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A CENTRAL LIMIT THEOREM FOR THE STATIONARY LINEAR PROCESS GENERATED BY AN ASSOCIATED PROCESS

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Abstract. A central limit theorem is obtained for stationary linear process $X_t = \sum_{j=0}^{\infty} a_j \epsilon_{t-j}$, where $\{\epsilon_t\}$ is a strictly stationary associated sequence with $E\epsilon_t = 0$, $E\epsilon_t^2 < \infty$ and $\{a_j\}$ is a sequence of real numbers with $\sum_{j=0}^{\infty} |a_j| < \infty$. A functional central limit theorem is also derived.

1. Introduction

In the last years there has been growing interest in concepts of positive dependence for families of random variables. Such concepts are of considerable use in deriving inequalities in probability and statistics. Lehmann (1966) introduced a simple and natural definition of positive dependence: Two random variables X and Y are said to be positive quadrant dependent (PQD) if for any real x, y

$$P(X > x,\ Y > y) \geq P(X > x)P(Y > y).$$

A much stronger concept than positive quadrant dependence was considered by Esary, Proschan and Walkup (1967).
A collection of random variables $\{\epsilon_1, \cdots, \epsilon_m\}$ is said to be associated if for any two coordinatewise nondecreasing functions f_1, f_2 on $\mathbb{R}^m$ such that $\tilde{f}_j = f_j(\epsilon_1, \cdots, \epsilon_m)$ has finite variance for $j = 1, 2$,

$$cov(\tilde{f}_1, \tilde{f}_2) \geq 0.$$

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An infinite collection $\{\epsilon_t,\ t \in \mathbb{Z}\}$, $\mathbb{Z} = \{0,\ \pm 1,\ \pm 2,\ \cdots\}$, is said to be associated if every finite subcollection is associated.
A large amount of papers has been concerned with limit theorems for associated processes (see, for example, Newman (1984)).
Newman (1980) established a central limit theorem for strictly stationary associated process $\{\epsilon_t\}$ with $E(\epsilon_t) = 0$ and $E(\epsilon_t^2) < \infty$; more precisely, if

$$0 < \sigma^2 = E\epsilon_1^2 + 2\sum_{t=2}^{\infty} E(\epsilon_1\ \epsilon_t) < \infty \tag{1}$$

then

$$n^{-\frac{1}{2}}(\epsilon_1 + \cdots + \epsilon_n) \xrightarrow{\mathcal{D}} N(0,\ \sigma^2). \tag{2}$$

Newman and Wright (1981) also improved this central limit theorem, that is they derived a functional central limit theorem: Let $\{\epsilon_t,\ t \in \mathbb{Z}\}$ be strictly stationary associated process with $E(\epsilon_t) = 0$ and $E(\epsilon_t^2) < \infty$. If (1) holds, then

$$W_n(t) = n^{-\frac{1}{2}}\sigma^{-1}\sum_{i=1}^{[nt]} \epsilon_i,\ 0 \le t \le 1$$

converges weakly to the Wiener measure W on $D[0,\ 1]$ the space of all functions on $[0,\ 1]$ which have left limits and are continuous from the right.
Let $\{X_t,\ t \in \mathbb{Z}_+\}$, $\mathbb{Z}_+ = \{1, 2, \cdots\}$, be a stationary linear process defined on a probability space $(\Omega, \mathcal{F}, P)$ of the form

$$X_t = \sum_{j=0}^{\infty} a_j \epsilon_{t-j}, \tag{3}$$

where $\{a_j\}$ is the sequence of coefficients satisfying a certain summability condition, that is, $\{a_j\}$ is a sequence of real numbers with $\sum_{j=0}^{\infty} |a_j| < \infty$, and $\{\epsilon_t\}$ is a strictly stationary sequence of associated random variables such that

$$E(\epsilon_t) = 0,\ E(\epsilon_t^2) < \infty. \tag{4}$$

Here we suppose that the coefficients a_j are geometrically bounded, i.e. there exist constants $A > 0$ and $0 < \rho < 1$ such that

$$|a_j| \le A\rho^j \text{ for all } j \ge 0.$$

This class of processes is a basic object in time series analysis.
A vast amount of literature has been devoted to the study of linear processes under various circumstances; Fakhre-Zakeri and Lee (1992) and Fakhre-Zakeri and Farshidi (1993) established central limit theorem under iid assumption on $\{\epsilon_t\}$ and Lee (1997) studied central limit theorem under strong mixing condition on $\{\epsilon_t\}$.

