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BEHAVIOR OF SOLUTIONS TO A PARABOLIC-ELLIPTIC SYSTEM MODELLING CHEMOTAXIS

TOSHITAKA NAGAI

ABSTRACT. A parabolic-elliptic system modelling chemotaxis is analysed. We study the behavior of solutions, especially the finite-time blowup of nonradial solutions, to the parabolic-elliptic system on $\mathbb{R}^n (n \geq 2)$.

1. Introduction

In this article, we consider the behavior of solutions, especially the finite-time blowup of solutions, to the following parabolic-elliptic system on $\mathbb{R}^n (n \geq 2)$:

$$\text{(P)} \qquad \begin{cases} u_t = \nabla \cdot (\nabla u - \chi u \nabla v) & \text{in } \mathbb{R}^n, \ t > 0, \\ 0 = \Delta v - v + \alpha u & \text{in } \mathbb{R}^n, \ t > 0, \\ u(\cdot, 0) = u_0 & \text{on } \mathbb{R}^n, \end{cases}$$

where α, χ are positive constants. Following [7], we always assume that

$$u_0 \geq 0 \text{ on } \mathbb{R}^n, \quad u_0 \in L^1(\mathbb{R}^n) \cap W^{1,p}(\mathbb{R}^n) \ (p > n).$$

Under the condition on u_0, it was shown in [7] that there exists some $T > 0$ such that (P) has a unique nonnegative solution (u, v) satisfying

(i) $u \in C([0,T] : W^{1,p}(\mathbb{R}^n)) \cap C^1((0,T] : L^p(\mathbb{R}^n))$,
$u(t) \in W^{2,p}(\mathbb{R}^n)$ for $0 < t \leq T$,

(ii) $v \in C([0,T] : W^{2,p}(\mathbb{R}^n))$.

Moreover, it holds that

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(i) $\int_{\mathbb{R}^n} u(x,t)\,dx = \int_{\mathbb{R}^n} u_0(x)\,dx$ for $0 < t \le T$,
(ii) (u, v) is smooth on $\mathbb{R}^n \times (0, T)$ and a classical solution of (P),
(iii) if $u_0 \not\equiv 0$, then $u(x,t) > 0$, $v(x,t) > 0$ on $\mathbb{R}^n \times (0, T]$,
(iv) if the maximal existence time T_{max} of (u, v) is finite, then

$$\limsup_{t \to T_{max}} \|u(t)\|_{L^\infty(\mathbb{R}^n)} = +\infty,$$

by which we mean that (u, v) blows up in finite time.

The system (P) is a simplified version of a parabolic system called the Keller-Segel model, proposed by Keller and Segel[15] in 1970, which is a mathematical model describing aggregation phenomena of organisms due to *chemotaxis*, i.e., the directed movement of organisms in response to the gradient of a chemical attractant.

One of interesting phenomena to the Keller-Segel model is the finite-time blowup of solutions exhibiting formation of singularities, which was conjectured in [4, 5, 21]. Recently, much attention has been paid to blowup problems for the Keller-Segel model on bounded domains Ω in $\mathbb{R}^n(n \ge 2)$ with smooth boundary, subject to homogeneous Neumann boundary conditions. As far as we know, in [14] they first showed the finite-time blowup of radial solutions to another parabolic-elliptic system of the Keller-Segel model on a disk Ω in $\mathbb{R}^2$, under the condition that $\int_\Omega u_0\,dx$ is large. In [16] he considered (P) on a ball Ω, and showed that radial solutions blow up in finite time if $n \ge 2$ and $\int_\Omega u_0(x)|x|^n\,dx$ is sufficiently small, under the condition $\int_\Omega u_0\,dx > 8\pi/(\alpha\chi)$ only for the case $n = 2$. For nonradial cases on two-dimensional bounded domains, we refer to recent results in [17]. In [9, 10, 11], they showed the existence of radial solutions on a disk in $\mathbb{R}^2$ exhibiting δ-function singularities at the origin and at the blowup time. For further study in three or more space dimensions, see [12, 13]. In recent results [19, 23], it is shown that finite-time blowup leads to the formation of singularities. We refer to [1, 2, 3, 18, 22] for results related to blowup problems, and to [2, 6, 7, 8, 16, 20, 25] for local or global existence.

