J. Korean Math. Soc. **37** (2000), No. 3, pp. 473–490

CONSENSUS N-TREES AND REMOVAL INDEPENDENCE

R. C. POWERS

ABSTRACT. Removal independence is a translation of Arrow's axiom of independence of irrelevant alternatives for social welfare functions to an axiom about consensus functions involving n-trees. It is shown that a consensus function is removal independent if and only if it is expressible as the union of three types of functions.

1. Introduction

In [6], Campbell and Kelly state the following fact about social welfare functions that involve preference relations on a finite set of alternatives. If a social welfare function f satisfies the axiom of independence of irrelevant alternatives ([1]), then there exists a partition of the set of alternatives such that f restricted to any class of this partition is either dictatorial, null, or this class contains exactly two members. Campbell and Kelly note that this fact follows directly from Theorem 5 in [9], and so they refer to it as (Part 1 of) Wilson's Partition Lemma. Thus, a social welfare function that satisfies the axiom of irrelevant alternatives can be thought of as a union of at most three "types" of functions: dictators (direct or inverse); null functions; social welfare functions restricted to two element subsets. The goal of this paper is to prove a similar result for consensus functions that involve n-trees: where the axiom of independence of irrelevant alternatives is replaced by the axiom of removal independence. We now develop the necessary definitions and notation.

Received September 22, 1999.

2000 Mathematics Subject Classification: 91B14.

Key words and phrases: consensus functions, n-trees, removal independence.

I have benefited from helpful comments by F. R. McMorris and M. F. Janowitz. This research was supported in part by ONR grant N00014-06-1-1201 to M. F. Janowitz.

Let S be a set with n elements. An *n-tree* on S is a set of subsets T of S satisfying: $S \in T$, $\phi \notin T$, $\{x\} \in T$ for all $x \in S$, and $X \cap Y \in \{\phi, X, Y\}$ for all $X, Y \in T$. An element X of an n-tree T such that $1 < |X| < n$ is called a *nontrivial cluster* of T. Thus, the *trivial clusters* of T are the singleton subsets and S. The n-tree with only trivial clusters is denoted by T_ϕ. A cluster in an n-tree is *maximal* if it is nontrivial and if it is not properly contained in any other nontrivial cluster. If X and Y are proper nonsingleton subsets of S such that $X \cap Y \in \{\phi, X, Y\}$, then we let $T_X = T_\phi \cup \{X\}$ and $T_{X,Y} = T_X \cup \{Y\}$. The set of all n-trees on S is denoted by $\mathcal{T}$.

A *consensus function* on $\mathcal{T}$ is just a mapping $C : \mathcal{T}^k \to \mathcal{T}$ where k is a positive integer. Elements of $\mathcal{T}^k$ are called *profiles* which are denoted by $P = (T_1, \ldots, T_k)$, $P' = (T_1', \ldots, T_k')$, and so on. In particular, for any proper nonsingleton subset A of S and $V \subseteq \{1, ..., k\}$,

$$P_{A;V} = (T_1, \ldots, T_k)$$

where $T_i = T_A$ whenever $i \in V$ and $T_i = T_\phi$ whenever $i \notin V$. If $V = \emptyset$, then we get

$$P_\phi = (T_\phi, \ldots, T_\phi).$$

In the sequel we will let K denote the index set $\{1, ..., k\}$. The image $C(P)$ is the *consensus n-tree* for the profile P.

In order for $C(P)$ to be a reasonable consensus of the profile P we need some restrictions on C. For example, a consensus function is said to satisfy the *Pareto* condition if, for any profile $P = (T_1, \ldots, T_k) \in \mathcal{T}^k$, $A \in T_i$ for all $i = 1, ..., k$ implies that $A \in C(P)$. Another reasonable condition for a consensus function C requires that whenever the profiles P and P' "agree" on a subset $X \subseteq S$, then so should $C(P)$ and $C(P')$ agree on X. There are many ways of defining this agreement as shown in [5]. In this paper we will focus on the following version of agreement.

If $T \in \mathcal{T}$ and $X \subseteq S$, then $T|_X$ denotes the n-tree whose nontrivial clusters are the nonempty distinct elements of $\{A \cap X : A$ is a nontrivial cluster of T and $1 < |A \cap X| < n\}$. In addition, $T|_X - X$ is the n-tree $T|_X$ without the cluster X. For any profile $P = (T_1, \ldots, T_k)$ and subset X of S,

$$P|_X = (T_1|_X, \ldots, T_k|_X)$$

and

$$P|_X - X = (T_1|_X - X, ..., T_k|_X - X).$$

A consensus function C is *removal independent* if, for every $X \subseteq S$ and profiles P, P';

(r) $P|_X - X = P'|_X - X$ implies $C(P)|_X - X = C(P')|_X - X$.

The axiom of removal independence was first proposed by Barthélemy et al. in [3].

There are two simple examples of removal independent consensus functions. The first example is a *constant function*, i.e., there exists an n-tree T such that $C(P) = T$ for all profiles P. In particular, C is *trivial* if $C(P) = T_\phi$ for all profiles P. The second example is a *projection*, i.e., there exists $j \in K$ such that for all $P = (T_1, \dots, T_k)$, $C(P) = T_j$. Notice that the first example does not satisfy Pareto whereas the second one does. In fact, projections are the only consensus functions that are removal independent and satisfy Pareto ([4]).

Observe that if $|X| \leq 2$, then $C(P)|_X - X = C(P')|_X - X$ holds for any function C and profiles P and P'. Similarly, if $X = S$, then $P|_X - X = P'|_X - X$ implies $P = P'$ and so $C(P)|_X - X = C(P')|_X - X$ holds for any function C. These comments illustrate why we may assume that $n \geq 4$. In addition, when we consider removal independence in the sequel we will implicitly assume that $3 \leq |X| < n$.

In [3], removal independence restricted to three element subsets is called *removal ternary independence*. This axiom touches upon the ternary relation associated with an n-tree (see [7]).

For each n-tree T there is an associated ternary relation R_T on S where $(a, b, c) \in R_T$ if and only if there exists $X \in T$, such that $a, b \in X$ and $c \notin X$. We will write $ab|c \in T$ instead of $(a, b, c) \in R_T$. Thus, we will think of an n-tree not only as a collection of clusters but also as a ternary relation on S. Note that $ab|c \in T$ if and only if $T|_{\{a,b,c\}} - \{a, b, c\} \doteq T_{\{a,b\}}$. Thus, if C is removal ternary independent and P, P' are two profiles such that $P|_{\{a,b,c\}} - \{a, b, c\} = P'|_{\{a,b,c\}} - \{a, b, c\}$, then $ab|c \in C(P)$ if and only if $ab|c \in C(P')$. Many of the arguments in the sequel only use this version of removal independence along with the following (see Proposition 2 in [2]):

LEMMA 1.1. *If $T \in \mathcal{T}$ and X is a proper subset of S, then $X \in T$ if and only if $ab|c \in T$ for all $a, b \in X$ and $c \notin X$.*

Finally, the proofs in this paper involve repeated applications of removal independence. Therefore, to simplify the discussion, we will often write only the conclusion of (r). The following is an example.