In this paper we derive a central limit theorem for a stationary linear process of the form (3), which is generated by a strictly stationary associated sequence $\{\epsilon_t\}$ with $E\epsilon_t = 0,\ E(\epsilon_t^2) < \infty$. And we also consider the functional central limit theorem.

2. Main results

We start this section by introducing the maximal inequality for a sequence $\{\epsilon_t\}$ of associated random variables.

LEMMA 2.1. *Suppose that $\{\epsilon_t,\ t \in \mathbb{Z}_+\}$ is a sequence of strictly stationary associated random variables with $E\epsilon_t = 0$ and $E\epsilon_t^2 < \infty$ and satisfying* (1). *Then for any $\epsilon > 0$*

$$P\Big(\max_{1\le k\le n} |\epsilon_1 + \cdots + \epsilon_k| > \epsilon\Big) \le \frac{n\sigma^2}{\epsilon^2} \tag{5}$$

where σ^2 is defined as in (1).

Proof. This lemma is a direct result of Newman and Wright (see, Theorem 2 of [8]);

$$\begin{aligned}
&P\Big(\max_{1\le k\le n} |\epsilon_1 + \cdots + \epsilon_k| > \epsilon\Big)\\
&\le E\big(\max_{1\le k\le n} |\epsilon_1 + \cdots + \epsilon_k|\big)^2/\epsilon^2\\
&\quad \text{by Chebyshev's inequality}\\
&\le E\big(\epsilon_1 + \cdots + \epsilon_n\big)^2/\epsilon^2\\
&\quad \text{by Theorem 2 of Newman and Wright (1981)}\\
&\le n\sigma^2/\epsilon^2 \ \text{ by (1).}
\end{aligned}$$

□

THEOREM 2.2. *Let $\{\epsilon_t,\ t \in \mathbb{Z}\}$ be a sequence of strictly stationary associated random variables defined on a probability space $(\Omega, \mathcal{F}, P)$*

with $E\epsilon_t = 0$, $E\epsilon_t^2 < \infty$ *and satisfying* (1).
Let us consider a stationary linear process of the form

$$X_t = \sum_{j=0}^{\infty} a_j \epsilon_{t-j}, \ t \in \mathbb{Z}^+, \tag{6}$$

where $\sum_{j=0}^{\infty} |a_j| < \infty$.
Then $\{X_t\}$ *fulfills the central limit theorem, that is,*

$$n^{-\frac{1}{2}} \sum_{t=1}^{n} X_t \xrightarrow{\mathcal{D}} N\left(0, \ \sigma^2 \left(\sum_{j=0}^{\infty} a_j\right)^2\right) \tag{7}$$

as $n \to \infty$.

Proof. Letting

$$\tilde{a}_j = \sum_{i=j+1}^{\infty} a_i$$

and

$$\tilde{\epsilon}_t = \sum_{j=0}^{\infty} \tilde{a}_j \epsilon_{t-j},$$

which is well-defined since $\sum_{j=0}^{\infty} |\tilde{a}_j| < \infty$, we have

$$\begin{aligned} X_t &= \sum_{j=0}^{\infty} a_j \epsilon_{t-j} \\ &= a_0 \epsilon_t + \sum_{j=1}^{\infty} a_j \epsilon_{t-j} \\ &= \left(\sum_{j=0}^{\infty} a_j\right) \epsilon_t - \tilde{a}_0 \epsilon_t + \sum_{j=1}^{\infty} (\tilde{a}_{j-1} - \tilde{a}_j) \epsilon_{t-j} \\ &= \left(\sum_{j=0}^{\infty} a_j\right) \epsilon_t + \tilde{\epsilon}_{t-1} - \tilde{\epsilon}_t, \end{aligned}$$

which implies that

$$\sum_{t=1}^{n} X_t = \left(\sum_{j=0}^{\infty} a_j\right)\left(\sum_{t=1}^{n} \epsilon_t\right) + \tilde{\epsilon}_0 - \tilde{\epsilon}_n. \tag{8}$$

