J. Korean Math. Soc. **37** (2000), No. 5, pp. 657–664

CRITERIA FOR A NEW CONCEPT OF STABILITY

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ABSTRACT. A new concept of stability that includes Lyapunov and orbital stabilities and leads to concepts in between them is discussed in terms of a given topology of the function space. The criteria for such new concepts to hold are investigated employing suitably Lyapunov-like functions and the comparison principle.

1. Introduction

Consider the dynamic system

$$x' = f(t, x),\ x(t_0) = x_0,\ t_0 \geq 0,\ \frac{d}{dt} = 1' \tag{1.1}$$

where $f \in C[R_+ \times R^n,\ R^n]$. Assume, for convenience, that the solutions $x(t) = x(t, t_0, x_0)$ of (1.1) exist and are unique for $t \geq t_0$. The original theorems of Lyapunov have been refined, extended and generalized in various directions. See [2; 3, 4, 5, 10] for details.

In the investigation of the initial value problems of differential equations, we have been partial to initial time all along in the sense that we only perturb or change the initial dependent variable or space variable and keep the initial time unchanged for all solutions. It appears, however, important to vary the initial time as well since it is impossible not to make errors in the starting time in any physical phenomena. If we do change the starting time for each solution along with the initial change of the dependent variable, we are faced with the problem of comparing any two solutions which differ in the starting time. There could be several ways of comparing and to each choice of measuring the difference, one may end up with a different set of conditions and a different result. In [6, 7, 8], this approach is initiated.

Received August 17, 1999. Revised January 30, 2000.
2000 Mathematics Subject Classification: 34D20, 34D30, 34D99.
Key words and phrases: a new concept of stability, criteria of stability.

Lyapunov stability compares, as we know, the phase-space positions of the unperturbed and perturbed solutions of (1.1) at exactly simultaneous times, namely,

$$\text{(LS)} \qquad |x(t,t_0,y_0) - x_0(t,t_0,x_0)| < \epsilon,\ t \geq t_0,$$

where $x_0(t,t_0,x_0)$ is the given solution of (1.1) and $x(t,t_0,y_0)$ is the perturbed solution of (1.1) with the same initial time t_0. If the solutions start at different times, then (LS) can be modified as

$$\text{(LS}^*\text{)} \qquad |x(t,\tau_0,y_0) - x_0(t-\eta,t_0,x_0)| < \epsilon,\ t \geq \tau_0,$$

where $\eta = \tau_0 - t_0$. In both cases, it is a too stringent demand.

Orbital stability, on the other hand, compares at any two unrelated instants of time, namely,

$$\text{(OS)} \qquad \inf_{s\in[t_0,\infty)} |x(t,t_0,y_0) - x_0(s,t_0,x_0)| < \epsilon,\ t \geq t_0.$$

As before, if the solutions start at different times as well, then we replace (OS) by

$$\text{(OS}^*\text{)} \qquad \inf_{s\in[\tau_0,\infty)} |x(t,\tau_0,y_0) - x_0(s-\eta,t_0,x_0) < \epsilon,\ t \geq \tau_0.$$

In this case, the measurement of time is entirely irregular and hence it is a too loose a requirement.

The foregoing considerations suggest a unification of the two concepts so that the notions in between these extreme situations, may have concepts of physical significance. We shall investigate this situation which was initiated by Messera[9] who discussed the meaning stability and gave some examples. Here we enlarge his notions and provide criteria.

2. A new concept of stability

Let E be the space of all functions from R_+ to R_+, each function $\sigma(t)$ representing a clock. We call $\sigma(t) = t$, the perfect clock. Let τ be any topology in E. Given the solution $x_0(t,t_0,x_0)$, let us define the following new concept of stability.

