BOUNDARY VALUE PROBLEMS FOR NONLINEAR PERTURBATIONS OF VECTOR P-LAPLACIAN-LIKE OPERATORS

RAÚL MANÁSEVICH AND JEAN MAWHIN

ABSTRACT. The aim of this paper is to obtain nonlinear operators in suitable function spaces whose fixed points coincide with the solutions of the nonlinear boundary value problems
$$(\phi(u'))' = f(t, u, u'), \quad l(u, u') = 0,$$
where $l(u, u') = 0$ denotes the Dirichlet, Neumann or periodic boundary conditions on $[0, T]$, $\phi : \mathbb{R}^N \to \mathbb{R}^N$ is a suitable monotone homeomorphism and $f : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function. The special case where $\phi(u)$ is the vector p-Laplacian $|u|^{p-2}u$ with $p > 1$, is considered, and the applications deal with asymptotically positive homogeneous nonlinearities and the Dirichlet problem for generalized Liénard systems.

1. Introduction

Let us consider the one-dimensional $p-Laplacian$ $operator$ $(\phi_p(u'))'$, where $p > 1$ and $\phi_p : \mathbb{R} \to \mathbb{R}$ is given by $\phi_p(s) = |s|^{p-2}s$ for $s \neq 0$ and $\phi_p(0) = 0$. Various separated two-point boundary value problems containing this operator have received a lot of attention lately with respect to existence and multiplicity of solutions. See for example, [2], [3], [5], [6], [11], [12], [13], [15], [16], [17], [18], [19], [20], [21], [23], [29], [32], [33], [34], [36], [37], [41], [42], [45], and the references therein.

Periodic boundary conditions for nonlinear perturbations of the p-Laplacian have been considered in [14], [19], [22], [29], and [30].

The case of separated two-point boundary conditions and ϕ_p replaced by a one dimensional but not longer homogeneous operator ϕ, has been recently dealt with in a series of papers, like [1], [4], [8], [24], [25], [26], [27],[28], [30], [31], [41], and [44].

The case of systems with $\phi(u) = \big(\phi_p(u_1), \dots, \phi_p(u_N)\big)$ and Dirichlet boundary conditions has been considered in [7] and [46].

Our aim in this paper is to study existence of solutions of various boundary value problems for some differential *systems* involving fairly general vector-valued operator ϕ. More explicitly, we will consider the *Dirichlet boundary value problem*

$$(1.1) \qquad (\phi(u'))' = f(t, u, u'), \quad u(0) = 0, \; u(T) = 0,$$

the *Neumann boundary value problem*

$$(1.2) \qquad (\phi(u'))' = f(t, u, u'), \quad u'(0) = 0, \; u'(T) = 0,$$

and the *periodic boundary value problem*

$$(1.3) \qquad (\phi(u'))' = f(t, u, u'), \quad u(0) = u(T), \; u'(0) = u'(T),$$

where the function $\phi : \mathbb{R}^N \to \mathbb{R}^N$ satisfies some monotonicity conditions which ensure in particular that ϕ is an homeomorphism onto $\mathbb{R}^N$. Our results apply to a large class of nonlinear operators $(\phi(u'))'$, which contains various vector versions of $p-$Laplacian operators like, for $x = (x_1, \cdots, x_N) \in \mathbb{R}^N$, and $|x|$ its Euclidean norm, $\phi(x) = \psi_p(x) \equiv |x|^{p-2}x$, for $x \neq 0$, $\psi_p(0) = 0$, $(p > 1)$, and $\phi(x) = \big(\phi_{p_1}(x_1), \cdots \phi_{p_N}(x_N)\big)$, with, for each $i = 1, \cdots, N$, $p_i > 1$, and $\phi_{p_i} : \mathbb{R} \to \mathbb{R}$ is the one dimensional p_i-Laplacian.

If $I = [0, T]$, $f : I \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is assumed to be a *Carathéodory* function. By a *solution* of (1.1), (1.2) or (1.3), we will understand a function $u : I \to \mathbb{R}^N$ of class C^1 with $\phi(u')$ absolutely continuous, which satisfies (1.1), (1.2) or (1.3) a.e. on I.

Throughout the paper $|\cdot|$ will denote absolute value, and the Euclidean norm on $\mathbb{R}^N$, while the inner product in $\mathbb{R}^N$ will be denoted by $\langle \cdot, \cdot \rangle$. Also for $N \geq 1$, we will set $C = C(I, \mathbb{R}^N)$, $C^1 = C^1(I, \mathbb{R}^N)$, $C_0 = \{u \in C \mid u(0) = u(T) = 0\}$, $C_0^1 = \{u \in C^1 \mid u(0) = u(T) = 0\}$, $C_N^1 = \{u \in C \mid u'(0) = u'(T) = 0\}$, $C_T = \{u \in C \mid u(0) = u(T)\}$, $C_T^1 = \{u \in C^1 \mid u(0) = u(T), u'(0) = u'(T)\}$, $L^p = L^p(I, \mathbb{R}^N)$, and $W^{1,p} = W^{1,p}(I, \mathbb{R}^N)$, $p \geq 1$. The norm in C and its subspaces will be

denoted by $||\cdot||_0$, the norm in C^1 and its subspaces by $||\cdot||_1$, and the norm in L^p by $||\cdot||_{L^p}$. We shall also, when appropriate, identify to $\mathbb{R}^N$ the subspace of constant functions over I.