"Since $ab|c \in C(P)$ it follows from (r) that $ab|c \in C(P')$." The reader should understand that it is either given or can be directly verified that $P|_{\{a,b,c\}} - \{a, b, c\} = P'|_{\{a,b,c\}} - \{a, b, c\}$.

2. The main result

In this section we give some techniques for generating removal independent consensus functions from other removal independent consensus functions and, at the end of this section, we state of our main result.

LEMMA 2.1. *If $C : \mathcal{T}^k \to \mathcal{T}$ is removal independent and X is a nonempty subset of S, then the function $D : \mathcal{T}^k \to \mathcal{T}$ given by*

$$D(P) = [C(P)|_X \cup \{X\}]$$

for every profile P is removal independent.

Proof. Suppose $P|_A - A = P'|_A - A$. Since C is removal independent, $C(P)|_A - A = C(P')|_A - A$. Without loss of generality there are two possibilities: $C(P)|_A = C(P')|_A$ or $C(P)|_A = C(P')|_A \cup \{A\}$. In the first case, $C(P)|_{A\cap X} = C(P')|_{A\cap X}$ and so $D(P)|_A - A = [C(P)|_{A\cap X} \cup \{A \cap X\}] - A = [C(P')|_{A\cap X} \cup \{A \cap X\}] - A = D(P')|_A - A$. In the second case, $C(P)|_{A\cap X} = C(P')|_{A\cap X} \cup \{A \cap X\}$ and it still follows that $D(P)|_A - A = D(P')|_A - A$. Hence D is removal independent. □

If we apply Lemma 2.1 to the case where C is a projection then there exists $j \in K$ such that for every profile P,

$$D(P) = T_j|_X \cup \{X\}.$$

If a consensus function D has the form given above, then D is called a *local projection.*

We now focus on a second type of removal independent consensus function. If $C : \mathcal{T}^k \to \mathcal{T}$ is a nonconstant removal independent consensus function such that C is not a local projection and there exists a three element subset $\{a, b, c\}$ of S where $C(P)|_{S-\{a,b,c\}} = T_\phi$ for every P, then C is called a *near constant.* This terminology is motivated by the following example.

EXAMPLE 2.2. *Let $\{a, b, c\}$ be a three element subset of S and let $j \in K$. Define $E : \mathcal{T}^k \to \mathcal{T}$ as follows:*

$$E(P) = \begin{cases} T_{\{a,b,c\},\{a,b\}} & \text{if } T_j|_{\{a,b,c\}} - \{a,b,c\} = T_\phi; \\ T_{\{a,b,c\}} & \text{otherwise.} \end{cases}$$

It is clear that E is well-defined. We claim that E is removal independent. Let $A \subset S$. If $c \notin A$, then $A \cap \{a,b\} = A \cap \{a,b,c\}$ and so $T_{\{a,b,c\},\{a,b\}}|_A - A = T_{\{a,b,c\}}|_A - A$. In this case, $E(P)|_A - A = E(P')|_A - A$ for all profiles P and P'. Assume that $c \in A$. If either $a \notin A$ or $b \notin A$, then $A \cap \{a,b\}$ is a trivial cluster or empty and so $T_{\{a,b,c\},\{a,b\}}|_A - A = T_{\{a,b,c\}}|_A - A$. Again, $E(P)|_A - A = E(P')|_A - A$ for all profiles P and P'. Assume that $a, b \in A$. So $\{a,b,c\} \subseteq A$. Now $P|_A - A = P'|_A - A$ implies that $T_j|_A - A = T'_j|_A - A$ and so $[T_j|_A - A]|_{\{a,b,c\}} = [T'_j|_A - A]|_{\{a,b,c\}}$. Since $A \cap \{a,b,c\} = \{a,b,c\}$ it follows that $T_j|_{\{a,b,c\}} - \{a,b,c\} = T'_j|_{\{a,b,c\}} - \{a,b,c\}$. Therefore, $E(P) = E(P')$ and so $E(P)|_A - A = E(P')|_A - A$ for all profiles P and P'. Hence E is removal independent.

Let $C_0, C_1, ..., C_t$ be consensus functions. Then $C_0, C_1, ..., C_t$ are said to be *compatible* if, for every profile P,

$$C_0(P) \cup C_1(P) \cup ... \cup C_t(P) \tag{1}$$

is an n-tree. If $C_0, C_1, ..., C_t$ are compatible, then we will write $C_0 \cup C_1 \cup ... \cup C_t$ for the consensus function whose output is given by equation (1) for each profile P. In the sequel, we will write $C_0 \cup C_1 \cup ... \cup C_t$ with the implicit assumption that $C_0, C_1, ..., C_t$ are compatible.

We now state a structure theorem for removal independent consensus functions.

THEOREM 2.3. *A function $C : \mathcal{T}^k \to \mathcal{T}$ is removal independent if and only if there exists $t \geq 0$ such that*

$$C = C_0 \cup C_1 \cup ... \cup C_t \tag{2}$$

where C_i is either a constant function (possibly trivial), a local projection, or a near constant for $i = 0, ..., t$.

If $C = C_0 \cup C_1 \cup ... \cup C_t$ where C_i is either a constant function, a local projection, or a near constant for $i = 0, ..., t$, then it is easy to verify that C is removal independent. A proof of the converse is achieved using two major results. The first is given in the next section under the assumption that $C(P_\phi) \neq T_\phi$. In this case, C acts like a constant with respect to maximal clusters. The second result is given in Section 4 under the assumption that $C(P_\phi) = T_\phi$. In this case C is either trivial or a projection. Finally, in Section 5, the proof of Theorem 2.3 is completed.

3. The case where $C(P_\phi) \neq T_\phi$

The aim of this section is to prove the following theorem.

THEOREM 3.1. *If $C : \mathcal{T}^k \to \mathcal{T}$ is removal independent and $C(P_\phi) \neq T_\phi$, then X is a maximal nontrivial cluster in $C(P^*)$ for some profile P^* if and only if X is a maximal nontrivial cluster in $C(P)$ for all profiles P.*

We will prove Theorem 3.1 through a sequence of lemmas many of which are interesting in their own right. The proofs of these lemmas involve the ternary relation discussed in the introduction. In addition, we will use the following facts (see [7]). If $T \in \mathcal{T}$ and $\{w, x, y, z\}$ is a four element subset of S, then:

(3) $\qquad xy|z \in T$ implies that either $xy|w \in T$ or $xw|z \in T$;

(4) $\qquad xy|z \in T$ and $xw|z \in T$ imply $wy|z \in T$.

LEMMA 3.2. *If $X \in C(P_\phi)$ has three or more elements, then $X \in C(P)$ for any profile P.*

Proof. We may as well assume that $X \neq S$. Let P be an arbitrary profile. We will show $X \in C(P)$ by using Lemma 1.1. Thus, we need to show that $ab|c \in C(P)$ for any $a, b \in X$ and $c \notin X$. Toward this end, let $a, b \in X$ and $c \notin X$. Since $|X| \geq 3$, there exists $d \in X - \{a, b\}$.