DEFINITION. The solution $x_0(t,t_0,x_0)$ of (1.1) is said to be

(1) τ-stable, if given $\epsilon > 0$, t_0, $\tau_0 \in R_+$ and a τ-neighborhood N of the perfect clock, there exists a $\delta = \delta(t_0,\tau_0,\epsilon) > 0$ such that for each

y_0 satisfying $|x_0 - y_0| < \delta$, there is a clock $\sigma \in N$ with $\sigma(\tau_0) = t_0$ satisfying

$$|x(t, \tau_0, y_0) - x_0(\sigma(t), t_0, x_0)| < \epsilon, \ t \geq \tau_0.$$

(2) τ-uniformly stable, if in (1) δ is independent of t_0, τ_0.

(3) τ-asymptotically stable, if (1) holds and there exists a $\delta_0 = \delta_0 (t_0, \tau_0) > 0$ such that $|x_0 - y_0| < \delta_0$ and for each $\epsilon > 0$ and τ-neighborhood N of the perfect clock, there exists a $T = T(t_0, \tau_0, \epsilon) > 0$, $\sigma \in N$ with $\sigma(\tau_0) = t_0$ satisfying

$$|x(t, \tau_0, y_0) - x_0(\sigma(t), t_0, x_0)| < \epsilon, \ t \geq \tau_0 + T.$$

(4) τ-uniformly asymptotically stable, if δ_0 and T in (3) are independent of t_0, τ_0.

We note that a partial ordering of topologies of E induces a corresponding partial ordering of stability concepts.

We shall consider the following topologies of E:

(τ_1) the discrete topology, where every set E is open;

(τ_2) the chaotic topology, where the open sets are only the empty set and the entire clock space E;

(τ_3) the topology defined by the base

$$U_{\sigma_0,\epsilon} = [\sigma, \sigma_0 \in C[R_+, R_+] : \sup_{t \in [\tau_0, \infty)} |\sigma(t) - \sigma_0(t)| < \epsilon];$$

(τ_4) the topology defined by the base

$$U_{\sigma_0,\epsilon} = [\sigma, \sigma_0 = C^1[R_+, R_+] : |\sigma(\tau_0) - \sigma_0(t_0)| < \epsilon$$

and

$$\sup_{t \in [\tau_0, \infty)} |\sigma'(t) - \sigma_0'(t)| < \epsilon].$$

(τ_5) the topology consisting of the three open sets, the empty set, the entire clock space, and the set of all continuous increasing functions from R_+ onto R_+.

It is clear that the topologies τ_3, τ_4, τ_5 lie between τ_1 and τ_2.

3. Comparison results

We need the following known result [1] to prove a comparison result in terms of Lyapunov-like function and comparison principle.

THEOREM 3.1. *Let $g \in C[R_+^3, R]$ and $g(t,u,v)$ be nondecreasing in v for each (t,u). Suppose that $r(t) = r(t,t_0,u_0)$ is the maximal solution of*

$$u' = g(t,u,u), \ u(t_0) = u_0 \geq 0, \tag{3.1}$$

existing on $[t_0,\infty)$ for each (t_0,u_0). Let $m \in C[R_+,R_+]$ and

$$D_-m(t) \leq g\big(t,m(t),v\big), \ t \geq t_0.$$

Then for all $v \leq r(t)$, we have

$$m(t) \leq r(t), \ t \geq t_0. \tag{3.2}$$

We can now prove the needed comparison result. Let $w \in C[R_+^2 \times R, R_+]$ and $r(t)$ be the maximal solution of (3.1). Consider the set Ω defined by

$$\Omega = \big[\sigma \in C^1[R_+,R] : w(t,\sigma,\sigma') \leq r(t), \ \ t \geq \tau_0\big]. \tag{3.3}$$