It was shown in [7, Theorem 3.1] that nonnegative solutions of (P) on $\mathbb{R}^2$ exists globally in time under the condition $\int_{\mathbb{R}^2} u_0\,dx < 8\pi/(\alpha\chi)$ by using symmetrization techniques, and in Theorem 4.1 and Corollary 4.1 that radial solutions on $\mathbb{R}^n(n \ge 2)$ blow up in finite time if $\int_{\mathbb{R}^n} u_0(x)|x|^n\,dx$ is sufficiently small, under the condition $\int_{\mathbb{R}^n} u_0\,dx >$

$8\pi/(\alpha\chi)$ only for the case $n = 2$. In this article, we show the finite-time blowup of solutions to (P) on $\mathbb{R}^n (n \geq 2)$ without assuming the radial condition on initial functions u_0. Our result is the following

THEOREM 1. *Let (u, v) be the solution of (P) corresponding to the initial function u_0, and $q \in \mathbb{R}^n$. Then (u, v) blows up in finite time, provided that*

(i) *in the case $n = 2$, $\int_{\mathbb{R}^2} u_0\, dx > 8\pi/(\alpha\chi)$ and $\int_{\mathbb{R}^2} u_0(x)|x-q|^2\, dx$ satisfies*

$$\int_{\mathbb{R}^2} u_0(x)|x-q|^2\, dx < \frac{1}{4}\Big(\int_{\mathbb{R}^2} u_0\, dx - \frac{8\pi}{\alpha\chi}\Big)^2 \Big(\int_{\mathbb{R}^2} u_0\, dx\Big)^{-1}, \tag{1}$$

(ii) *in the case $n \geq 3$, $\int_{\mathbb{R}^n} u_0(x)|x-q|^n\, dx$ satisfies*

$$\begin{aligned} &2(n-1)\Big(\int_{\mathbb{R}^n} u_0\, dx\Big)^{2/n}\Big(\int_{\mathbb{R}^n} u_0(x)|x-q|^n\, dx\Big)^{(n-2)/2} \\ &\quad + \alpha\chi\beta_n\Big(\int_{\mathbb{R}^n} u_0\, dx\Big)^{(2n-1)/n}\Big(\int_{\mathbb{R}^n} u_0(x)|x-q|^n\, dx\Big)^{1/n} \\ &\quad < \frac{\alpha\chi\beta_n}{2}\Big(\int_{\mathbb{R}^n} u_0\, dx\Big)^2, \end{aligned} \tag{2}$$

where

$$\beta_n = \frac{2^{3-2n}\pi^{(1-n)/2}}{\Gamma(\frac{n-1}{2})}\int_0^\infty e^{-\xi}\xi^{n-2}\, d\xi.$$

In order to prove Theorem 1, we employ the method of moments which is used in [1, 3, 16] to prove the finite-time blowup of solutions to parabolic-elliptic systems. In the case $n = 2$, from Theorem 1 and [7, Theorem 3.1] we conclude that $8\pi/(\alpha\chi)$ is the critical number on whether solutions of (P) on $\mathbb{R}^2$ exist globally in time.

2. Fundamental solution of $-\Delta + 1$ on $\mathbb{R}^n$

Let $n \geq 2, 1 \leq p < \infty$, and given $f \in L^p(\mathbb{R}^n)$ consider the problem

$$-\Delta w + w = f \text{ in } \mathbb{R}^n.$$

The solution w is represented as

$$w(x) = \int_{\mathbb{R}^n} G(x-y)f(y)\, dy,$$

where $G(x)$ is the fundamental solution of $-\Delta + 1$ on $\mathbb{R}^n$. $G(x)$ is the Bessel function which can be expressed as

$$G(x) = \gamma_n e^{-|x|} \int_0^\infty e^{-|x|s} \Big(s + \frac{s^2}{2}\Big)^{(n-3)/2} ds \tag{3}$$

with the constant γ_n given by

$$\gamma_n = (2\pi)^{(1-n)/2} \Big\{ 2\Gamma(\frac{n-1}{2}) \Big\}^{-1}. \tag{4}$$

For (3), see a book of Stein [24, Ch. 5, Sec. 6.5].

LEMMA 1. *It holds that for $x, y \in \mathbb{R}^n (x \neq y)$,*

$$(|x|^{n-2}x - |y|^{n-2}y) \cdot \nabla G(x-y) \leq -\beta_n e^{-|x-y|}, \tag{5}$$

where

$$\beta_n = 2^{(7-3n)/2} \gamma_n \int_0^\infty e^{-\xi} \xi^{n-2}\, d\xi \tag{6}$$

and γ_n is the one given by (4).