Since $(\sum_{j=0}^{\infty} a_j)\epsilon_t$ *'s* are associated by Newman's central limit theorem we have

$$\left(\sum_{j=0}^{\infty} a_j\right)\left(n^{-\frac{1}{2}}\sum_{t=1}^{n}\epsilon_t\right) \to N\left(0,\ \left(\sum_{j=0}^{\infty} a_j\right)^2 \sigma^2\right). \tag{9}$$

Note that $(\sum_{j=0}^{\infty} a_j)^2$ is finite by the assumption $\sum_{j=0}^{\infty} |a_j| < \infty$.
Thus if

$$\tilde{\epsilon}_0/\sqrt{n} \xrightarrow{P} 0 \text{ as } n \to \infty$$

and

$$\tilde{\epsilon}_n/\sqrt{n} \xrightarrow{P} 0 \text{ as } n \to \infty, \tag{10}$$

then

$$\sum_{t=1}^{n} X_t \xrightarrow{\mathcal{D}} N\left(0,\ \left(\sum_{j=0}^{\infty} a_j\right)^2 \sigma^2\right) \tag{11}$$

will follow.
To prove (10), it suffices to prove that

$$\tilde{\epsilon}_0/\sqrt{n} \to 0 \text{ a.s. as } n \to \infty \tag{12}$$

and

$$\tilde{\epsilon}_n/\sqrt{n} \longrightarrow 0 \text{ a.s. as } n \to \infty. \tag{13}$$

But (12) and (13) follow from $E(\tilde{\epsilon}_n)^2 < \infty$ (see Remark) and the fact that for any $\delta > 0$

$$\sum_{n=1}^{\infty} P(|\tilde{\epsilon}_n|/\sqrt{n} > \delta) = \sum_{n=1}^{\infty} P(|\tilde{\epsilon}_0| > \sqrt{n}\delta) < \infty,$$

because for an arbitrary random variable Z

$$EZ^2 < \infty \Leftrightarrow \sum_{n=1}^{\infty} P(|Z| > \sqrt{n}\delta) < \infty.$$

REMARK.

$$\begin{aligned}
E(\tilde{\epsilon}_t)^2 &= E\Big(\sum_{j=0}^{\infty}\tilde{a}_j\epsilon_{t-j}\Big)^2 \\
&= \sum_{j=0}^{\infty}\tilde{a}_j^2 E\epsilon_{t-j}^2 + \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}{}_{k\neq l}\ \tilde{a}_k\tilde{a}_l E(\epsilon_{t-k}\epsilon_{t-l}) \\
&\leq \Big(\sum_{j=0}^{\infty}\tilde{a}_j^2\Big)E\epsilon_t^2 + \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}{}_{(k\neq l)}\ |\tilde{a}_k||\tilde{a}_l|E(\epsilon_{t-k}\epsilon_{t-l}) \\
&\leq \Big(\sum_{j=0}^{\infty}\tilde{a}_j^2\Big)E\epsilon_1^2 + 2\sum\sum\nolimits_{k<l}|\tilde{a}_k||\tilde{a}_l|(\sum_{t=2}^{\infty}E(\epsilon_1\epsilon_t)) \\
&\leq \Big(\sum_{j=0}^{\infty}\tilde{a}_j^2\Big)\Big(E\epsilon_1^2 + 2\sum_{t=2}^{\infty}E\epsilon_1\epsilon_t\Big) < \infty \text{ by (1).}
\end{aligned}$$

Let $\tau^2 = \sigma^2(\sum_{j=0}^{\infty})^2$. Define, for $n \geq 1$, the stochastic process

$$\xi_n(u) = n^{-\frac{1}{2}}\tau^{-1}\sum_{t=1}^{[nu]}X_t,\ 0 \leq u \leq 1, \tag{14}$$

where $[x]$ is the integer part of x.