This paper is organized as follows. In Section 2, we introduce the monotone type conditions on the function ϕ we will consider and show some important examples of functions ϕ which verify those conditions.

In Section 3, we introduce a notion of mean-value of a function associated to ϕ which will be useful, in Section 4, to solve non-homogeneous p-Laplacian-like systems with various boundary conditions.

In Section 5, we reduce (1.1), (1.2), and (1.3) to fixed point problems in suitable subspaces of C^1. In the Dirichlet and Neumann case, those results extend to the vector case earlier ones of [25] and [24]. In the periodic case, they were first given in [35]. They generalize to our situation some well known continuation theorem [38, 39, 40, 43], obtained in the framework of Leray-Schauder or coincidence degree for nonlinear perturbations of linear differential operators with various boundary conditions. Indeed our approach can be viewed as an extension of coincidence degree to some quasilinear problems obtained by using classical Leray-Schauder degree theory, instead of the more sophisticated degree theory for mappings of type $(S)_+$ used in [29].

In Section 6, combining Leray-Schauder degree theory with the results of Section 5, we state and prove some existence theorems for problems (1.1), (1.2) and (1.3) when f is an asymptotically autonomous, odd and $(p-1)$-positive homogeneous system.

2. A class of monotone mappings

Let $\phi : \mathbb{R}^N \to \mathbb{R}^N$ be a continuous function which satisfies the following two conditions:

(H_1) *(Strict monotonicity). For any $x_1, x_2 \in \mathbb{R}^N$, $x_1 \neq x_2$,*

$$\langle \phi(x_1) - \phi(x_2), x_1 - x_2 \rangle > 0,$$

(H_2) *(Coercivity). there exists a function $\alpha : [0, +\infty[\to [0, +\infty[$, $\alpha(s) \to +\infty$ as $s \to +\infty$, such that*

$$\langle \phi(x), x \rangle \geq \alpha(|x|)|x|, \quad for\ all \quad x \in \mathbb{R}^N.$$

It is well known that under these two conditions ϕ is an homeomorphism from $\mathbb{R}^N$ onto $\mathbb{R}^N$, satisfies (H_1) and that $|\phi^{-1}(y)| \to +\infty$ as $|y| \to +\infty$ (see [9], Ch. 3).

Let us first give some examples of simple operators ϕ for which conditions (H_1) and (H_2) are satisfied.

EXAMPLE 2.1. Let ϕ be an homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. Then ϕ is either increasing or decreasing. Clearly in the first case, ϕ satisfies (H_1) and (H_2) while in the second case, $-\phi$ does.

EXAMPLE 2.2. For $p > 1$, let $\psi_p : \mathbb{R}^N \to \mathbb{R}^N$ be given by

$$\psi_p(x) = |x|^{p-2}x \text{ for } x \neq 0, \quad \psi_p(0) = 0.$$

Then ψ_p is an homeomorphism from $\mathbb{R}^N$ onto $\mathbb{R}^N$ with inverse

$$\psi_{p^*}(x) = |x|^{p^*-2}x,$$

where

$$p^* = \frac{p}{p-1}.$$

Let now $x, y \in \mathbb{R}^N$; from the inequality

$$\langle \psi_p(x) - \psi_p(y), x - y \rangle \geq (|x|^{p-1} - |y|^{p-1})(|x| - |y|) \geq 0,$$

it follows immediately that $\langle \psi_p(x) - \psi_p(y), x - y \rangle = 0$ implies $x = y$, and thus (H_1) holds. Also (H_2) follows from $\langle \psi_p(x), x \rangle = |x|^p = |x|^{p-1}|x|$.

EXAMPLE 2.3. More generally, we can consider any $\phi = \nabla\Phi$, with $\Phi : \mathbb{R}^N \to \mathbb{R}$ of class C^1 and strictly convex, satisfying (H_2). An interesting example of this class is given by $\Phi(x) = e^{|x|^2} - |x|^2 - 1$, for which $\langle \nabla\Phi(x), x \rangle = 2\left(e^{|x|^2} - 1\right)|x|^2$, and (H_2) is satisfied.

EXAMPLE 2.4. Further examples can be obtained from the following proposition, which is proved in [35].

PROPOSITION 2.1. For $i = 1, \cdots k$, let $N_i \in \mathbb{N}$ and $\psi_i : \mathbb{R}^{N_i} \to \mathbb{R}^{N_i}$ be a function which satisfies the following conditions.