Proof. Differentiating (3) in x gives that for $x \neq 0$,

$$\nabla G(x) = -\gamma_n \frac{x}{|x|} e^{-|x|} \int_0^\infty e^{-|x|s} (1+s) \Big(s + \frac{s^2}{2}\Big)^{(n-3)/2} ds,$$

from which it follows that

$$\begin{aligned} &(|x|^{n-2}x - |y|^{n-2}y) \cdot \nabla G(x-y) \\ &= -\gamma_n \frac{(|x|^{n-2}x - |y|^{n-2}y) \cdot (x-y)}{|x-y|} e^{-|x-y|} I, \end{aligned} \tag{7}$$

where

$$I = \int_0^\infty e^{-|x-y|s} (1+s) \Big(s + \frac{s^2}{2}\Big)^{(n-3)/2} ds.$$

By the following equalities and inequality

$$\begin{aligned} 2x \cdot (x-y) &= (|x|^2 - |y|^2) + |x-y|^2, \\ 2y \cdot (x-y) &= (|x|^2 - |y|^2) - |x-y|^2, \\ |x|^{n-2} + |y|^{n-2} &\geq 2^{3-n} |x-y|^{n-2}, \end{aligned}$$

we observe that

$$
\begin{aligned}
&(|x|^{n-2}x - |y|^{n-2}y) \cdot (x-y) \\
&= \frac{1}{2}(|x|^{n-2} + |y|^{n-2})|x-y|^2 + \frac{1}{2}(|x|^{n-2} - |y|^{n-2})(|x|^2 - |y|^2) \\
&\geq \frac{1}{2}(|x|^{n-2} + |y|^{n-2})|x-y|^2 \\
&\geq 2^{2-n}|x-y|^n.
\end{aligned} \tag{8}
$$

Using the inequality

$$(1+s)\Big(s + \frac{s^2}{2}\Big)^{(n-3)/2} \geq 2^{(3-n)/2} s^{n-2} \text{ for } \ s > 0,$$

we have

$$
\begin{aligned}
I &\geq 2^{(3-n)/2} \int_0^\infty e^{-|x-y|s} s^{n-2}\, ds \\
&= 2^{(3-n)/2} |x-y|^{1-n} \int_0^\infty e^{-\xi} \xi^{n-2}\, d\xi.
\end{aligned} \tag{9}
$$

Then, putting together (7)–(9) yields that

$$
\begin{aligned}
&(|x|^{n-2}x - |y|^{n-2}y) \cdot G(x-y) \\
&\leq -\gamma_n 2^{(7-3n)/2} e^{-|x-y|} \int_0^\infty e^{-\xi}\xi^{n-2}\, d\xi \\
&= -\beta_n e^{-|x-y|}.
\end{aligned}
$$

Hence, the proof is complete. □

3. Proof of Theorem 1

In order to prove Theorem 1, we begin with the following lemma, which is shown by Hölder's inequality.

LEMMA 2 (The moment inequality). *Let* $0 < p_1 < p_2 < \infty$ *and* $|x|^{p_2} f \in L^1(\mathbb{R}^n)$. *Then,*

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |f(x)||x|^{p_1}\, dx \\
&\leq \Big(\int_{\mathbb{R}^n} |f(x)|\, dx\Big)^{(p_2-p_1)/p_2} \Big(\int_{\mathbb{R}^n} |f(x)||x|^{p_2}\, dx\Big)^{p_1/p_2}.
\end{aligned}
$$

The following lemma is a key one to prove Theorem 1.

LEMMA 3. *Let (u,v) be the solution of (P) corresponding to the initial function u_0 and q be a point in $\mathbb{R}^n$. Assume that $\int_{\mathbb{R}^n} u_0(x)|x-q|^n\,dx < +\infty$. Then $M(t) := \int_{\mathbb{R}^n} u(x,t)|x-q|^n\,dx < +\infty$ for $0<t<T_{max}$, and it holds that*

$$M(t) \le M(0) + \int_0^t F(M(s))\,ds \quad \text{for } \ 0<t<T_{max},$$

where $M \mapsto F(M)$ is the increasing function on $[0,\infty)$ defined by

$$\begin{aligned} F(M) = 2n(n-1)&\Big(\int_{\mathbb{R}^n} u_0\,dx\Big)^{2/n} M^{(n-2)/n} \\ &- \frac{n\alpha\chi\beta_n}{2}\Big(\int_{\mathbb{R}^n} u_0\,dx\Big)^2 \\ &+ n\alpha\chi\beta_n\Big(\int_{\mathbb{R}^n} u_0\,dx\Big)^{(2n-1)/n} M^{1/n} \end{aligned}$$

and β_n is the one given by (6).