THEOREM 3.2. *Assume that $V \in C[R_+^2 \times R^n, R_+]$, $V(t,\sigma,x)$ is locally Lipschitzian in x, and*

$$D_-V(t,\sigma,x-y) = \liminf_{h\to 0^-} \frac{1}{h}\big[V(t+h,\sigma(t+h),x-y+h(f(t,x)$$

$$-f(\sigma(t),y)\sigma'(t)) - V(t,\sigma(t),x-y)\big] \leq g\big(t,V(t,\sigma(t),x-y),w(t,\sigma,\sigma')\big),$$

where $g \in C[R_+^3,R]$ and $g(t,u,v)$ is nondecreasing in v for each (t,u). Then $\sigma(\tau_0) = t_0$ and $u_0 = V(\tau_0,\sigma(\tau_0),y_0-x_0)$ implies

$$V\big(t,\sigma(t),x\big(t,\tau_0,y_0\big) - x_0\big(\sigma(t),t_0,x_0\big) \leq r\big(t,\tau_0,v(\tau_0,\sigma(\tau_0),y_0-x_0)\big) \tag{3.4}$$

for $t \geq \tau_0$.

Proof. Let $x(t,\tau_0,y_0)$ and $x_0(t,t_0,x_0)$ be the solutions of (1.1) through (τ_0,y_0) and (t_0,x_0) existing on $[\tau_0,\infty)$ and $[t_0,\infty)$ respectively. We set $m(t) = V(t,\sigma(t),x(t,\tau_0,y_0) - x_0(\sigma(t),t_0,x_0))$ for $\sigma \in \Omega$. Then the standard computation [1] yields the differential inequality

$$D_-m(t) \leq g\big(t,m(t),w(t,\sigma,\sigma')\big), \ t \geq \tau_0.$$

Since $\sigma \in \Omega$, this implies because of the monotone character of $g(t,u,v)$ in v, the inequality

$$D_{-}m(t) \leq g\big(t, m(t), r(t)\big),\ t \geq \tau_0,$$

where $r(t) = r(t, \tau_0, u_0)$ is the maximal solution of (3.1). Theorem 3.1 now gives the stated results proving the theorem. □

4. Stability criteria

With the help of comparison theorems, we can now provide criteria for τ_3, τ_4 stabilities.

In τ_1-topology, one can use the neighborhood consisting solely of the perfect clock $\sigma(t) = t$ and consequently we get right away Lyapunov stability from the existing results.

In τ_2-topology, we proceed as follows. We set $B(t_0, x_0) = x_0([t_0, \infty) - \eta,\ t_0, x_0)$ $\eta = \tau_0 - t_0$, and obtain using standard results, the stability of the set $B(t_0, x_0)$ assuming that $B(t_0, x_0)$ is closed, namely

$$d\big[y_0, B(t_0, x_0)\big] < \delta \quad \text{implies} \quad d\big[x(t, \tau_0, y_0), B(t_0, x_0)\big] < \epsilon,\ t \geq \tau_0.$$

Since

$$d\big[x(t, \tau_0, y_0), B(t_0, x_0)\big] = \inf_{s \in [\tau_0, \infty)} |x(t, \tau_0, y_0) - x_0(s - \eta, t_0, x_0)|,$$

denoting the infimum for each $t \geq \tau_0$ by s_t and defining $\sigma(t) = s_t - \eta$ for each $t > \tau_0$, we see that $\sigma \in E$ in τ_2-topology. Thus, we obtain orbital stability of $x_0(t, t_0, x_0)$ in terms of τ_2-topology.

Next we shall provide criteria for τ_3-stability.

THEOREM 4.1. *Assume that condition* (1) *of Theorem 3.2 is satisfied. Suppose further that*

(a) $b(|x|) \leq V(t, \sigma, x) \leq a(t, \sigma, |x|)$,
(b) $d(|t - \sigma|) \leq w(t, \sigma, \sigma')$, *where* $b(\cdot)$, $d(\cdot)$ *and* $a(t, \sigma, \cdot) \in \mathcal{K}$, *and* $a \in C[R_+^3, R_+]$ *and* $\mathcal{K} = \big[a \in C[R_+, R_+] : a(0) = 0$ *and* $a(u)$ *is increasing in* $u\big]$.

