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THE NUMBER OF LINEAR SYSTEMS COMPUTING THE GONALITY

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ABSTRACT. Let C be a smooth k-gonal curve of genus g. We study the number of pencils of degree k on C. In case $g \geq k(k-1)/2$ we state a conjecture based on a discussion on plane models for C. From previous work it is known that if C possesses a large number of pencils then C has a special plane model. From this observation the conjectures are split up in two cases: the existence of some types of plane curves should imply the existence of curves C with a given number of pencils; the non-existence of plane curves should imply the non-existence of curves C with some given large number of pencils. The non-existence part only occurs in the range $k(k-1)/2 \leq g \leq k(k-1)/2 + [(k-2)/2]$ if $k \geq 7$. In this range we prove the existence part of the conjecture and we also prove some non-existence statements. Those result imply the conjecture in that range for $k \leq 10$. The cases $k \leq 6$ are handled separately.

1. Introduction

(1.1). In this paper C always denotes a smooth connected complete curve of genus g over the field $\mathbf{C}$ of the complex numbers. Remember the definition of the gonality of C: there exists a linear system g^1_k but no linear system g^1_{k-1} on C. Equivalently, k is the minimal degree of a covering $C \to \mathbf{P}^1$. From Brill-Noether Theory it follows that $k \leq (g+3)/2$. In this paper we investigate the possible number of linear systems g^1_k on a k-gonal curve.

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In case $k = (g+3)/2$ (with g being odd) then C has infinitely many g^1_k and in case $k = (g+2)/2$ (with g being even) then a general curve of genus g has exactly $\frac{g!}{(g-k+2)!(g-k+1)!}$ linear systems g^1_k. Both claims follow from Brill-Noether Theory. In case $k = (g+2)/2$ the curve C can have less than $\frac{g!}{(g-k+2)!(g-k+1)!}$ linear systems g^1_k. In that case some of them are limits of at least two different g^1_k in a family of curves, so they should be counted with some multiplicity. In this way, if $g = 2k-2$ and C has finitely many linear systems g^1_k then the number of g^1_k on C, counted with suited multiplicities, always is equal to $\frac{g!}{(g-k+2)!(g-k+1)!}$. Further on we only consider linear systems that have to be counted with multiplicity 1.

Assume a smooth curve C has two different base point free linear systems g^1_k: call them g_1 and g_2. Take a general element $F_i \in g_i$. Then $F_1 + g_2$ and $F_2 + g_1$ are lines in $|g_1 + g_2|$ intersecting at $F_1 + F_2$. They span a linear system g^2_{2k}. This linear system has no base points and defines a morphism $\varphi : C \to \mathbf{P}^2$. Let Γ be the image. If $\varphi : C \to \Gamma$ is not a birational equivalence then there exists a smooth curve C' (the normalisation of Γ), a non-trivial morphism $f : C \to C'$ of some degree $a \geq 2$ and two linear systems $g^1_{k'}$ on C' (call them g'_1 and g'_2) with $f^*(g'_i) = g_i$ (hence $k = a.k'$). In particular we find no restriction on g in this case. More general, starting with a curve of some gonality k' having many linear systems $g^1_{k'}$, and using coverings of some degree $a \geq 2$ we find $k = ak'$-gonal curves of arbitrary large genus having many linear systems g^1_k. Further on we are going to exclude such possibility. If $\varphi : C \to \Gamma$ is a birational morphism then C is birational equivalent to a plane curve of degree $2k$ having at least 2 singular points of multiplicity k (namely the image on Γ of the points in F_1 and in F_2). This implies $g \leq (k-1)^2$. So in this case we have a restriction on g.

(1.2) To make the second restriction more precise we introduce the following definition.

Definition. Two base point free linear systems g^1_k on C (call them g_1 and g_2) are called dependent if there exists a non-trivial morphism $f : C \to C'$ of some degree $a \geq 2$ and two linear systems $g^1_{k'}$ on C' (call them g'_1 and g'_2) such that $f^*(g_i') = g_i$. Otherwise they are called independent. If $g_1, \cdots, g_m$ are m different base point free linear systems g_k^1 on C then they are called mutually independent if for each $1 \leq i < j \leq m$ the linear systems g_i and g_j are independent.