Proof. We may assume that q is the origin by the translation $x \mapsto x-q$. For $m = 1,2,3,\ldots$ let ψ_m be a function in $C^2([0,\infty))$ such that

$$0 \le \psi_m \le 1, \quad \psi_m(r) = \begin{cases} 1 & \text{for } \ r \le m, \\ 0 & \text{for } \ r \ge m+1, \end{cases}$$

$$\psi_m' \le 0, \quad |\psi_m'| \le C\psi_m, \quad |\psi_m''| \le C\psi_m,$$

where C is a constant independent of m. Put

$$M_m(t) = \int_{\mathbb{R}^n} u(x,t)|x|^n\psi_m(|x|)\,dx.$$

Multiply $u_t = \nabla\cdot(\nabla u - \chi u\nabla v)$ by $|x|^n\psi_m(|x|)$ and integrate over $\mathbb{R}^n$. Integrating by parts gives

$$\begin{aligned} &\frac{d}{dt}M_m(t) \\ &= \int_{\mathbb{R}^n} u\Delta(|x|^n\psi_m)\,dx + \chi\int_{\mathbb{R}^n} u\nabla v\cdot\nabla(|x|^n\psi_m)\,dx. \end{aligned} \tag{10}$$

Noting

$$\nabla|x|^n = n|x|^{n-2}x, \quad \Delta|x|^n = 2n(n-1)|x|^{n-2}$$

and using $\psi_m' \le 0,\ |\psi_m''| \le C\psi_m$, we have

$$\begin{aligned}(11)\qquad &\int_{\mathbb{R}^n} u\Delta(|x|^n\psi_m)\,dx \\ &\le 2n(n-1)\int_{\mathbb{R}^n} u|x|^{n-2}\psi_m\,dx + C\int_{m\le|x|\le m+1} u|x|^n\,dx.\end{aligned}$$

The moment inequality and $\int_{\mathbb{R}^n} u(x,t)\,dx = \int_{\mathbb{R}^n} u_0(x)\,dx$ give us that

$$\begin{aligned}&\int_{\mathbb{R}^n} u|x|^{n-2}\psi_m\,dx \\ &\le \Big(\int_{\mathbb{R}^n} u\psi_m\,dx\Big)^{2/n}\Big(\int_{\mathbb{R}^n} u|x|^n\psi_m\,dx\Big)^{(n-2)/n} \\ &= \Big(\int_{\mathbb{R}^n} u_0\,dx\Big)^{2/n}\{M_m(t)\}^{(n-2)/n},\end{aligned}$$

which together with (11) yields that

$$\begin{aligned}&\int_{\mathbb{R}^n} u\Delta(|x|^n\psi_m)\,dx \\ &\le 2n(n-1)\Big(\int_{\mathbb{R}^n} u_0\,dx\Big)^{2/n}\{M_m(t)\}^{(n-2)/n} + C\int_{m\le|x|\le m+1} u|x|^n\,dx.\end{aligned}$$

We next estimate the second integral on the right-hand side of (10):

$$\begin{aligned}&\int_{\mathbb{R}^n} u\nabla v\cdot\nabla(|x|^n\psi_m)\,dx \\ &= n\int_{\mathbb{R}^n} u|x|^{n-2}(x\cdot\nabla v)\psi_m\,dx + \int_{\mathbb{R}^n} u|x|^{n-1}(x\cdot\nabla v)\psi_m'\,dx.\end{aligned}$$

Let $T \in (0, T_{max})$. Since $v \in C([0,T] : W^{2,p}(\mathbb{R}^n))$ and $W^{1,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ with continuous inclusion because of $p > n$, by $|\psi_m'| \le C\psi_m$ we observe that for any $t \in (0,T)$,

$$\begin{aligned}&\int_{\mathbb{R}^n} u|x|^{n-1}(x\cdot\nabla v)\psi_m'\,dx \\ &\le C\|\nabla v\|_{L^\infty(\mathbb{R}^n\times(0,T))}\int_{m\le|x|\le m+1} u|x|^n\,dx.\end{aligned}$$