Then the stability properties of the trivial solution of (3.1) *imply the corresponding* τ_3*-stability properties of* (1.1) *respectively.*

Proof. Let $x_0(t, t_0, x_0)$ be the given solution of (1.1) and let $\epsilon > 0$, $t_0, \tau_0 \in R_+$ and a τ_3-neighborhood of the perfect clock $\sigma(t) = t$, namely, $N = \left[\sigma \in C^1[R_+, R_+] : |t - \sigma(t)| < \gamma, t \geq \tau_0\right]$ be given for some $\gamma = \gamma(\epsilon) > 0$. Assume that the trivial solution of (3.1) is stable. Then given $b(\epsilon) > 0$ and $\tau_0 \in R_+$ there exists a $\delta_1 = \delta_1(\tau_0, \epsilon) > 0$ such that

$$0 \leq u_0 < \delta_1 \text{ implies } u(t, \tau_0, u_0) < b(\epsilon),\ t \geq \tau_0, \tag{4.1}$$

where $u(t, \tau_0, u_0)$ is any solution of (3.1). Set $u_0 = V(\tau_0, \sigma(\tau_0), y_0 - x_0)$. Then choosing $\delta = \delta(t_0, \tau_0, \epsilon) > 0$ and $\eta = \eta(\epsilon) > 0$ satisfying

$$a(\tau_0, t_0, \delta) < \delta_1 \text{ and } \gamma = d^{-1}(b(\epsilon)), \tag{4.2}$$

we have, using (b) and the fact $\sigma \in \Omega$,

$$\begin{aligned} d\big[|t - \sigma(t)|\big] \leq w(t, \sigma, \sigma') &\leq r\big(t, \tau_0, V(\tau_0, \sigma(\tau_0), y_0 - x_0)\big) \\ &\leq r(t, \tau_0, \delta_1) < b(\epsilon). \end{aligned}$$

It then follows that $|t - \sigma(t)| < \gamma$ and hence $\sigma \in N$. We now claim that whenever $|y_0 - x_0| < \delta$ and $\sigma \in N$, it follows that

$$\big|x(t, \tau_0, y_0) - x_0(\sigma(t), t_0, x_0)\big| < \epsilon,\ t \geq \tau_0.$$

If not, there exist a solution $x(t, \tau_0, y_0)$ of (1.1) and a $t_1 > \tau_0$ such that

(4.3)

$$\big|x(t_1, \tau_0, y_0) - x_0(\sigma(t_1), t_0, x_0)\big| = \epsilon \text{ and } \big|x(t, \tau_0, y_0) - x_0(\sigma(t), t_0, x_0)\big| \leq \epsilon$$

for $\tau_0 \leq t \leq t_1$. Then by Theorem 3.2, we get

$$V(t, \sigma(t), x(t, \tau_0, y_0) - x(\sigma(t), t_0, x_0) \leq r(t, \tau_0, V(\tau_0, \sigma(\tau_0), y_0 - x_0))$$

for $\tau_0 \leq t \leq t_1$. It then follows from (a), (4.1) and (4.3),

$$\begin{aligned} b(\epsilon) = b\Big(\big|x(t_1, \tau_0, y_0) - x(\sigma(t_1), t_0, x_0)\big|\Big) &\leq V\big(t_1, \sigma(t_1, \tau_0, y_0) \\ -x_0(\sigma(t_1), t_0, x_0)\big) &\leq r\big(t_1, \tau_0, V(\tau_0, \sigma(\tau_0), y_0 - x_0)\big) \\ &\leq r(t_1, \tau_0, \delta_1) < b(\epsilon), \end{aligned}$$

a contradiction, which proves τ_3-stability of (1.1).