Let $P' = P|_{\{a,b,c\}}$ and let $P'' = P'|_{\{a,c,d\}}$. Note that $P'' = P_{\{a,c\};V}$ for some $V \subseteq K$. A major goal in this proof is to show that $ad|c \in C(P'')$. We consider two cases.

Case 1. $bd|a \notin C(P_\phi)$.

It follows from (r) that $C(P'')|_{\{c,b,d\}} - \{c, b, d\} = C(P_\phi)|_{\{c,b,d\}} - \{c, b, d\} = T_{\{b,d\}}$ and $C(P'')|_{\{a,b,d\}} - \{a, b, d\} = C(P_\phi)|_{\{a,b,d\}} - \{a, b, d\}$. It follows that $bd|c \in C(P'')$ and $bd|a \notin C(P'')$. It follows from equation (3) that $ad|c \in C(P'')$.

Case 2. $bd|a \in C(P_\phi)$.

It follows from (r) that $C(P_{\{b,c\};V})|_{\{a,c,d\}} - \{a, c, d\} = C(P_\phi)|_{\{a,c,d\}} - \{a, c, d\} = T_{\{a,d\}}$ and $C(P_{\{b,c\};V})|_{\{a,b,d\}} - \{a, b, d\} = C(P_\phi)|_{\{a,b,d\}} - \{a, b, d\} = T_{\{b,d\}}$. Thus $ad|c \in C(P_{\{b,c\};V})$ and $ad|b \notin C(P_{\{b,c\};V})$. It follows from equation (3) that $bd|c \in C(P_{\{b,c\};V})$. Since $bd|c \in C(P_{\{b,c\};V})$ it follows from (r) that $bd|c \in C(P_{\{a,b,c\};V})$. Since $ab|c \in C(P_\phi)$ it follows from (r) that $ab|c \in C(P_{\{a,b,c\};V})$. It follows from equation (4) that

$ad|c \in C(P_{\{a,b,c\};V})$. Since $ad|c \in C(P_{\{a,b,c\};V})$ it follows from (r) that $ad|c \in C(P'')$.

Since $ad|c \in C(P'')$ it follows from (r) that $ad|c \in C(P')$. A similar argument will show that $bd|c \in C(P')$. It follows from equation (4) that $ab|c \in C(P')$ and so, by (r), $ab|c \in C(P)$. □

Example 2.2 shows that Lemma 3.2 need not be true if $|X| = 2$. The next two results act as partial converses to Lemma 3.2. From now on we will apply equations (3) and (4) without explicitly stating it.

LEMMA 3.3. *Let $\{x, y\}$ be a two element subset of S and let $V \subseteq K$. If $X \in C(P_{\{x,y\};V})$ has three or more elements, then $X \in C(P_\phi)$.*

Proof. As in the proof of Lemma 3.2, let $a, b, d \in X$ and let $c \notin X$. We want to show that $ab|c \in C(P_\phi)$.

If $|\{x, y\} \cap \{a, b, c\}| \leq 1$ then $P_{\{x,y\};V}|_{\{a,b,c\}} - \{a, b, c\} = P_\phi|_{\{a,b,c\}} - \{a, b, c\}$ and it follows that $ab|c \in C(P_\phi)$.

If $|\{x, y\} \cap \{a, b, c\}| = 2$ then either $\{x, y\} = \{a, b\}$, $\{x, y\} = \{a, c\}$, or $\{x, y\} = \{b, c\}$. We consider each of these cases separately. For the remainder of this proof, we let $P = (P_{\{x,y\};V})$.

Case 1. $\{x, y\} = \{a, b\}$.

Since $ad|c$ and $bd|c$ belong to $C(P)$ it follows from (r) that $ad|c$ and $bd|c$ belong to $C(P_\phi)$. Thus $ab|c \in C(P_\phi)$.

Case 2. $\{x, y\} = \{a, c\}$.

We consider two subcases.

Subcase 2a. $bd|a \notin C(P)$.

Since $bd|c \in C(P)$ and $bd|a \notin C(P)$ it follows from (r) that $bd|c \in C(P_\phi)$ and $bd|a \notin C(P_\phi)$. Thus $ab|c \in C(P_\phi)$.

Subcase 2b. $bd|a \in C(P)$.

Since $ad|c \in C(P)$ and $bd|c \in C(P)$ it follows from (r) that $ad|c \in C(P_{\{a,b,c\};V})$ and $bd|c \in C(P_{\{a,b\};V})$. It follows from (r) that $C(P_{\{a,b,c\};V})|_{\{a,b,d\}} - \{a, b, d\} = C(P_{\{a,b\};V})|_{\{a,b,d\}} - \{a, b, d\}$. If $ab|c \in C(P_{\{a,b,c\};V})$ then, by (r), $ab|c \in C(P_\phi)$. If $ab|c \notin C(P_{\{a,b,c\};V})$, then, since $ad|c \in C(P_{\{a,b,c\};V})$, $ad|b \in C(P_{\{a,b,c\};V})$. Thus $ad|b \in C(P_{\{a,b\};V})$ and, since $bd|c \in C(P_{\{a,b\};V})$, $ad|c \in C(P|_{\{a,b\};V})$. Since $ad|c$ and $bd|c$ belong to $C(P_{\{a,b\};V})$ it follows from (r) that $ad|c$ and $bd|c$ belong $C(P_\phi)$. Thus $ab|c \in C(P_\phi)$. This completes the proof of Case 2.

Case 3. $\{x, y\} = \{b, c\}$.

This proof is symmetric to the proof given for Case 2. □

LEMMA 3.4. *Let P be a profile with $X \in C(P)$ a maximal cluster. If $C(P_\phi) \neq T_\phi$ and X has three or more elements, then $X \in C(P_\phi)$.*

Proof. As in the proof of Lemma 3.3, let $a, b, d \in X$ and let $c \notin X$. We want to show that $ab|c \in C(P_\phi)$.

Let $P' = P|_{\{a,b,c\}} - \{a, b, c\}$. Since $ab|c \in C(P)$, it follows that $ab|c \in C(P')$. We now consider two cases.

Case 1. $ab|d \notin C(P')$.

First, since $ab|c \in C(P')$ and $ab|d \notin C(P')$, it follows that $\{ad|c, bd|c\} \subseteq C(P')$. Next, let $P'' = P'|_{\{a,c,d\}}$ and $P''' = P'|_{\{b,c,d\}}$. Then $P'' = P_{\{a,c\};V}$ and $P''' = P_{\{b,c\};W}$ for some subsets V and W of K. Since $\{ad|c, bd|c\} \subseteq C(P')$ it follows from (r) that $ad|c \in C(P'')$ and $bd|c \in C(P''')$. It follows from (r) that $C(P'')|_{\{a,b,d\}} - \{a, b, d\} = C(P''')|_{\{a,b,d\}} - \{a, b, d\}$. Therefore, either $ad|b \notin C(P'')$ or $bd|a \notin C(P''')$. Assume without loss of generality that $ad|b \notin C(P'')$. Then, since $ad|c \in C(P'')$, there exists $Y \in C(P'')$ such that $a, b, d \in Y$ and $c \notin Y$. Since $|Y| \geq 3$ and $P'' = P_{\{a,c\};V}$ it follows from Lemma 3.3 that $Y \in C(P_\phi)$. Hence $ab|c \in C(P_\phi)$.