(i) $\langle \psi_i(z) - \psi_i(w), z - y \rangle_i \geq 0$, (with $\langle \cdot, \cdot \rangle_i$ denoting the inner product in $\mathbb{R}^{N_i}$) for any $z, y \in \mathbb{R}^{N_i}$, with equality holding true if and only if $z = y$;

(ii) *there exists a function* $\alpha_i : [0, +\infty) \to [0, +\infty)$, $\alpha_i(s) \to +\infty$ *as* $s \to +\infty$, *such that* $\langle \psi_i(z), z \rangle_i \geq \alpha_i(|z|)|z|$, *for all* $z \in \mathbb{R}^{N_i}$.

Then the function

$$\Psi : \prod_{i=1}^{k} \mathbb{R}^{N_i} \to \prod_{i=1}^{k} \mathbb{R}^{N_i}, \ x = (x^1, \cdots, x^k) \mapsto \Psi(x) = (\psi_1(x^1), \cdots, \psi_k(x^k)),$$

satisfies conditions (H_1) *and* (H_2) *with* $N = \sum\limits_{i=1}^{k} N_i$.

3. Generalized mean values

The following system of nonlinear equations associated to a monotone mapping ϕ of the type considered in Section 2 plays a big role in the study of various boundary value problems associated to the Laplacian-type operator defined through ϕ. For fixed $l \in C$, let us define

$$(3.4) \qquad\qquad G_l(a) = \frac{1}{T} \int_0^T \phi^{-1}\left(a + l(t)\right) \, dt.$$

The following proposition is proved in [35].

PROPOSITION 3.1. *If ϕ satisfies conditions (H_1) and (H_2), then the function G_l has the following properties :*

(i) *For any fixed $l \in C$, the equation*

$$(3.5) \qquad\qquad G_l(a) = 0$$

    has a unique solution $-Q_\phi(l)$.

(ii) *The function $Q_\phi : C \to \mathbb{R}^N$, defined in (i), is continuous and sends bounded sets into bounded sets.*

An easy but useful property of Q_ϕ is the following one.

PROPOSITION 3.2. *If, in addition, ϕ is odd, then the mapping Q_ϕ defined by Proposition 3.1 is odd.*

Proof. If ϕ is odd, the same is true for ϕ^{-1}. By definition, we have

$$\int_0^T \phi^{-1}\left(-Q_\phi(-l) - l(t)\right) \, dt = 0,$$

and hence

$$\int_0^T \phi^{-1}\left(Q_\phi(-l) + l(t)\right) dt = 0.$$

By the uniqueness of the solution of $G_l(a) = 0$, this implies that

$$Q_\phi(-l) = -Q_\phi(l). \qquad \square$$

When $\phi = I$, equation (3.5) reduces to

$$\int_0^T [a + l(t)] \, dt = 0,$$

and has the unique solution

$$a = -\frac{1}{T} \int_0^T l(s) \, ds = -Q(l),$$

where

$$Q : L^1 \to \mathbb{R}^N, \quad l \mapsto \frac{1}{T} \int_0^T l(s) \, ds,$$

is the *mean-value* operator. So, $Q_I = Q$ and Q_ϕ can be seen as a generalized mean-value operator associated to the mapping ϕ. This is illustrated by the following results, dealing with the special case $\phi = \psi_p$, which are proved in [35].

PROPOSITION 3.3. *For each $u \in C$, there exists a unique $\overline{u_p} = Q_{\psi_{p^*}}(u) \in \mathbb{R}^N$ such that the function $\widetilde{u_p} := u - \overline{u_p}$ satisfies the relation*

$$\int_0^T \psi_p(\widetilde{u_p}(t)) \, dt = 0.$$

Furthermore, the mapping $u \mapsto \overline{u_p}$ is continuous and takes bounded sets of C into bounded sets of $\mathbb{R}^N$.

REMARK 3.1. *For $p = 2$, $\overline{u_p}$ reduces to the usual mean value $\overline{u} = Qu$ of u. Therefore, we can refer to $\overline{u_p}$ as the* p-mean value of u.

The following *Sobolev-type inequality* associated to the *p-mean value of a scalar function u* extends standard ones for $p = 2$, and is proved in [35].

PROPOSITION 3.4. *If $u \in W^{1,p}(I, \mathbb{R}^N)$, then one has the inequality*

$$(3.6) \qquad \|\widetilde{u_p}\|_{L^\infty} \le T^{\frac{1}{p^*}} \|u'\|_{L^p},$$

where

$$\|\widetilde{u_p}\|_{L^\infty} = \max_{1 \le j \le N} \|\widetilde{u_{j,p}}\|_{L^\infty}$$

and

$$\|u'\|_{L^p} = \left[\sum_{j=1}^N \int_0^T |u_j'|^p \right]^{1/p}$$

4. Non-homogeneous Dirichlet, Neumann and periodic boundary value problems

Let us first consider the *nonhomogeneous Dirichlet boundary value problem*

$$(4.7) \qquad (\phi(u'))' = h(t), \quad u(0) = 0, \ u(T) = 0,$$

where $h \in L^1$. We define $H : L^1 \to C$ by

$$(4.8) \qquad H(h)(t) = \int_0^t h(s)ds,$$

and $\Phi^{-1} : C \to C$ by

$$(4.9) \qquad \Phi^{-1}(v)(t) = \phi^{-1}(v(t)) \quad (t \in I).$$

It is clear that Φ^{-1} is continuous and sends bounded sets into bounded sets.

