such that*

$$\|\tilde{u}\|_{L^2} < \tilde{M}, \tag{3.3}$$

*where $\tilde{M}$ is given Lemma 3.3.*

*Proof.* Let

$$g(\alpha_0\phi(x_0)+\tilde{u}_0)=\min_{\substack{x\in\bar{\Omega}\\ |\alpha|\le\tilde{r}\\ |\tilde{u}|\le\tilde{M}}} g(\alpha\phi(x)+\tilde{u}).$$

Define

$$\Delta(G(h))=\{u\in L^2(Q)|\alpha_0\phi(x)<\alpha\phi(x)<\tilde{r}_0\phi(x)\text{ for }x\in\Omega,\|\tilde{u}\|_{L^2}<\tilde{M}\}$$

where $\tilde{r}_0(h)>\tilde{r}$ which is given in Lemma 3.3.

If $u\in\partial\Delta G(h)$, then necessary $u=\alpha_0\phi(x)+\tilde{u}$ or $u=\tilde{r}_0\phi(x)+\tilde{u}$. If $u=\alpha_0\phi(x)+\tilde{u}$ with $\|\tilde{u}\|_{L^2}<\tilde{M}$, then, by taking inner product with $\phi$ on the both sides of $(3.1_h^\mu)$, we have

$$\iint_Q g(\alpha_0\phi(x)+\tilde{u}(t,x))\phi(x)dtdx=\iint_\Omega h\phi(x)dtdx$$

which, from (3.2) and (3.3), is impossible. If $u=\tilde{r}_0\phi(x)+\tilde{u}$ with $\|\tilde{u}\|_{L^2}<\tilde{M}$, then, by the choice of $\tilde{r}_0>0$, we have

$$\iint_Q g(\tilde{r}_0\phi(x)+\tilde{u})\phi(x)dtdx>\iint_\Omega h\phi(x)dtdx$$

which is also impossible. Thus for $\mu\in[0,1]$, $D_L(L+N_h^\mu,\Delta G(h))$ is well defined and

$$D_L(L+N_h^\mu,\Delta G(h))=D_B(JP_2N_h^\mu,\Delta G(h)\cap\ker L,0)$$

where $D_B$ ia Brouwer degree and $P_2N_h^\mu:L^2(Q)\to\ker L$ is an operator defined by

$$(P_2N_h^\mu u)(t,x)=\mu[\iint_Q g(u(t,x))\phi(x)dtdx-\iint_\Omega hdtdx]\phi(x).$$

Now let $T:\ker L\to R$ be defined by

$$T(\alpha\phi(x))=\alpha.$$

Then, for $\mu = 1$,

$$D_L(L + N_h^1, \Delta G(h)) = D_B(JP_2N_h^1, \Delta G(h) \cap \ker L, 0)$$
$$= D_B(T(JP_2N_h^1)T^{-1}, T(\Delta G(h)) \cap \ker L), 0).$$

If we let $J : \mathrm{Im}P_2 \to \ker L$ be the identity map, then the operator $\Phi = T(JP_2N_h^1)T^{-1}$ will be defined by

$$\Phi(\alpha) = \iint_Q g(\alpha\phi(x))\phi(x)dtdx - \iint_Q h\phi(x)dtdx.$$

Thus, we have

$$\Phi(\alpha_0) = \iint_Q g(\alpha_0\phi(x))\phi(x)dtdx - \iint_Q h\phi(x)dtdx < 0$$

and by the choice of $\tilde{r}_0$, we have

$$\Phi(\tilde{r}_0) = \iint_Q g(\tilde{r}_0\phi(x))\phi(x)dtdx - \iint_Q h\phi(x)dtdx > 0.$$

Hence, the coincidence degree exists and the corresponding value

$$|D_L(L - N, \Delta G(h))| = |D_B[JP_2N_h^1, \Delta \cap \mathrm{Ker}L, , 0]| = 1.$$

Therefore, the equation $(3, 1_1)$ has at least one solution in $\Delta G(h)$

Choose $G \supsetneq \Delta G(h)$, where $G$ is defined in Lemma 3.3. By the additivity of degree, we have

$$0 = D_L(L + N_h^1, G) = D_L(L + N_h^1, \Delta G(h))) + D_L(L + N_h^1, G - \overline{\Delta G(h)})$$

and hence

$$|D_L(L + N_h^1, G - \overline{\Delta G(h)})| = 1.$$

Therefore $(3.1_h^1)$ has another solution in $G - \overline{\Delta G(h)}$. This proves our assertion. □

## References


[1] A. Ambrosetti and G. Prodi, *On the inversion of some differentiable mappings with singularities between Banach space*, Ann. Mat. Pura. Appl. **93** (1972), 231-247.