Case 2. $ab|d \in C(P')$.

Let $P^* = P'|_{\{a,b,d\}}$. Then $P^* = P_{\{a,b\};L}$ for some $L \subseteq K$. If there exists $A \in C(P^*)$ such that $a, b \in A$ and $3 \leq |A| < n$ then, by Lemmas 3.2 and 3.3, $A \in C(P_\phi)$ and $A \in C(P)$. Since $X \in C(P)$ is a maximal nontrivial cluster with respect to set inclusion and $A \cap X \neq \emptyset$, it follows that $A \subseteq X$. Thus $c \notin A$ and so $ab|c \in C(P_\phi)$.

Since $ab|d \in C(P')$ it follows from (r) that $ab|d \in C(P^*)$. Given the previous paragraph, we may as well assume that $\{a, b\}$ is a maximal cluster in $C(P^*)$.

Recall the hypothesis: $C(P_\phi) \neq T_\phi$. Let $Z \in C(P_\phi)$ be a nontrivial cluster. We consider various possibilities for Z.

Subcase 2a. $\{a, b\} \subseteq Z$.

If $c \in Z$, then $Z \in C(P_\phi)$ contains at least three elements and so, by Lemma 3.2, $Z \in C(P)$. But $X \cap Z \neq \emptyset$ and Z not a subset of X contradicts the maximality of X in $C(P)$. So $c \notin Z$ and hence $ab|c \in C(P_\phi)$.

Subcase 2b. $|Z \cap \{a, b\}| = 1$.

Assume without loss of generality that $Z \cap \{a, b\} = \{a\}$. If $|Z| \geq 3$, then $Z \in C(P^*)$ by Lemma 3.2. But $\{a, b\}$ and Z can not both belong to $C(P^*)$. Thus $Z = \{a, u\}$ for some $u \in S - \{a, b\}$. Let $w \in S - \{a, b, u\}$. Since $au|w \in C(P_\phi)$ it follows from (r) that $au|w \in C(P^*)$. But $au|w \in$

$C(P^*)$ contradicts the fact that $\{a, b\}$ is a maximal cluster in $C(P^*)$. In sum, it is impossible to have $|Z \cap \{a, b\}| = 1$.

Subcase 2c. $Z \cap \{a, b\} = \emptyset$.

Let $u, v \in Z$ and let $P^\vee = P_{\{a,b,u\};L}$. For any $w \in S - \{a, b, u\}$, since $ab|w \in C(P^*)$, it follows from (r) that $ab|w \in C(P^\vee)$. Assume that $ab|u \notin C(P^\vee)$. Then, by Lemma 1.1, $\{a, b, u\} \in C(P^\vee)$. Since $v \in S - \{a, b, u\}$ it follows that $au|v \in C(P^\vee)$ (using $\{a, b, u\}$). It follows from (r) that $au|v \in C(P_{\{a,u\};L})$. Since $uv|b \in C(P_\phi)$ it follows from (r) that $uv|b \in C(P_{\{a,u\};L})$. Since $\{au|v, uv|b\} \subseteq C(P_{\{a,u\};L})$ it follows that there exists a cluster $A \in C(P_{\{a,u\};L})$ such that $a, u, v \in A$ and $b \notin A$. Since $|A| \geq 3$ it follows from Lemma 3.3 that $A \in C(P_\phi)$ and $|A \cap \{a, b\}| = 1$. We showed that this is impossible in Subcase 2b. Thus $ab|u \in C(P^\vee)$ and so, by (r), $ab|u \in C(P_\phi)$. It follows from Subcase 2a that $ab|c \in C(P_\phi)$ and this completes the proof. □

It follows from Lemmas 3.2 and 3.4 that if X is a maximal nontrivial cluster in $C(P^*)$ for some profile P^* and $|X| \geq 3$, then X is a maximal nontrivial cluster in $C(P)$ for every profile P. Our last result of this section considers the case where $|X| = 2$.

LEMMA 3.5. *Let $\{a, b\}$ be a two element subset of S. If $C(P_\phi) \neq T_\phi$ and $\{a, b\}$ is a maximal cluster in $C(P^*)$ for some profile P^*, then $\{a, b\}$ is a maximal cluster in $C(P)$ for every P.*

Proof. First, note that it is impossible to find a profile P such that $\{a, b\}$ is a proper subset of a nontrivial cluster in $C(P)$. Otherwise, there exists a maximal cluster X in $C(P)$ for some P such that $|X| \geq 3$ and $\{a, b\} \subseteq X$. It follows from Lemmas 3.2 and 3.4 that $X \in C(P^*)$ contrary to the maximality of $\{a, b\}$ in $C(P^*)$. Therefore, for any profile P, $\{a, b\}$ is a maximal cluster in $C(P)$ if and only if there exists $z \in S - \{a, b\}$ such that $ab|z \in C(P)$.

Let $c, d \in S - \{a, b\}$ and set $P' = P^*|_{\{a,b,c\}}$ and $P'' = P'|_{\{a,b,d\}}$. By (r), $ab|c \in C(P^*)$ implies $ab|c \in C(P')$. By the first paragraph, $\{a, b\} \in C(P')$. Since $ab|d \in C(P')$ it follows from (r) that $ab|d \in C(P'')$. So $\{a, b\} \in C(P'')$. Now $P'' = P_{\{a,b\};V}$ for some $V \subseteq K$. By (r), $ab|d \in C(P'')$ implies $ab|d \in C(P_{\{a,b,c\};V})$. So $\{a, b\} \in C(P_{\{a,b,c\};V})$. By (r), $ab|c \in C(P_{\{a,b,c\};V})$ implies $ab|c \in C(P_{\{a,b,c\};W})$ for any $W \subseteq K$. So $\{a, b\} \in C(P_{\{a,b,c\};W})$ for any $W \subseteq K$. By (r), $ab|d \in C(P_{\{a,b,c\};W})$ implies $ab|d \in C(P_{\{a,b\};W})$ for any $W \subseteq K$. So $\{a, b\} \in C(P_{\{a,b\};W})$ for any $W \subseteq K$.