LEMMA 4.1. *For each $h \in L^1$, (4.7) has a unique solution given by*

$$(4.10) \qquad u = H \circ \Phi^{-1} \circ (I - Q_\phi) \circ H(h) := \mathcal{K}_D(h).$$

Proof. By integrating from 0 to $t \in I$, we find that (4.7) is equivalent to

$$(4.11) \qquad \phi(u'(t)) = a + H(h)(t), \quad u(0) = 0, \ u(T) = 0,$$

where H is defined by (4.8) and $a \in \mathbb{R}^N$ is a constant. (4.11) is in turn equivalent to

$$u'(t) = \phi^{-1}(a + H(h)(t)), \quad u(0) = 0, \ u(T) = 0,$$

and hence to

$$(4.12) \qquad u(t) = \int_0^t \phi^{-1}\left(a + H(h)(s)\right) \, ds, \quad u(T) = 0.$$

The remaining boundary condition at T provides the system of equations

$$(4.13) \qquad \int_0^T \phi^{-1}\left(a + H(h)(s)\right) \, ds = 0,$$

which, by Lemma 3.1, has the unique solution

$$a = -Q_\phi(H(h)),$$

with $Q_\phi \circ H : L^1 \to \mathbb{R}^N$ a continuous mapping sending bounded sets of L^1 into bounded sets of $\mathbb{R}^N$.

Thus (4.7) has the unique solution

$$u(t) = \int_0^t \phi^{-1}\left[-Q_\phi(H(h)) + H(h)(s)\right] \, ds$$

$$(4.14) \qquad = \left[H \circ \Phi^{-1} \circ (I - Q_\phi) \circ H(h)\right](t). \qquad \square$$

Using arguments quite similar to those of Lemma 2.1 of [35], one can prove the following compactness result.

LEMMA 4.2. *The operator* $\mathcal{K}_D$ *is continuous and sends equi-integrable sets in* L^1 *into relatively compact sets in* C_0^1.

When $\phi = I$, simple integrations by part show that (4.14) is equivalent to

$$u(t) = \int_0^T G(t,s)h(s) \, ds,$$

where

$$G(t,s) = \begin{cases} \left(\frac{t}{T} - 1\right)s & \text{for} \quad 0 \le s \le t \\ \left(\frac{s}{T} - 1\right)t & \text{for} \quad t \le s \le T \end{cases}$$

is the classical Green function.

Let us now consider the *non-homogeneous Neumann boundary value problem*

$$(4.15) \qquad (\phi(u'))' = h(t), \quad u'(0) = 0, \; u'(T) = 0,$$

where $h \in L^1$.

We define the projector P by

(4.16) $$P : C \to C, \ u \mapsto u(0).$$

The necessary condition for the solvability of (4.15)

(4.17) $$\int_0^T h(s) \, ds = 0$$

is obtained by integrating both members of (4.15) over I and using the boundary conditions. The following result shows that it is also sufficient.

LEMMA 4.3. *For each $h \in L^1$ satisfying* (4.17), *the solutions of* (4.15) *are given by*

(4.18) $$u = Pu + H \circ \Phi^{-1} \circ [\phi(0) + H(h)] := Pu + \mathcal{K}_N(h).$$

Proof. If (4.17) holds, then (4.15) is equivalent to

$$\phi(u'(t)) - \phi(0) = H(h)(t),$$

hence to

$$u'(t) = \phi^{-1}\left(\phi(0) + H(h)(t)\right),$$

whose solutions are given by

$$u(t) = u(0) + \int_0^t \phi^{-1}\left(\phi(0) + H(h)(s)\right) ds,$$

(4.19) $$= Pu + H \circ \Phi^{-1} \circ \left[\phi(0) + H(h)\right](t). \qquad \square$$

REMARK 4.1. If we assume that

$$\phi(0) = 0,$$

then (4.18) takes the simpler form

$$u = Pu + H \circ \Phi^{-1} \circ H(h).$$

Following the lines of the proof of Lemma 2.1 of [35], one can prove the following compactness result.

LEMMA 4.4. *The operator $\mathcal{K}_N$ is continuous and sends equi-integrable sets in L^1 into relatively compact sets in C_N^1.*

Let us finally consider the *non-homogeneous periodic boundary value problem*

$$(4.20) \qquad (\phi(u'))' = h(t), \quad u(0) = u(T), \ u'(0) = u'(T),$$

where $h \in L^1$. By integrating both members of (4.20) over I and using boundary conditions, we find that (4.17) is again a necessary condition for the solvability of (4.20). The following result shows that it is also sufficient.