Let us next suppose that the trivial solution of (3.1) is asymptotically stable. Then it is stable and given $\epsilon > 0$ and $\tau_0 \geq 0$, there exist $\delta_{01} = \delta_{01}(\tau_0) > 0$ and a $T = T(\tau_0, \epsilon) > 0$ such that

$$0 \leq u_0 < \delta_{01} \quad \text{implies} \quad u(t, \tau_0, u_0) < b(\epsilon), t \geq \tau_0 + T. \tag{4.4}$$

The τ_3-stability gives taking $\epsilon = \rho > 0$ and designating $\delta_0 = \delta(t_0, \tau_0, \rho)$,

$$|y_0 - x_0| < \delta_0 \quad \text{implies} \quad |x(t, \tau_0, y_0) - x_0(\sigma(t), t_0, x_0)| < \rho, t \geq \tau_0,$$

for every σ satisfying $|t-\sigma(t)|<\gamma(\rho)$. Thus, Theorem 3.2, we have

$$V\big(t,\sigma(t),x(t,\tau_0,y_0)-x_0(\sigma(t),t_0,x_0)\big)\le r(t,\tau_0,\delta_{10}),t\ge\tau_0. \tag{4.5}$$

Since by (4.4), we see that $r(t,\tau_0,\delta_{10})<b(\epsilon)$, $t\ge\tau_0+T$, we get

$$d\big[|t-\sigma(t)|\big]\le w(t,\sigma,\sigma')\le r(t,\tau_0,\delta_{10})<b(\epsilon),t\ge\tau_0+T.$$

Thus $|t-\sigma(t)|<d^{-1}(b(\epsilon))=\gamma(\epsilon)$, $t\ge\tau_0+T$. Hence there exists a $\sigma\in N$ satisfying

$$b\big[|x(t,\tau_0,y_0)-x_0(\sigma(t),t_0,x_0)|\big]\le V\big(t,\sigma(t),x(t,\tau_0,y_0)-x_0(\sigma(t),t_0,x_0)\big)$$
$$\le r(t,\tau_0,\delta_{10})<b(\epsilon),t\ge\tau_0+T,$$

which yields for $\sigma\in N$,

$$\big|x(t,\tau_0,y_0)-x_0(\sigma(t),t_0,x_0)\big|<\epsilon,t\ge\tau_0+T.$$

This completes the proof of τ_3-asymptotic stability. □

To obtain sufficient conditions for τ_4-stability, we need to make the following changes in Theorem 4.1.

THEOREM 4.2. *Let the conditions of Theorem 4.1 hold except that (b) is changed to*

$$\text{(b*)}\qquad d\big(|1-\sigma'(t)|\big)\le w(t,\sigma,\sigma'),\ d\in\mathcal{K}.$$

Then the stability properties of the trivial solution of (3.1) *imply the corresponding τ_4-stability properties* (1.1) *respectively.*

One can construct the proof of Theorem 4.2 following the proof of Theorem 4.1. We omit the details.

For uniform stability concepts, we need to modify condition (a) of Theorem 4.1 as follows:

$$\text{(a*)}\qquad b(|x|)\le V(t,\sigma,x)\le a_0(|x|)+a_1\big(|t-\sigma|\big),\ a_0,\ a,\ b\in\mathcal{K}.$$

REMARK. The function $g(t,u,v)=-\sigma u+\lambda v$, $\lambda-\alpha=\beta>0$ is admissible to give $r(t)=u_0e^{-\beta(t-\tau_0)}$, which yields τ_3-exponential asymptotic stabiliyt by Theorem 4.1. Also, the function $g(t,u,v)=\lambda(t)v$, $\lambda\in L^1[R_+,R_+]$ is admissible to give $r(t)=u_0\exp[\int_{\tau_0}^t\lambda(s)ds]\le u_0e^q$, where $\int_0^\infty\lambda(s)ds\le q$. Thus we get τ_3-stability from Theorem 4.1.

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