Let P be an arbitrary profile. Let $P^{\vee} = P|_{\{a,b,c\}}$ and $P^{\vee\vee} = P^{\vee}|_{\{a,b,d\}}$. Note that $P^{\vee\vee} = P_{\{a,b\};W}$ for some $W \subseteq K$. So $\{a,b\} \in C(P^{\vee\vee})$. By (r), $ab|d \in C(P^{\vee\vee})$ implies $ab|d \in C(P^{\vee})$. So $\{a,b\} \in C(P^{\vee})$. By (r), $ab|c \in C(P^{\vee})$ implies $ab|c \in C(P)$. Hence $\{a,b\}$ is a maximal cluster in $C(P)$. □

Theorem 3.1 now follows from Lemmas 3.2 through 3.5.

4. The case where $C(P_\phi) = T_\phi$

The aim of this section is to prove the following theorem.

THEOREM 4.1. *If $C : \mathcal{T}^k \to \mathcal{T}$ is removal independent and $C(P_\phi) = T_\phi$, then C is either a projection or trivial.*

The proof of Theorem 4.1 is established using a sequence of lemmas and it follows, in many ways, the classical proof of Arrow's Impossibility Theorem for preference relations. The first lemma, for example, establishes the notion of a decisive set.

LEMMA 4.2. *If $C(P_\phi) = T_\phi$, then $C(P_{\{a,b\};V}) \subseteq T_{\{a,b\}}$ for all $\{a,b\} \subseteq S$ and $V \subseteq K$.*

Proof. Notice that in the proof of Lemma 3.3 we did not use the fact that $C(P_\phi) \neq T_\phi$. So the statement of Lemma 3.3 holds under the assumption that $C(P_\phi) = T_\phi$. Therefore, since $C(P_\phi) = T_\phi$, the only possible nontrivial cluster in $C(P_{\{a,b\};V})$ contains exactly two elements. Assume $\{x,y\} \in C(P_{\{a,b\};V})$ and $\{x,y\} \neq \{a,b\}$. If $\{x,y\} \cap \{a,b\} = \emptyset$, then $xy|a \in C(P_{\{a,b\};V})$ and so, by (r), $xy|a \in C(P_\phi)$. This contradicts $C(P_\phi) = T_\phi$. If $|\{x,y\} \cap \{a,b\}| = 1$, then we can assume without loss of generality that $\{x,y\} = \{a,y\}$ where $y \in S - \{a,b\}$. Let $z \in S - \{a,b,y\}$. Since $ay|z \in C(P_{\{a,b\};V})$ it follows from (r) that $ay|z \in C(P_\phi)$. In either case we contradict the fact that $C(P_\phi) = T_\phi$. □

We will say that a subset V of K is *decisive for* $\{a,b\}$ if $C(P_{\{a,b\};V}) = T_{\{a,b\}}$. It turns out that if V is decisive for some $\{a,b\}$, then V is decisive for all $\{x,y\}$. Thus, we will say that V is *decisive.*

LEMMA 4.3. *If $C(P_\phi) = T_\phi$ and $C(P_{\{a,b\};V}) = T_{\{a,b\}}$ for some $\{a,b\} \subseteq S$ and $V \subseteq K$, then $C(P_{\{x,y\};V}) = T_{\{x,y\}}$ for all $\{x,y\} \subseteq S$.*

Proof. Let $c, d \in S - \{a, b\}$. Since $ab|d \in C(P_{\{a,b\};V})$ it follows from (r) that $ab|d \in C(P_{\{a,b,c\};V})$. If $ab|c \in C(P_{\{a,b,c\};V})$, then, by (r), $ab|c \in C(P_\phi)$ contrary to $C(P_\phi) = T_\phi$. Thus $bc|d \in C(P_{\{a,b,c\};V})$ and so, by (r), $bc|d \in C(P_{\{b,c\};V})$. Therefore, by Lemma 4.2, $C(P_{\{b,c\};V}) = T_{\{b,c\}}$. By repeated use of this argument it follows that $C(P_{\{x,y\};V}) = T_{\{x,y\}}$ for all $\{x, y\} \subseteq S$. □

Let A and B be two proper nonsingleton subsets of S such that $A \cap B \in \{\emptyset, A, B\}$. If V and W are subsets of K, then

$$P_{A,B;V,W} = (T_1, ..., T_k)$$

where $T_i = T_{A,B}$ whenever $i \in V \cap W$; $T_i = T_A$ whenever $i \in V \cap (K - W)$; $T_i = T_B$ whenever $i \in W \cap (K - V)$; $T_i = T_\phi$ whenever $i \in K - (V \cup W)$. In addition, we can drop the assumption that $A \cap B \in \{\emptyset, A, B\}$ if we restrict V and W so that $V \cap W = \emptyset$. The above notation will be used in the sequel.

For the remaining lemmas in this section we will assume that $C(P_\phi) = T_\phi$ and $C(P_{\{a,b\};V}) = T_{\{a,b\}}$ for some $\{a, b\} \subseteq S$ and $V \subseteq K$, i.e., there exists a decisive set V.

LEMMA 4.4. *$C(P_{\{x,y\};W}) = T_{\{x,y\}}$ for all $\{x, y\} \subseteq S$ and $V \subseteq W \subseteq K$.*

Proof. Let $c, d \in S - \{a, b\}$ and consider the profile $P' = P_{\{a,b,c\},\{b,c\};V,W}$. Since $ab|d \in C(P_{\{a,b\};V})$ it follows from (r) that $ab|d \in C(P')$. By Lemma 4.3, $ac|d \in C(P_{\{a,c\};V})$ and so, by (r), $ac|d \in C(P')$. Thus $bc|d \in C(P')$. By (r), $bc|d \in C(P_{\{b,c\};W})$. By Lemma 4.2, $C(P_{\{b,c\};W}) = T_{\{b,c\}}$ and so, by Lemma 4.3, $C(P_{\{x,y\};W}) = T_{\{x,y\}}$ for all $\{x, y\} \subseteq S$ and $V \subseteq W \subseteq K$. □

It follows from Lemma 4.4 that the collection of decisive sets is an order filter with respect to set inclusion. Next, we establish the existence of a decisive set with exactly one element.

LEMMA 4.5. *There exists $j \in K$ such that $C(P_{\{x,y\};\{j\}}) = T_{\{x,y\}}$ for all $\{x, y\} \subseteq S$.*

Proof. Let M be a minimal set, with respect to set inclusion, such that $C(P_{\{x,y\};M}) = T_{\{x,y\}}$ for all $\{x, y\} \subseteq S$. Assume that $|M| > 1$. Let $j \in M$ and set $M_1 = \{j\}$, $M_2 = M - \{j\}$, and $M_3 = K - M$. Let $P = P_{\{a,b\},\{a,b,c\};M_1,M_2}$ and let $d \in S - \{a, b, c\}$.

Since $ab|d \in C(P_{\{a,b\};M})$ it follows from (r) that $ab|d \in C(P)$. Either $ab|c \in C(P)$ or $bc|d \in C(P)$. If $ab|c \in C(P)$ then, by (r), $ab|c \in C(P_{\{a,b\};M_1})$. If $bc|d \in C(P)$ then, by (r), $bc|d \in C(P_{\{b,c\};M_2})$. In either case we contradict the minimality of M. □

For the remainder of this section let $\{j\}$ be a minimal decisive set.