[2] H. Brezis and L. Nirenberg, *Chacterization of range of some nonlinear operators and applications to boundary value problems*, Annali Scu. Norm. Sup. Pisa. **4** (1978), 225-323.

[3] R. Chiappinelli, J. Mawhin and R. Nugari, *Generalized Ambrosetti-Prodi conditions for nonlinear two-point boundary value problems*, J. Diff. Eq. **69** (1987), no. 3, 422-434.

[4] R. Courant and D. Hilbert, *Method of Mathematical Physics*, Inter. Pub. John Wiley and Sons **II** (1962).

[5] S. H. Ding and J. Mawhin, *A multiplicity result for periodic solutions of higher order orinary Differential equations*, Differential and Integral Equations **1** (1988), no. 1, 31-40.

[6] C. Fabry, J. Mawhin, and M. Nkashama, *A multiplicity result for periodic solutions of forced nonlinear second order ordinary differential equations*, Bull. London Math. Soc. **18** (1986), 173-180.

[7] S. Fucik and J. Mawhin, *Generalized periodic solutions of nonlinear telegraph equations*, Nonlinear Analysis, T.M.A. **2** (1978), no. 5, 609-617.

[8] R. G. Gains and J. Mawhin, *Coincidence Degree and Nonlinear Differential Equations*, Lecture Note in Math., Springer-Verlag **568** (1997).

[9] N. Hirano and W. S. Kim, *Periodic-Dirichlet boundary value problem for semilinear dissipative hyperbolic equations*, J. Math. Anal. Appli. **148** (1990), no. 2, 371-377.

[10] ———, *Multiplicity and stability result for semilinear for semilinear parabolic equations*, Discrete and Continus Dynamical Systems **2** (1996), no. 2, 271-280.

[11] ———, *Existence of stable and ubstable solutions for semilinear parabolic problems with a jumping nonlinearity*, Nonlinear Analysis **26** (1996), no. 6, 1143-1160.

[12] ———, *Multiple existence of periodic solutions for Lienard system*, Diff. Int. Eq. **8** (1995), no. 7, 1805–1811.

[13] W. S. Kim, *Boundary value problems for nonlinear telegraph equations with superlinear growth*, Nonlinear Analysis, T. M. A. **12** (1988), no. 12, 1371-1376.

[14] W. S. Kim and O. Y. Woo, *Boundary value problem for non-linear dissipative hyperbolic equations with superlinear growth nonlinearity*, Comm. KMS **4** (1989), no. 1, 47-57.

[15] W. S. Kim, *Periodic-Dirichlet boundary value problem for nonlinear dissipative hyperbolic equations at rersonance*, Bull. KMS **26** (1989), no. 2, 221-229.

[16] ———, *Double-periodic boundary value problem for non-linear dissipative hyperbolic equations*, J. Math. Anal. Appli. **145** (1990), no. 1, 1-16.

[17] ———, *The asymptotic behavior of non-linear dissipative hyperbolic equations*, Bull. KMS **29** (1992), no. 1, 371-377.

[18] ———, *Existence of periodic solutions for nonlinear Lienard systems*, Int. J. Math. **18** (1995), no. 2, 265–272.

[19] ———, *Multiplicity results for Doubly periodic solutions of nonlinear dissipative hyperbolic equations*, J. Math. Anal. Appli. **197** (1996), 735-748.

[20] ———, *Multiplicity result for semilinear dissipative hyperbolic equations*, J. Math. Anal. Appli. **231** (1999), 34-46.

[21] A. C. Lazer and P. J. Mckenna, *Multiplicity results for a class of semi-linear elliptic and parabolic boundary value problems*, J. Math. Anal. **107** (1985), 371-395.

[22] J. Mawhin, *Periodic solutions of nonlinear telegraph equations*, in Dynamical Systems, Bednark and Cesari, eds, Academic Press, 1977.

[23] M. N. Nkashma and M. Willem, *Time periodic solutions of boundary value problems for nonlinear heat, telegraph and beam equations*, Seminarire de mathematique, universite Catholique de Louvain **Rapport no. 54** (1984).

[24] O. Vejvoda, *Partial Differential Equations: time-periodic solution*, Martinus Nijhoff Pub. 1982.

Department of Mathematics
Hanyang University
Seoul 133-791, Korea
*E-mail*: wanskim@email.hanyang.ac.kr