(1.3) Now we make the first restriction more precise. A base point free linear system g_k^1 on C (call it g) is said to be the limit of two different linear systems g_k^1 in a family of curves if there exists a 1-parameter family of curves $\pi : C \to \Delta(\Delta = \{z \in \mathbf{C} : |z| < 1\}$ with $\pi^{-1}(0) = C$ and two families G_1 and G_2 of linear systems g_k^1 on this family (see [2] for the definition) such that $(G_1)_0 = (G_2)_0 = g$ on $C = \pi^{-1}(0)$ and $(G_1)_t \neq (G_2)_t$ on $C_t = \pi^{-1}(t)$ for $t \neq 0$. In this case we should count g with some multiplicity. In [7] it is proved that g is the limit of two different g_k^1 in a family of curves if and only if $\dim(|2g|) \geq 3$.

DEFINITION. We say that g (a base point free g_k^1 on C) is of type I if $\dim(|2g|) = 2$ (i.e., g is not the limit of two different g_k^1 in a family of curves, hence g is counted with multiplicity 1).

REMARK: The linear system g corresponds to some point x in $W_k^1 \subset J(C)$. Then g is of type I if and only if x is an isolated point of W_k^1 and as a scheme W_k^1 is reduced at x.

(1.4) Let M_g be the coarse moduli space of smooth curves of genus g. Let $M_{g,k}$ be the k-gonal locus.

DEFINITION. Let $m \in \mathbf{Z}_{\geq 1}$. Then $M_{g,k}(m)$ is the set of k-gonal curves C having exactly m linear systems g_k^1. Moreover those linear systems g_k^1 are mutually independent and each one is of type I.

The precise question considered in this paper is the following: given g; k (with $g \geq 2k-2$) determine all values $m \in \mathbf{Z}_{\geq 1}$ such that $M_{g,k}(m)$ is not empty.

(1.5) As already noted, in case $g = 2k-2$, then $M_{g,k}(m)$ is not empty if and only if $m = \frac{g!}{(g-k+2)!(g-k+1)!}$. So, from now on we assume $g \geq 2k-1$.

A general element C of $M_{g,k}$ has a unique g_k^1 and it is of type I, hence $M_{g,k}(1)$ is not empty for $g \geq 2k-1$. This follows from [3] and [18]. Also $M_{g,k}(2)$ is not empty if and only if $2k-1 \leq g \leq (k-1)^2$ and $(g;k) \neq (7;4)$. This is proved in [12].

In Section 2, using plane curves, we obtain elements in $M_{g,k}(m)$ for some values of $g; k$ and m. In case $g \geq k(k-1)/2$ there is a good upper bound on m for the condition $M_{g,k}(m) \neq \emptyset$. This bound is due to Accola. In section 3, using this bound and the examples from Section 2, we state a conjecture concerning the non-emptiness of $M_{g,k}(m)$ in the range $g \geq k(k-1)/2$. Finally in Section 4 we consider gonality $k \leq 10$, giving some evidence for the conjectures. Some related results are contained in my paper ([13]).

(1.6) PROBLEM. Given a smooth k-gonal curve C and m' mutually independent linear systems g^1_k (call them $g_1; \cdots ; g_{m'}$). Is it true that either at least one of $g_1; \cdots ; g_{m'}$ moves in a one dimensional family of mutually independent linear systems g^1_k on C (hence C has infinitely many linear systems g^1_k) or C is the limit of a family of curves C_t in $M_{g,k}(m)$ for some $m \geq m'$ such that each g_i $(1 \leq i \leq m')$ is a limit of a g^1_k on C_t.

If the answer to this problem is yes, then the restriction to curves in $M_{g,k}(m)$ is harmless. Up to now examples indicate that the answer should be yes. In particular this is the case for $g = 2k-2$; the discussion on 4-gonal curves in [6] shows it is true for 4-gonal curves; the discussion on 5-gonal curves in [13] shows it is true in case $g \geq 10$, from the discussion of 5-gonal curves of genus 9 in a forthcoming paper it follows that the answer is yes if $k = 5$. The author likes to thank Young Rock Kim and the referee for their suggestions for improvements.

2. Examples

(2.1) DEFINITION. Let Γ be an integral plane curve and $s \in \Gamma$. We say that Γ has multiplicity m at s of simple type if all infinitesimally near points of s on Γ are smooth on Γ. This is equivalent to the following statement. If $\pi : X \to \mathbf{P}^2$ is the blowing-up of $\mathbf{P}^2$ at s with exceptional divisor E and if $\tilde{\Gamma}$ is the proper transform of Γ then each point of $E \cap \tilde{\Gamma}$ is a smooth point of $\tilde{\Gamma}$.