LEMMA 4.5. *For each $h \in L^1$ such that (4.17) holds, the solutions of (4.20) are given by*

$$(4.21) \quad u = Pu + H \circ \Phi^{-1} \circ (I - Q_\phi) \circ H(h) := Pu + \mathcal{K}_P(h).$$

Proof. Integrating (4.20) from 0 to $t \in I$, we find that it is equivalent to

$$(4.22) \qquad\qquad \phi(u'(t)) = a + H(h)(t),$$

where $a \in \mathbb{R}^N$ is a constant. The boundary conditions imply that

$$\frac{1}{T} \int_0^T \phi^{-1}(a + H(h)(t)) \, dt = 0,$$

so that, by Lemma 3.1, $a = -Q_\phi \circ H(h)$. By solving for u' in (4.22) and integrating, we find

$$(4.23) \qquad u(t) = u(0) + H \left\{ \phi^{-1} \left[-Q_\phi(H(h)) + H(h) \right] \right\} (t),$$

which is clearly equivalent to (4.21). $\square$

The following compactness result is Lemma 2.1 of [35].

LEMMA 4.6. *The operator $\mathcal{K}_P$ is continuous and sends equi-integrable sets in L^1 into relatively compact sets in C_T^1.*

5. Fixed point formulations for the nonlinear boundary value problems

Let now $f : I \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ be Carathéodory. We shall denote by $N_f : C^1 \to L^1$ the *Nemitsky operator* associated to f defined by

$$\left[N_f(u) \right](t) = f\left(t, u(t), u'(t) \right).$$

We first consider the *nonlinear Dirichlet problem*

$$(5.24) \qquad (\phi(u'))' = f(t, u, u'), \quad u(0) = 0, \ u(T) = 0.$$

THEOREM 5.1. *Problem (5.24) is equivalent to the fixed point problem*

$$(5.25) \qquad u = H \circ \Phi^{-1} \circ (I - Q_\phi) \circ H \circ N_f(u) := \mathcal{G}_f^D(u)$$

in the space C_0^1, and $\mathcal{G}_f^D$ is a completely continuous mapping of C_0^1 into itself.
Furthermore, if $\phi(0) = 0$, then $\mathcal{G}_0^D = 0$. Finally, if ϕ is odd and

$$(5.26) \qquad f(t, -u, -v) = -f(t, u, v)$$

for all $(u, v) \in \mathbb{R}^N \times \mathbb{R}^N$ and almost all $t \in I$, then $\mathcal{G}_f^D$ is odd.

Proof. The result follows immediately from (4.10), Lemma 4.1, and Proposition 3.2. □

A similar fixed point operator was given in [25] and [32] in the scalar case, and in [46] for the special case of $\phi(u) = \left(\phi_p(u_1), \dots, \phi_p(u_N) \right)$.

Let us consider now the *nonlinear Neumann problem*

$$(5.27) \qquad (\phi(u'))' = f(t, u, u'), \quad u'(0) = 0, \ u'(T) = 0.$$

THEOREM 5.2. *Problem (5.27) is equivalent to the fixed point problem*
$$(5.28)$$
$$u = Pu + QN_f(u) + H \circ \Phi^{-1} \circ [\phi(0) + H(I - Q)N_f(u)] := \mathcal{G}_f^N(u)$$

in the space C_N^1, and $\mathcal{G}_f^N$ is a completely continuous mapping of C_N^1 into itself.
If $\phi(0) = 0$, $\mathcal{G}_f^N$ takes the simpler form

$$Pu + \left[Q + H \circ \Phi^{-1} \circ H(I - Q) \right] \circ N_f(u).$$

Furthermore, if condition (5.26) holds and ϕ is odd, then $\mathcal{G}_f^N$ is odd.

Proof. Problem (5.27) can be written, equivalently,

$$(5.29) \quad (\phi(u'))' = (I - Q)N_f(u), \ u'(0) = 0, \ u'(T) = 0, \ QN_f(u) = 0,$$

and hence, by the discussion of Section 4, is equivalent to the operator equations in C_N^1

$$(5.30) \quad u - Pu - H \circ \Phi^{-1} \circ [\phi(0) + H(I - Q)N_f(u)] = 0, \ QN_f(u) = 0.$$

But, as the two equations in (5.30) take values in supplementary subspaces of C_N^1, they are in turn equivalent to the unique fixed point problem (5.28). The remaining conclusions follow from Lemma 4.2, Remark 4.1, and Proposition 3.2. $\qquad\qquad\square$

A similar fixed point operator was given in [24] in the scalar case.

Let us consider finally the *nonlinear periodic problem*

$$(5.31) \qquad (\phi(u'))' = f(t, u, u'), \quad u(0) = u(T), \ u'(0) = u'(T).$$

THEOREM 5.3. *Problem (5.31) is equivalent to the fixed point problem*
(5.32)
$$u = Pu + \left[Q + H \circ \Phi^{-1} \circ (I - Q_\phi) \circ H(I - Q) \right] \circ N_f(u) := \mathcal{G}_f^P(u)$$

in the space C_T^1, and $\mathcal{G}_f^P$ is a completely continuous mapping of C_T^1 into itself. Furthermore, if (5.26) holds and ϕ is odd, then $\mathcal{G}_f^D$ is odd.