Then, for any $t \in (0,T)$ we have

$$
\begin{aligned}
&\frac{d}{dt}M_m(t)\\
&\le 2n(n-1)\Big(\int_{\mathbb{R}^n} u_0\,dx\Big)^{2/n}\{M_m(t)\}^{(n-2)/n}\\
&\quad + n\chi\int_{\mathbb{R}^n} u|x|^{n-2}(x\cdot\nabla v)\psi_m\,dx\\
&\quad + C\|\nabla v\|_{L^\infty(\mathbb{R}^n\times(0,T))}\int_{m\le|x|\le m+1} u|x|^n\,dx.
\end{aligned}
\tag{12}
$$

To obtain that the integral $\int_{\mathbb{R}^n} u(x,t)|x|^n\,dx$ is finite, we estimate the second term on the right-hand side of (12) as follows: for $0<t<T$,

$$
\begin{aligned}
&\int_{\mathbb{R}^n} u|x|^{n-2}(x\cdot\nabla v)\psi_m\,dx\\
&\le \|\nabla v\|_{L^\infty(\mathbb{R}^n\times(0,T))}\int_{\mathbb{R}^n} u|x|^{n-1}\psi_m\,dx\\
&\le \|\nabla v\|_{L^\infty(\mathbb{R}^n\times(0,T))}\Big(\int_{\mathbb{R}^n} u_0\,dx\Big)^{1/n}\{M_m(t)\}^{(n-1)/n}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\frac{d}{dt}M_m(t)\\
&\le 2n(n-1)\Big(\int_{\mathbb{R}^n} u_0\,dx\Big)^{2/n}\{M_m(t)\}^{(n-2)/n}\\
&\quad + \chi\|\nabla v\|_{L^\infty(\mathbb{R}^n\times(0,T))}\Big(\int_{\mathbb{R}^n} u_0\,dx\Big)^{1/n}\{M_m(t)\}^{(n-1)/n}\\
&\quad + C\big(\|\nabla v\|_{L^\infty(\mathbb{R}^n\times(0,T))}+1\big)\int_{m\le|x|\le m+1} u|x|^n\,dx.
\end{aligned}
$$

By Young's inequality, it follows from this differential inequality that

$$
\frac{d}{dt}M_m(t) \le C(M_m(t)+1) \quad \text{for } 0<t<T,
$$

which implies that $M_m(t)\le C$ for $0\le t\le T$ with a positive constant C depending on T. Letting $m\to\infty$, by Fatou's lemma, we have

$$
M(t) = \int_{\mathbb{R}^n} u(x,t)|x|^n\,dx \le C \quad \text{for } \ 0\le t\le T,
$$

which implies that $\int_{\mathbb{R}^n} u(x,t)|x|^n\,dx$ is finite for any $t \in (0, T_{max})$.

Next, integrating (12) from 0 to t and letting $m \to \infty$, we have

$$
\begin{aligned}
&M(t) - M(0) \\
&\le 2n(n-1)\Big(\int_{\mathbb{R}^n} u_0\,dx\Big)^{2/n}\int_0^t \{M(s)\}^{(n-2)/n}\,ds \\
&\quad + n\chi\int_0^t\int_{\mathbb{R}^n} u|x|^{n-2}(x\cdot\nabla v)\,dxds.
\end{aligned}
\tag{13}
$$

To estimate the second term on the right-hand side of (13), we use the following representation of v:

$$
v(x,t) = \alpha\int_{\mathbb{R}^n} G(x-y)u(y,t)\,dy.
$$

Then,

$$
\begin{aligned}
I &= \int_{\mathbb{R}^n} u(x,t)|x|^{n-2}x\cdot\nabla v(x,t)\,dx \\
&= \alpha\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x,t)u(y,t)|x|^{n-2}x\cdot\nabla G(x-y)\,dydx \\
&= \frac{\alpha}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x,t)u(y,t)(|x|^{n-2}x - |y|^{n-2}y)\cdot\nabla G(x-y)\,dydx.
\end{aligned}
$$

Here, we used the following symmetry properties of the integral:

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x,t)u(y,t)|x|^{n-2}x\cdot\nabla G(x-y)\,dydx \\
&= -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x,t)u(y,t)|y|^{n-2}y\cdot\nabla G(x-y)\,dydx.
\end{aligned}
$$