LEMMA 4.6. *$C(P_{\{x,y\};W}) = T_{\{x,y\}}$ for all $\{x,y\} \subseteq S$ if and only if $j \in W$.*

Proof. Assume that there exists $W \subseteq K$ such that $C(P_{\{a,b\};W}) = T_{\{a,b\}}$ for some $\{a,b\} \subseteq S$ and $j \notin W$. Let $P = P_{\{a,b\},\{b,c\};\{j\},W}$. It follows from (r) and the above that $\{ab|d, bc|d\} \subseteq C(P)$. Thus $ac|d \in C(P)$ and so, by (r), $ac|d \in C(P_\phi)$ contrary to the fact that $C(P_\phi) = T_\phi$. Hence $C(P_{\{a,b\};W}) = T_{\{a,b\}}$ implies $j \in W$. The converse follows from Lemma 4.4 and the choice of j. □

LEMMA 4.7. *For any profile $P = (T_1, ..., T_k)$, $ab|c \in C(P)$ whenever $ab|c \in T_j$.*

Proof. Let $P = (T_1, ..., T_k)$ be an arbitrary profile such that $ab|c \in T_j$. Let $P' = P|_{\{a,b,c\}} - \{a,b,c\}$ and $P'' = P'|_{\{a,b,d\}}$. Note that $P'' = P_{\{a,b\};W}$ for some $W \subseteq K$ with $j \in W$. Thus $ab|d \in C(P'')$ and so, by (r), $ab|d \in C(P')$. If $ac|d \in C(P')$ then, by (r), $ac|d \in C(P'|_{\{a,c,d\}})$ where $P'|_{\{a,c,d\}} = P_{\{a,c\};Z}$ for some $Z \subseteq K$ such that $j \notin Z$. This contradicts Lemma 4.6. Thus $ab|c \in C(P')$ and so, by (r), $ab|c \in C(P)$. □

It follows from Lemmas 1.1 and 4.7 that $A \in C(P)$ whenever $A \in T_j$. Thus, $C(P) \subseteq T_j$. In this situation, we call C a *dictatorship.* The final step in the proof of Theorem 4.1 is to show that a dictatorship is actually a projection.

PROOF OF THEOREM 4.1. Assume $C(P) \neq T_\phi$ for some profile $P = (T_1, ..., T_k)$. Then $ab|c \in C(P)$ for some three element subset $\{a,b,c\}$ of S. Let $P' = P|_{\{a,b,c\}} - \{a,b,c\}$. It follows from (r) that $ab|c \in C(P')$. Let $d \in S - \{a,b,c\}$. Either $ad|c \in C(P')$ or $ab|d \in C(P')$. If $ad|c \in C(P')$ then, by (r), $ad|c \in C(P'|_{\{a,c,d\}})$. But $P'|_{\{a,c,d\}} = P_{\{a,c\};V}$ for some $V \subseteq K$ and so, $ad|c \notin C(P'|_{\{a,c,d\}})$ by Lemma 4.2. So $ab|d \in C(P')$. It follows from (r) that $ab|d \in C(P'|_{\{a,b,d\}})$. Now $P'|_{\{a,b,d\}} = P_{\{a,b\};W}$ for some $W \subseteq K$. Since $ab|d \in C(P_{\{a,b\};W})$ it follows from Lemma 4.2 that $C(P_{\{a,b\};W}) = T_{\{a,b\}}$. Thus W is a decisive set. As usual, let $\{j\}$ be a minimal decisive set. Then, by Lemma 4.6, $C(P_{\{a,b\};Z}) = T_{\{a,b\}}$ if and only if $j \in Z$. In particular, $j \in W$. Since $P_{\{a,b\};W} = [P|_{\{a,b,c\}} - \{a,b,c\}]|_{\{a,b,d\}}$ it follows that $ab|c \in T_j$. Therefore, for any profile $P = (T_1, ..., T_k)$ and subset $\{a,b,c\}$ of S, if $ab|c \in C(P)$ then $ab|c \in T_j$. The converse follows from Lemma 4.7. Hence, by Lemma 1.1, C is a projection. □

The next result, which was stated without proof in [5], is a consequence of Theorems 3.1 and 4.1.

THEOREM 4.8. *If* $C : \mathcal{T}^k \to \mathcal{T}$ *is removal independent then, for any profiles* P, P' *and subset* X *of* S,

$$(5) \qquad P|_X = P'|_X \text{ implies } C(P)|_X = C(P')|_X.$$

Proof. If $C(P_\phi) = T_\phi$, then C is either a projection or trivial. In either case, C satisfies equation (5).

Now suppose $C(P_\phi) \neq T_\phi$. Suppose $P|_X = P'|_X$ where P and P' are profiles and X is a proper subset of S containing two or more elements. If $X \in C(P)|_X$ then there exists a maximal cluster Z in $C(P)$ such that $X \subseteq Z$. By Theorem 3.1, $Z \in C(P')$ and so $Z \in C(P')|_X$. Thus, $X \in C(P)|_X$ if and only if $X \in C(P')|_X$. Since $P|_X - X = P'|_X - X$ it follows from (r) that $C(P)|_X - X = C(P')|_X - X$. Hence $C(P)|_X = C(P')|_X$. □

As a conclusion to this section, suppose $C : \mathcal{T}^k \to \mathcal{T}$ is removal independent and satisfies Pareto. Then C is not trivial and, as a consequence of Theorem 3.1, $C(P_\phi) = T_\phi$. It follows from Theorem 4.1 that C is a projection (see [4]).

5. Return of the structure theorem

We now return our attention to proving Theorem 2.3.

Let $C : \mathcal{T}^k \to \mathcal{T}$ be removal independent and let $\mathcal{I} = \{X | 1 < |X| < n$ and $X \in C(P)$ for every $P\}$. Let $X \in \mathcal{I}$ such that $|X| \geq 3$. Let $\mathcal{T}(X)$ denote the set of all m-trees where $m = |X|$. There is a natural way to identify an m-tree L in $\mathcal{T}(X)$ with an n-tree $\alpha(L)$ in $\mathcal{T}$. Specifically, let

$$\alpha(L) = L \cup T_\phi$$

where T_ϕ is, as usual, the n-tree with only trivial clusters. Conversely, there is a natural way to identify an n-tree T in $\mathcal{T}$ with an m-tree $\beta(T)$ in $\mathcal{T}(X)$. Specifically, let

$$\beta(T) = T|_X \cup \{X\} - \{S, \{a\} | a \in S - X\}.$$

The mappings α and β are illustrated in Figure 1 where $S = \{1, 2, 3, 4, 5\}$ and $X = \{1, 2, 3\}$.