Proof. We write (5.31), equivalently,
(5.33)
$$(\phi(u'))' = (I - Q)N_f(u), \ QN_f(u) = 0, \ u(0) = u(T), \ u'(0) = u'(T).$$

Using the discussion of Section 4, we see that (5.33) is equivalent to the operator equations in C_T^1
(5.34)
$$u - Pu - \left[H \circ \Phi^{-1} \circ (I - Q_\phi) \circ H(I - Q) \right] \circ N_f(u) = 0, \ QN_f(u) = 0.$$

Like in the Neumann problem, the two equations in (5.34) take values in supplementary subspaces of C_T^1, and hence they are equivalent to the unique fixed point problem (5.32). The other properties follow immediately from Lemma 4.3 and Proposition 3.2. $\qquad\qquad\square$

This operator was first introduced in [35].

Notice finally that, in the above problems, the Nemitsky operator N_f could be replaced by a continuous abstract nonlinear operator $\mathcal{N}$ from $C_0^1 \times [0,1]$ (resp. $C_N^1 \times [0,1]$ or $C_T^1 \times [0,1]$) into L^1, which sends bounded sets into equi-integrable sets. One then obtains the family of abstract differential equations

$$(\phi(u'))' = \mathcal{N}(u, \lambda), \quad \lambda \in [0,1],$$

and the corresponding fixed point operators are completely continuous from $C_0^1 \times [0,1]$ (resp. $C_N^1 \times [0,1]$ or $C_T^1 \times [0,1]$) into C_0^1 (resp. C_N^1 or C_T^1).

6. Applications to asymptotically homogeneous systems

As an application of the results of Section 5, let us consider the problem

(6.35) $(\psi_p(u'))' = h(u, u') + e(t, u, u'), \quad l(u, u') = 0,$

where $h : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous, $e : I \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ is Carathéodory, and

$$l(u, u') = 0$$

denotes the Dirichlet, Neumann or periodic boundary conditions over $[0, T]$.

THEOREM 6.1. *Assume that the following conditions hold.*
(1) $h(ku, kv) = k^{p-1}h(u, v)$ *for all $k > 0$ and all $(u, v) \in \mathbb{R}^N \times \mathbb{R}^N$.*
(2) $h(-u, -v) = -h(u, v)$ *for all $(u, v) \in \mathbb{R}^N \times \mathbb{R}^N$.*
(3) $\lim_{|u|+|v| \to \infty} \frac{e(t,u,v)}{(|u|+|v|)^{p-1}} = 0$, *uniformly a.e. in $t \in I$.*
(4) *The problem*

$$(\psi_p(y'))' = h(y, y'), \quad l(y, y') = 0,$$

has only the trivial solution $y = 0$.
Then problem (6.35) has at least one solution.

Proof. We consider the homotopy

(6.36) $\begin{aligned} (\psi_p(u'))' &= h(u, u') + \lambda e(t, u, u'), \\ l(u, u') &= 0, \quad \lambda \in [0,1], \end{aligned}$

and show that there exists some $R > 0$ such that, for each $\lambda \in [0,1]$ and each possible solution u of (6), one has $\|y\|_1 < R$, with

$\|y\|_1 = \|y\|_0 + \|y'\|_0$. If this is the case, then the Leray-Schauder degree $d_{LS}[I - \mathcal{G}(\cdot, \lambda), B(R), 0]$ (see e.g. [9]) is well-defined and independent of λ, where $\mathcal{G}$ denotes the fixed point operator associated in Section 5 to the boundary value problem (6). Furthermore, $\mathcal{G}(\cdot, 0)$ is odd, and hence, by Borsuk-Ulam theorem [9],

$$d_{LS}[I - \mathcal{G}(\cdot, 0), B(R), 0] = 1 \pmod 2.$$

The result will then follow from Leray-Schauder's existence theorem [9].

If the claim about the possible solutions does not hold, one can find a sequence $\{\lambda_n\}$ in $[0, 1]$ and a sequence $\{u_n\}$ of solutions of (6) with $\lambda = \lambda_n$ such that $\|u_n\|_1 \to \infty$ when $n \to \infty$. If we set

$$y_n = \frac{u_n}{\|u_n\|_1}, \quad n = 1, 2, \dots,$$

it follows from assumption (1) that

$$(6.37) \qquad (\psi_p(y_n'))' = h(y_n, y_n') + \lambda_n \frac{e(t, \|u_n\|_1 y_n, \|u_n\|_1 y_n')}{\|u_n\|_1^{p-1}},$$

$$l(y_n, y_n') = 0, \ n = 1, 2, \dots.$$

As $\|y_n\|_1 = 1$ for all n, we can assume, going if necessary to a subsequence, that $y_n \to y$ uniformly in I, for some $y \in C$ satisfying respectively, in the Dirichlet and periodic cases, the boundary conditions

$$y(0) = 0, \ y(T) = 0; \quad y(0) = y(T).$$

Letting $z_n = \psi_p(y_n')$, it is clear that $\{z_n\}$ is bounded in C and it follows from equation (6.37) and from assumption (3) that $\{z_n'\}$ is bounded in C as well. Thus, up to a further subsequence, we can assume that $\{z_n\}$ converges uniformly on I to some $z \in C$, which, in the Neumann and periodic cases respectively satisfies the boundary conditions