By (5), I is estimated as

$$\begin{aligned}
I &\le -\frac{\alpha\beta_n}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x,t)u(y,t)e^{-|x-y|}\,dydx \\
&= -\frac{\alpha\beta_n}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x,t)u(y,t)\,dydx \\
&\quad + \frac{\alpha\beta_n}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x,t)u(y,t)(1-e^{-|x-y|})\,dydx \quad . \\
&\le -\frac{\alpha\beta_n}{2}\Big(\int_{\mathbb{R}^n} u(x,t)\,dx\Big)^2 \\
&\quad + \frac{\alpha\beta_n}{2}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x,t)u(y,t)|x-y|\,dydx \\
&= -\frac{\alpha\beta_n}{2}\Big(\int_{\mathbb{R}^n} u_0(x)\,dx\Big)^2 + \frac{\alpha\beta_n}{2} II,
\end{aligned}$$

where

$$II = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x,t)u(y,t)|x-y|\,dydx.$$

Using the moment inequality, we estimate II as follows:

$$\begin{aligned}
II &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u(x,t)u(y,t)(|x|+|y|)\,dydx \\
&= 2\Big(\int_{\mathbb{R}^n} u(y,t)\,dy\Big)\Big(\int_{\mathbb{R}^n} u(x,t)|x|\,dx\Big) \\
&\le 2\Big(\int_{\mathbb{R}^n} u(y,t)\,dy\Big)\Big(\int_{\mathbb{R}^n} u(x,t)\,dx\Big)^{(n-1)/n}\Big(\int_{\mathbb{R}^n} u(x,t)|x|^n\,dx\Big)^{1/n} \\
&= 2\Big(\int_{\mathbb{R}^n} u_0(x)\,dx\Big)^{(2n-1)/n}\{M(t)\}^{1/n}.
\end{aligned}$$

Hence,

$$\begin{aligned}
I &\le -\frac{\alpha\beta_n}{2}\Big(\int_{\mathbb{R}^n} u_0(x)\,dx\Big)^2 \\
&\quad + \alpha\beta_n\Big(\int_{\mathbb{R}^n} u_0(x)\,dx\Big)^{(2n-1)/n}\{M(t)\}^{1/n}.
\end{aligned} \tag{14}$$

Putting (14) into (13) yields that

$$M(t) - M(0) \le \int_0^t F\big(M(s)\big)\,dx,$$

which concludes the proof of the lemma. □

We are now in a position to prove Theorem 1.

Proof of Theorem 1. We prove the theorem by contradiction. Suppose $T_{max} = +\infty$, that is, the solution (u, v) exists globally in time. Define the function $H(t)$ on $[0, \infty)$ by

$$H(t) = M(0) + \int_0^t F\big(M(s)\big)\, ds.$$

It follows from Lemma 3 that $M(t) \leq H(t)$ for $t > 0$. Since $F\big(M(t)\big) \leq F\big(H(t)\big)$ because of the monotonicity of $F(M)$, we have

$$H'(t) \leq F\big(H(t)\big) \quad \text{for} \quad 0 < t < \infty. \tag{15}$$

In the case $n = 2$, since $\beta_2 = 1/(2\pi)$,

$$F(M) = 4 \int_{\mathbb{R}^2} u_0\, dx - \frac{\alpha\chi}{2\pi}\Big(\int_{\mathbb{R}^2} u_0\, dx\Big)^2 + \frac{\alpha\chi}{\pi}\Big(\int_{\mathbb{R}^2} u_0\, dx\Big)^{3/2} M^{1/2}.$$

Note that $F(0) = 4\int_{\mathbb{R}^2} u_0\, dx - \alpha\chi/(2\pi)\Big(\int_{\mathbb{R}^2} u_0\, dx\Big)^2 < 0$ by virtue of $\int_{\mathbb{R}^2} u_0\, dx > 8\pi/(\alpha\chi)$, and $F\big(H(0)\big) = F\big(M(0)\big) < 0$ by virtue of (1). Hence, the differential inequality (15) gives that there is some $T_0 > 0$ such that $H(T_0) = 0$, i.e., $M(T_0) = 0$. This is a contradiction to the positivity of $M(t)$, since $u(x, t) > 0$ on $\mathbb{R}^2 \times (0, \infty)$. Hence, $T_{max} < \infty$.