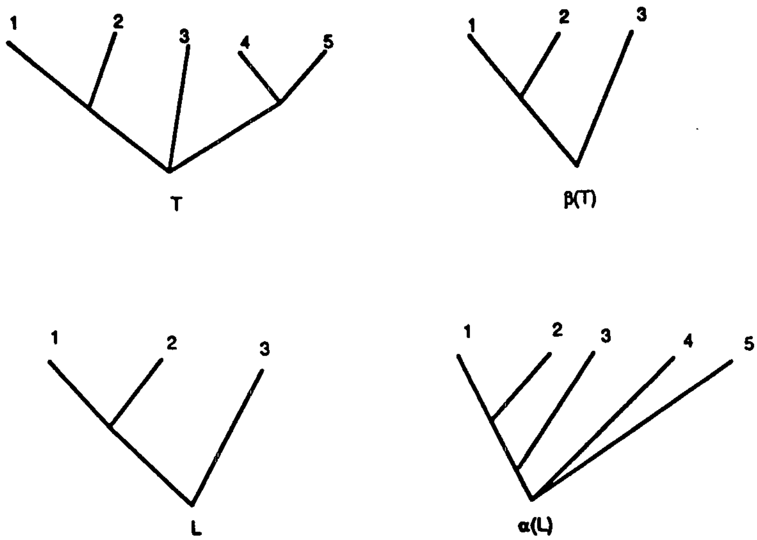

Figure 1

For any m-tree L in $\mathcal{T}(X)$ and for any T in $\mathcal{T}$ we have

(6) $$\beta\alpha(L) = \beta(L \cup T_\phi) = L$$

and

(7) $$\alpha\beta(T) = T|_X \cup \{X\}.$$

For any profiles $Q = (L_1, ..., L_k) \in \mathcal{T}(X)^k$ and $P = (T_1, ..., T_k) \in \mathcal{T}^k$ we let $\alpha(Q) = (\alpha(L_1), ..., \alpha(L_k))$ and $\beta(P) = (\beta(T_1), ..., \beta(T_k))$. Thus, $\alpha(Q) \in \mathcal{T}^k$ and $\beta(P) \in \mathcal{T}(X)^k$. Finally, for each profile Q in $\mathcal{T}(X)^k$, we define

$$C_X(Q) = \beta[C(\alpha(Q))].$$

Note that C_X is a well-defined function from $\mathcal{T}(X)^k$ into $\mathcal{T}(X)$.

LEMMA 5.1. *If L_ϕ denotes the trivial m-tree in $\mathcal{T}(X)$ and $Q_\phi = (L_\phi, ..., L_\phi) \in \mathcal{T}(X)^k$, then $C_X(Q_\phi) = L_\phi$ if and only if $C(P_\phi)|_X = T_X$.*

Proof. ($\Rightarrow$) By definition, $\beta[C(\alpha(Q_\phi)] = L_\phi$. So $\alpha\beta[C(\alpha(Q_\phi))] = \alpha(L_\phi)$. Observe that $\alpha(L_\phi) = T_X$. It follows from (7) that $\alpha\beta(C(P_{X;K})) = C(P_{X;K})|_X \cup \{X\}$. Since $X \in \mathcal{I}$ it follows that $C(P_{X;K})|_X \cup \{X\} =$

$C(P_{X;K})|_X$. It follows from (r) that $C(P_{X;K})|_X - X = C(P_\phi)|_X - X$. Again, since $X \in \mathcal{I}$, it follows that $C(P_{X;K})|_X = C(P_\phi)|_X$. Hence $C(P_\phi)|_X = T_X$.

($\Leftarrow$) As above, $C(P_{X;K})|_X = C(P_\phi)|_X$. Since $\alpha(L_\phi) = T_X$ and $C(P_\phi)|_X = T_X$ it follows that $C(\alpha(Q_\phi))|_X = \alpha(L_\phi)$. So $\beta[C(\alpha(Q_\phi))|_X] = \beta\alpha(L_\phi)$. It follows from (6) and the definition of β that $\beta[C(\alpha(Q_\phi)] = L_\phi$. Hence $C_X(Q_\phi) = L_\phi$. □

LEMMA 5.2. *C_X is removal independent.*

Proof. Suppose $Q|_A - A = Q'|_A - A$ where $Q, Q' \in \mathcal{T}(X)^k$ and $A \subseteq X$. Then $\alpha(Q|_A - A) = \alpha(Q'|_A - A)$. By using the definition of α we get $\alpha(Q)|_A - A = \alpha(Q')|_A - A$. Since C is removal independent, $C(\alpha(Q))|_A - A = C(\alpha(Q'))|_A - A$. Then $\beta(C(\alpha(Q))|_A - A) = \beta(C(\alpha(Q'))|_A - A)$. By using the definition of β we get $\beta(C(\alpha(Q)))|_A - A = \beta(C(\alpha(Q')))|_A - A$. Hence $C_X(Q)|_A - A = C_X(Q')|_A - A$. □

Let $P = (T_1, ..., T_k) \in \mathcal{T}^k$ and let X be a nonempty subset of S. Then $P \cup \{X\}$ denotes the profile $(T_1 \cup \{X\}, ..., T_k \cup \{X\})$.

LEMMA 5.3. *If $C(P_\phi)|_X = T_X$ for some $X \in \mathcal{I}$ and $|X| \geq 4$, then either $C(P)|_X = T_X$ for every P or there exists $j \in K$ such that $C(P)|_X = T_j|_X \cup \{X\}$ for every P.*

Proof. Consider $C_X : \mathcal{T}(X)^k \to \mathcal{T}(X)$. By the previous lemmas we know that C_X is removal independent and $C_X(Q_\phi) = L_\phi$. Therefore, by Theorem 4.1, C_X is either trivial or a projection.

Consider the case where C_X is trivial. Let $P \in \mathcal{T}^k$. Then $C_X(\beta(P)) = L_\phi$. So $\alpha(C_X(\beta(P))) = \alpha(L_\phi)$. Thus $[\alpha\beta C\alpha\beta](P) = T_X$. So $C(P|_X \cup \{X\})|_X = T_X$. It follows from (r) and the fact that $X \in \mathcal{I}$ that $C(P)|_X = C(P|_X \cup \{X\})|_X$. Hence $C(P)|_X = T_X$ for every P.

Consider the case where C_X is a projection. So there exists $j \in K$ such that $C_X(Q) = L_j$ for every $Q = (L_1, ..., L_k) \in \mathcal{T}(X)^k$. Let $P = (T_1, ..., T_k) \in \mathcal{T}^k$. Then $C_X(\beta(P)) = \beta(T_j)$. So $\alpha[C_X(\beta(P))] = \alpha\beta(T_j)$. It follows that $C(P|_X \cup \{X\})|_X = T_j|_X \cup \{X\}$. It follows from (r) and the fact that $X \in \mathcal{I}$ that $C(P)|_X = C(P|_X \cup \{X\})|_X$. Hence $C(P)|_X = T_j|_X \cup \{X\}$ for every P. □

Let $X \in \mathcal{I}$ and let $\mathcal{J} = \{Y | 1 < |Y| < |X| \text{ and } Y \in C_X(Q) \text{ for every } Q\}$. Let $[X) = \{Z | Z \subset X\}$. Then

$$\mathcal{J} = \mathcal{I} \cap [X). \tag{8}$$

To see why equation (8) is true, let $Y \in \mathcal{J}$. Then $Y \in [X)$. For any profile $P \in \mathcal{T}^k$ we have $Y \in C_X(\beta(P)) = \beta C \alpha\beta(P) = \beta[C(P|_X \cup \{X\})]$. It follows from the definition of β and (r) that $Y \in C(P|_X \cup \{X\})|_X = C(P)|_X$. Since $X \in C(P)$ it follows that $Y \in C(P)$. So $Y \in \mathcal{I}$. Thus $\mathcal{J} \subseteq \mathcal{I} \cap [X)$.