$$z(0) = 0, \ z(T) = 0; \quad z(0) = z(T).$$

Notice that, then, $\{y_n'\}$ converges uniformly on I to $\psi_{p^*}(z)$, and that

$$(6.38) \qquad \|y\|_0 + \|\psi_{p^*}(z)\|_0 = 1.$$

Now, problem (6.37) is equivalent to

$$y_n' = \psi_{p^*}(z_n)$$

$$(6.39) \qquad z_n' = h(y_n, \psi_{p^*}(z_n)) + \lambda_n \frac{e(t, \|u_n\|_1 y_n, \|u_n\|_1 \psi_{p^*}(z_n))}{\|u_n\|_1^{p-1}},$$

$$m(y_n, z_n) = 0, \ n = 1, 2, \dots,$$

with $m(y, z) = (y(0), y(T))$ in the Dirichlet case, $m(y, z) = (z(0), z(T))$ in the Neumann case, and $m(y, z) = (y(0) - y(T), y'(0) - y'(T))$ in the periodic case. Using the above convergence results and an integrated form of (6.39), it is easy to see that (y, z) will be a solution of the problem

$$y' = \psi_{p^*}(z), \quad z' = h(y, \psi_{p^*}(z))$$
$$l(y, z) = 0,$$

and hence y will be a solution of the problem

$$(\psi_p(y'))' = h(y, y'), \quad l(y, y') = 0.$$

But then using assumption (4), it follows that $y = 0$, and hence $\psi_{p^*}(z) = 0$, a contradiction to (6.38). $\qquad\qquad\square$

By using more sophisticated properties of Leray-Schauder's degree for S^1-invariant operators, the following extension of Theorem 6.1 has been proved in [35].

THEOREM 6.2. *Assume that the following conditions hold.*

(1) $h(ku, kv) = k^{p-1}h(u, v)$ *for all* $k > 0$ *and all* $(u, v) \in \mathbb{R}^N \times \mathbb{R}^N$.
(2) $d_B[h(\cdot, 0), b(R_0), 0] \neq 0$ *for some* $R_0 > 0$.
(3) $\lim_{|u|+|v|\to\infty} \frac{e(t,u,v)}{(|u|+|v|)^{p-1}} = 0$, *uniformly a.e. in* $t \in I$.
(4) *The problem*

$$(\psi_p(y'))' = h(y, y'), \quad y(0) = y(T), \ y'(0) = y'(T),$$

has only the trivial solution $y = 0$.

Then problem (6.35), with periodic boundary conditions, has at least one solution.

We also refer to [35] for other continuation theorems in the periodic case and various applications to existence conditions.

As a special case of Theorem 6.1, we get the following result, which is classical in the scalar case.

Recall that $\mu \in \mathbb{R}$ is an *eigenvalue* of minus the p-Laplacian with the boundary conditions $l(u, u') = 0$ if the problem

$$(\psi_p(u'))' + \mu\psi_p(u) = 0, \quad l(u, u') = 0,$$

has a non-trivial solution.

COROLLARY 6.1. *If μ is not an eigenvalue of minus the p-Laplacian with boundary conditions $l(u, u') = 0$, then, for each $e \in L^1$, the problem*

$$(\psi_p(u'))' + \mu\psi_p(u) = e(t), \quad l(u, u') = 0,$$

has at least one solution.

7. Liénard-type perturbations of the vector p-Laplacian with Dirichlet conditions

Let us give some improvement of Corollary 6.1 in the case of Dirichlet boundary conditions.

Let $F : \mathbb{R}^N \to \mathbb{R}$ be of class C^2.

THEOREM 7.1. *If the $(n \times n)$-matrix A is such that*

$$(7.40) \qquad \qquad \langle Au, u \rangle \leq a|u|^2$$

for all $u \in \mathbb{R}^N$ and some

$$(7.41) \qquad \qquad a < \left(\frac{\pi_p}{T}\right)^p,$$

where

$$(7.42) \qquad \qquad \pi_p = 2(p-1)^{1/p}\frac{\pi/p}{\sin(\pi/p)},$$

then the Dirichlet problem

$$(7.43) \qquad [\psi_p(u') + \nabla F(u)]' + A\psi_p(u) = e(t), \quad u(0) = 0, \; u(T) = 0$$

has at least one solution for each $e \in L^1$.