If the process $\{\xi_n,\ n \in \mathbb{Z}_+\}$ converges weakly to Wiener measure on $D[0,1]$, the space of all functions on $[0,1]$ which have left limits and are continuous from the right, then we say that the sequence $\{\xi_n,\ n \in \mathbb{Z}_+\}$ fulfills the functional central limit theorem.

Finally, we close this section by deriving that the linear process $\{X_t\}$ generated by the associated sequence $\{\epsilon_t\}$ fulfills the functional central limit theorem.

THEOREM 2.2. *Let $\{\epsilon_t,\ t \in \mathbb{Z}\}$ be a sequence of strictly stationary associated random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ with $E\epsilon_t = 0$ and $E\epsilon_t^2 < \infty$ and satisfying (1). Let X_t be defined as in (6). Then the stochastic process $\{\xi_n(u),\ n \in \mathbb{Z}_+\}$ fulfills the functional central limit theorem, that is, $\{\xi_n(u),\ 0 \leq u \leq 1\}$ converges weakly to the Wiener measure W.*

Proof. As in Theorem 2.1 let

$$\tilde{a}_j = \sum_{i=j+1}^{\infty} a_i$$

and

$$\tilde{\epsilon}_t = \sum_{j=0}^{\infty} \tilde{a}_j \epsilon_{t-j}$$

with

$$\sum_{j=0}^{\infty} |a_j| < \infty.$$

Then we have

$$X_t = \left(\sum_{j=1}^{\infty} a_j\right)\epsilon_t + \tilde{\epsilon}_{t-1} - \tilde{\epsilon}_t \tag{15}$$

and

$$\sum_{t=1}^{n} X_t = \left(\sum_{j=0}^{\infty} a_j\right)\left(\sum_{t=1}^{n} \epsilon_t\right) + \tilde{\epsilon}_0 - \tilde{\epsilon}_n.$$

Thus

$$\begin{aligned} \xi_n(u) &= n^{-\frac{1}{2}}\tau^{-1} \sum_{t=1}^{[nu]} X_t \\ &= n^{-\frac{1}{2}}\tau^{-1} \left(\sum_{j=0}^{\infty} a_j\right) \sum_{t=1}^{[nu]} \epsilon_t \\ &\quad + n^{-\frac{1}{2}}\tau^{-1}\tilde{\epsilon}_0 - n^{-\frac{1}{2}}\tau^{-1}\tilde{\epsilon}_{[nu]}. \end{aligned} \tag{16}$$

First note that $(\sum_{j=0}^{\infty} a_j)\epsilon_t$ *'s* are associated since ϵ_t *'s* are associated. According to (12) and (13) the second term and the third term in the right-hand side of (16) converge in probability to zero, i.e.,

$$n^{-\frac{1}{2}}\tau^{-1}\tilde{\epsilon}_{[nu]} \xrightarrow{P} 0 \text{ and } n^{-\frac{1}{2}}\tau^{-1}\tilde{\epsilon}_0 \xrightarrow{P} 0.$$

Hence by Theorem 4.2 of Billingsley (1968) it remains to prove

$$n^{-\frac{1}{2}}\tau^{-1}\left(\sum_{j=0}^{\infty} a_j\right)\sum_{t=1}^{[nu]} \epsilon_t \xrightarrow{\mathcal{D}} W, \tag{17}$$

where W is a Wiener measure on $D[0,1]$ the space of all functions on $[0,1]$ which have left limits and are continuous from the right.
Since ϵ_t 's are associated it is well known that (18) holds according to Newman and Wrights' functional central limit theorem (see Theorem 3 of [8]). Hence the proof is complete. □

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