Conversely, let $Z \in \mathcal{I} \cap [X)$. Then $Z \in C(P)|_X$ for every P. In particular, $Z \in C_X(Q)$ for every Q. So $Z \in \mathcal{J}$ and equation (8) is established.

LEMMA 5.4. *If $A \in \mathcal{I}$ and $B \in C(P)$ for some $A \subseteq B \subseteq S$ and profile P, then $B \in \mathcal{I}$.*

Proof. If B is maximal in $C(P)$ then, by Theorem 3.1, $B \in \mathcal{I}$. Otherwise, there exists $X \in \mathcal{I}$ with the properties: B is a proper subset of X; $X \in \mathcal{I}$; if $B \subset Y$ and $Y \in \mathcal{I}$ then $X \subseteq Y$. Note that we may assume that $|B| \geq 3$ and $|X| \geq 4$.

Consider $C_X : \mathcal{T}(X)^k \to \mathcal{T}(X)$. For the profile P in the hypothesis of this lemma we have $B \in C_X(\beta(P))$. Let Z be a maximal nontrivial cluster in $C_X(\beta(P))$ such that $B \subseteq Z$. Since $A \in \mathcal{I} \cap [X)$, it follows from (8) that $A \in \mathcal{J}$. Thus $C_X(L_\phi) \neq L_\phi$. It follows from Theorem 3.1, applied to C_X, that $Z \in \mathcal{J}$. Therefore, by (8), $Z \in \mathcal{I}$. Since $B \subseteq Z \subset X$ and $Z \in \mathcal{I}$ it follows from our choice of X that $Z = B$. Hence $B \in \mathcal{I}$. □

LEMMA 5.5. *Let $X \in \mathcal{I}$ such that $|X| \geq 4$. Then X is minimal in $\mathcal{I}$ if and only if $C_X(Q_\phi) = L_\phi$.*

Proof. (⇒) Let $\mathcal{J} = \{Y | 1 < |Y| < |X| \text{ and } Y \in C_X(Q) \text{ for every } Q\}$. Since X is minimal in $\mathcal{I}$ it follows from (8) that $\mathcal{J} = \emptyset$. It follows from Theorem 3.1 that $C_X(L_\phi) = L_\phi$.

(⇐) If X is not minimal in $\mathcal{I}$ then there exists $Y \in \mathcal{I} \cap [X)$. It follows from (8) that $Y \in \mathcal{J}$. Thus $Y \in C_X(L_\phi)$. Hence $C_X(L_\phi) \neq L_\phi$. □

We are now ready to complete the proof of Theorem 2.3.

PROOF OF THEOREM 2.3. Let $C : \mathcal{T}^k \to \mathcal{T}$ be removal independent. If $\mathcal{I} = \emptyset$, then $C(T_\phi) = T_\phi$. By Theorem 4.1, C is either trivial or a projection. Thus, either $C = C_0$ or $C = C_0 \cup C_1$ where C_0 is the trivial constant function and C_1 is a projection.

If $\mathcal{I} \neq \emptyset$, then let $X_1, ..., X_t$ be the minimal elements of $\mathcal{I}$ such that $|X_i| \geq 3$ for $i = 1, ..., t$. Next, for $i = 1, ..., t$, define $C_i : \mathcal{T}^k \to \mathcal{T}$ by $C_i(P) = C(P)|_{X_i}$ for every P. In addition, define $C_0 : \mathcal{T}^k \to \mathcal{T}$ by

$C_0(P) = T_\phi \cup \mathcal{I}$ for every P. It is straightforward to verify that

$$C_0(P) \cup C_1(P) \cup ... \cup C_t(P) \subseteq C(P)$$

for every P.

To get the reverse inclusion, let P' be a profile and let $Y \in C(P')$. If $Y \in \mathcal{I}$, then $Y \in C_0(P')$. Otherwise, there exists $X \in \mathcal{I}$ with the properties: Y is a proper subset of X; if $Y \subset Z$ and $Z \in \mathcal{I}$, then $X \subseteq Z$. We consider two possibilities for X.

If $|X| = 3$, then, since $Y \subset X$ and $Y \notin \mathcal{I}$, X is minimal in $\mathcal{I}$. So $X = X_i$ for some $i \in \{1, ..., t\}$. It follows that $Y = Y \cap X_i \in C(P'))|_{X_i} = C_i(P')$.

Now consider the case where $|X| \geq 4$. Assume X is not minimal in $\mathcal{I}$. Then, by Lemma 5.5, $C_X(Q_\phi) \neq L_\phi$. Note that $Y \in C_X(\beta(P'))$. Let Z be a maximal nontrivial cluster in $C_X(\beta(P'))$ such that $Y \subseteq Z$. It follows from Theorem 3.1 that $Z \in \mathcal{J}$. Therefore, by (8), $Z \in \mathcal{I}$. It follows from our choice of X that $Y = Z$ contrary to $Y \notin \mathcal{I}$. Therefore, X is minimal in $\mathcal{I}$. So $X = X_i$ for some $i \in \{1, ..., t\}$ and hence $Y = Y \cap X_i \in C_i(P')$.

At this stage we have

$$C_0(P) \cup C_1(P) \cup ... \cup C_t(P) = C(P)$$

for every P.

To complete the proof we need to show that each C_i is either a constant function, a local projection, or a near constant for $i = 0, ..., t$. First, note that C_0 is a constant function. Next, it follows from Lemma 2.1 that C_i is removal independent for $i = 1, ..., t$. In the case where $|X_i| = 3$, if C_i is neither a constant nor a projection, then, by definition, C_i is a near constant. Finally, suppose $|X_i| \geq 4$. It follows from Lemmas 5.1 and 5.5 that $C(P_\phi)|_{X_i} = T_{X_i}$. It follows from Lemma 5.3 that either $C(P)|_{X_i} = T_{X_i}$ for every P or there exists $j \in K$ such that $C(P)|_{X_i} = T_j|_{X_i} \cup \{X_i\}$ for every P. Therefore, in this case, C_i is either a constant function or a local projection. This completes the proof of the structure theorem. □

Finally, there is hope that results like Wilson's Partition Lemma, Theorem 2.3 of this paper, and Theorem 5 in [8], will be obtained for other discrete structures and versions of independence of irrelevant alternatives.

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Department of Mathematics
University of Louisville
Louisville, KY 40292
USA
E-mail: rcpowe01@athena.louisville.edu