Proof. Consider the homotopy

$$(7.44) \qquad [\psi_p(u') + \lambda\nabla F(u)]' + A\psi_p(u) = \lambda e(t),$$
$$u(0) = 0, \; u(T) = 0, \quad \lambda \in [0, 1],$$

and, for $\lambda \in [0, 1]$ fixed, let u be a possible solution of (7.44). Then, taking the inner product of the differential system by u, we obtain

$$\langle [\psi_p(u') + \lambda\nabla F(u)]', u \rangle + \langle A\psi_p(u), u \rangle = \lambda\langle e, u \rangle,$$

and hence

$$\langle \psi_p(u') + \lambda\nabla F(u), u \rangle' - \langle \psi_p(u'), u' \rangle - \lambda\langle \nabla F(u), u' \rangle + |u|^{p-2}\langle Au, u \rangle = \lambda\langle e, u \rangle.$$

Integrating this identity over I and using the boundary conditions and assumption (7.41), we get

$$(7.45) \qquad \int_0^T |u'|^p - a \int_0^T |u|^p \le \|e\|_{L^1} \|u\|_{L^\infty}.$$

As observed by del Pino [10], the spectrum of minus the vector p-Laplacian with Dirichlet boundary conditions is the same as the spectrum of minus the scalar p-Laplacian with the same boundary conditions, and hence (see e.g. [12]) the smallest eigenvalue λ_1 is given by

$$\lambda_1 = \left(\frac{\pi_p}{T}\right)^p,$$

where π_p is defined in (7.42). By the variational characterization of λ_1, we have the generalized Poincaré's inequality

$$\lambda_1 \int_0^T |u|^p \le \int_0^T |u'|^p,$$

which, introduced in (7.45) implies that

$$(7.46) \qquad \left(1 - \frac{a}{\lambda_1}\right) \|u'\|_{L^p}^p \ \le\ \|e\|_{L^1} \|u\|_{L^\infty}.$$

But now Sobolev inequality for functions $u \in W_0^{1,p}$ implies the existence of a constant $S > 0$ such that

$$\|u\|_{L^\infty} \le S \|u'\|_{L^p},$$

which, combined with (7.46), and using assumption (7.41), implies that

$$(7.47) \quad \|u'\|_{L^p} \le \left(\frac{S\|e\|_{L^1}}{1 - \frac{a}{\lambda_1}}\right)^{\frac{1}{p-1}}, \quad \|u\|_{L^\infty} \le \left(\frac{S^p\|e\|_{L^1}}{1 - \frac{a}{\lambda_1}}\right)^{\frac{1}{p-1}}.$$

Those inequalities imply immediately the existence of $R' > 0$ such that

$$(7.48) \qquad \int_0^T \left|\lambda(\nabla F(u))' + A\left(|u|^{p-2}u\right) - \lambda e\right| \le R'.$$

As, for each $j \in \{1, \cdots, N\}$, the function u'_j has mean value zero, there is a $t_j^* \in [0, T]$ such that $u'_j(t_j^*) = 0$. Then by integrating the jth component of the equation in (7.44) from t_j^* to $t \in I$, we obtain

$$|u'(t)|^{p-2}|u'_j(t)| \le \left|\int_{t_j^*}^t \left|\lambda\left(\nabla F(u)\right)' + A\left(|u|^{p-2}u\right) - \lambda e\right|\right| \le R'',$$

which by elementary means implies that $|u'(t)|^{p-1} \leq R'''$. Thus there is a positive constant M such that

$$|u'(t)| \leq M \text{ for all } t \in I,$$

and hence there is $R > 0$, such that $\|u\|_1 < R$ for all possible solutions of (7.44). Consequently, if $\mathcal{G}^D(\cdot, \lambda)$ is the fixed point operator equivalent to (7.44), it follows that for sufficiently large $R > 0$, the Leray-Schauder degree $d_{LS}[I - \mathcal{G}^D(\cdot, \lambda), B(R), 0]$ is well-defined and independent of λ. and hence equal to $d_{LS}[I - \mathcal{G}^D(\cdot, 0), B(R), 0]$. As $\mathcal{G}^D(\cdot, 0)$ is odd, this last Leray-Schauder degree is an odd number, and hence $\mathcal{G}^D$ has a fixed point and problem (7.43) has at least one solution. $\square$

REMARK 7.1. Interesting special cases of Theorem 7.1 correspond to the choice of $F(u) = c|u|^q/q$, $(q > 1,\ c \in \mathbb{R})$, for which the equation in (7.43) becomes

$$\left[\psi_p(u') + c\psi_q(u)\right]' + A\psi_p(u) = e(t),$$

and the choice of $F(u) = (1/2)\langle Cu, u \rangle$, ($C$ a symmetric matrix), for which the equation in (7.43) becomes

$$(\psi_p(u'))' + Cu' + A\psi_p(u) = e(t).$$

The case of a scalar equations reads as follows.

COROLLARY 7.1. *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and if $a \in \mathbb{R}$ satisfies condition (7.41), then the Dirichlet problem*

$$(7.49) \qquad (\phi_p(u'))' + f(u)u' + a\phi_p(u) = e(t), \quad u(0) = 0,\ u(T) = 0$$

has at least one solution for each $e \in L^1$.

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Raúl Manásevich
Departamento de Ingenieria Matemática
Universidad de Chile
Santiago, Chile
E-mail: manasevi@dim.uchile.cl

Jean Mawhin
Institut mathématique
Université de Louvain
B-1348 Louvain-la-Neuve, Belgium
E-mail: mawhin@amm.ucl.ac.be