In the case $n \geq 3$, observe that $F(0) < 0$ only under the condition $\int_{\mathbb{R}^n} u_0\, dx > 0$. Since $F\big(M(0)\big) < 0$ by virtue of (2), the same argument above gives $T_{max} < \infty$. Thus, the proof of the theorem is complete. □

References

[1] P. Biler, *Existence and nonexistence of solutions for a model of gravitational interactions of particles, III*, Colloq. Math. **68** (1995), 229–239.

[2] ———, *Local and global solvability of some parabolic systems modelling chemotaxis*, Adv. Math. Sci. Appl. **8** (1998), 715–743.

[3] P. Biler and T. Nadzieja, *Existence and nonexistence of solutions for a model of gravitational interactions of particles, I*, Colloq. Math. **66** (1994), 319–334.

[4] S. Childress, *Chemotactic collapse in two dimensions, Lecture Notes in Biomath.*, Springer, Berlin **55** (1984), 61–66.

[5] S. Childress and J. K. Percus, *Nonlinear aspects of chemotaxis, Math. Biosci.* **56** (1981), 217–237.

[6] J. I. Diaz and T. Nagai, *Symmetrization in a parabolic-elliptic system related to chemotaxis*, Adv. Math. Sci. Appl. **5** (1995), 659–680.

[7] J. I. Diaz, T. Nagai and J. M. Rakotoson, *Symmetrization techniques on unbounded domains: Application to a chemotaxis system on* $\mathbb{R}^N$, J. Differential Equations **145** (1998), 156–183.
[8] H. Gajewski and K. Zacharias, *Global behavior of a reaction-diffusion system modelling chemotaxis*, Math. Nachr. **195** (1998), 77–114.
[9] M. A. Herrero and J. J. L. Velázquez, *Singularity patterns in a chemotaxis model*, Math. Ann. **306** (1996), 583–623.
[10] ———, *Chemotaxis collapse for the Keller-Segel model*, J. Math. Biol. **35** (1996), 177–194.
[11] ———, *A blow-up mechanism for a chemotactic model*, Ann. Scuola Normale Sup. Pisa **XXIV** (1997), 633–683.
[12] M. A. Herrero, E. Medina and J. J. L. Velázquez, Finite-time aggregation into a single point in a reaction-diffusion system, Nonlinearity **10** (1997), 1739–1754.
[13] ———, *Self-similar blow-up for a reaction-diffusion system*, J . Comp. Appl. Math. **97** (1998), 99–119.
[14] W. Jäger and S. Luckhaus, *On explosions of solutions to a system of partial differential equations modelling chemotaxis*, Trans. Amer. Math. Soc. **329** (1992), 819–824.
[15] E. F. Keller and L. A. Segel, *Initiation of slime mold aggregation viewed as an instability*, J. Theor. Biol. **26** (1970), 399–415.
[16] T. Nagai, *Blow-up of radially symmetric solutions to a chemotaxis system*, Adv. Math. Sci. Appl. **5** (1995), 581–601.
[17] ———, *Blowup of nonradial solutions to a parabolic-elliptic system modelling chemotaxis in two-dimensional domains*, to appear in J. Inequal. Appl.
[18] T. Nagai and T. Senba, *Global existence and blow-up of radial solutions to a parabolic-elliptic system of chemotaxis*, Adv. Math. Sci. Appl. **8** (1998), 145–156.
[19] T. Nagai, T. Senba and T. Suzuki, *Chemotactic collapse in a parabolic system of mathematical biology*, to appear in Hiroshima Math. J.
[20] T. Nagai, T. Senba and K. Yoshida, *Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis, Funckcial. Ekvac.* **40** (1997), 411–433.
[21] V. Nanjundiah, *Chemotaxis, signal relaying, and aggregation morphology*, J. Theor. Biol. **42** (1973), 63–105.
[22] T. Senba, *Blow-up of radially symmetric solutions to some system of partial differential equations modelling chemotaxis*, Adv. Math. Sci. Appl. **7** (1997), 79–92.
[23] T. Senba and T. Suzuki, *Chemotactic collapse in a parabolic-elliptic system of chemotaxis*, to appear in Adv. Differential Equations.
[24] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, New Jersey, 1970.
[25] A. Yagi, *Norm behavior of solutions to the parabolic system of chemotaxis*, Math. Japonica **45** (1997), 241–265.

Department of Mathematics
Graduate School of Science
Hiroshima University
Higashi-Hiroshima, 739-8526
Japan
E-mail: nagai@math.sci.hiroshima-u.ac.jp