(2.2) LEMMA. *Let Γ be an integral plane curve of degree d and let s be a point of multiplicity m of simple type on Γ. Let C be the normalization of Γ. Let $\mathbf{P}$ be a pencil of plane curves of degree e having no fixed component and no element of $\mathbf{P}$ contains Γ. Let $\Gamma_1; \Gamma_2$ be two* *general elements of* $\mathbf{P}$. *Let* g^1_{de} *be the linear system on* C *induced by* $\mathbf{P}$ *and let* F *be the subdivisor of the fixed divisor of* g^1_{de} *supported on the inverse image of* $s \in \Gamma$ *on* C. *Then* $i(\Gamma_1, \Gamma_2; s) \geq \deg(F) - [m^2/4]$ *(as usual* [ ] *means integral part;* $i(\Gamma_1, \Gamma_2; s)$ *is the intersection multiplicity of* Γ_1 *and* Γ_2 *at* s*).*

REMARK. In case $m = 1$ (s a smooth point on Γ) we obtain Namba's Lemma: $i(\Gamma_1, \Gamma_2; s) \geq \deg(F) = i(\Gamma_1, \Gamma; s)$ (for a short proof see e.g. [15], the end of the introduction). In case $m = 2$ (ordinary nodes and cusps) this is proved in [14].

Proof. Let $\pi : X \to \mathbf{P}^2$ be the blowing-up at s; let E be the exceptional divisor and let $\tilde{\Gamma}; \tilde{\Gamma}_1; \tilde{\Gamma}_2$ be the proper transforms of resp. $\Gamma; \Gamma_1; \Gamma_2$. Let (as a set) $\tilde{\Gamma} \cap E = \{s_1; \cdots; s_x\}$ ($x \leq m$); they are smooth points on $\tilde{\Gamma}$. Let $n_i = i(\tilde{\Gamma}_1, \tilde{\Gamma}; s_i)$ for $1 \leq i \leq x$, then Namba's Lemma implies $i(\tilde{\Gamma}_1, \tilde{\Gamma}_2; s_i) \geq n_i$, hence $i(\Gamma_1, \Gamma_2; s) \geq [\text{mult}_s(\Gamma_1)]^2 + (n_1 + \cdots + n_x)$. On the other hand $\deg(F) = m.\text{mult}_s(\Gamma_1) + (n_1 + \cdots + n_x)$, hence $i(\Gamma_1, \Gamma_2; s) \geq [\text{mult}_s(\Gamma_1)]^2 + \deg(F) - m.\text{mult}_s(\Gamma_1) \geq \deg(F) - [m^2/4]$.□

(2.3) PROPOSITION. *Let* Γ *be an integral plane curve of degree* d*; let* $m \in \mathbf{Z}_{>0}; u \in \mathbf{Z}_{\geq 0}$ *and* $\mu \in \mathbf{Z}_{\geq 3}$ *and assume* Γ *has* m *singular points of multiplicity* μ *of simple type;* u *singular points of multiplicity* 2 *of simple type and no other singularities. Let* C *be the normalisation of* Γ. *If* $(d-2)^2 \geq 2m\mu(\mu-1) + 4(u-\mu+1)$ *and* $d > m[\mu^2/4] + u + 4 - \mu$ *then* C *is* $(d-\mu)$*-gonal and each* $g^1_{d-\mu}$ *is induced by a pencil of lines through a point of multiplicity* μ *of* Γ.

Proof. Let g^1_k be a base point free linear system on C for some $k \leq d - \mu$. On Γ we obtain a so-called generalized linear system $g^1_{k+\delta}$ (here $\delta = m\frac{\mu(\mu-1)}{2} + u$) (see [11], Lemma 1.4), hence a free g^1_n for some $n \leq k + \delta$. If $n \leq f(d-f)$ then there exists a pencil $\mathbf{P}$ of plane curves of degree at most $f - 1$ without fixed components inducing g^1_k on C (see [11], Theorem 3.2.1; see also [4]). So, if $d - \mu + m\frac{\mu(\mu-1)}{2} + u \leq f(d-f)$ then g^1_k is induced by a pencil $\mathbf{P}$ of curves of degree at most $f - 1$ without fixed components. We can take $f \leq d/2$ because $(d-2)^2 \geq 2m\mu(\mu-1) + 4(u-\mu+1)$. So let $\mathbf{P}$ be a pencil of plane curves of some degree at most $e \leq (d-2)/2$ without fixed components inducing g^1_k on C, i.e. intersections of Γ with curves in $\mathbf{P}$ gives a linear system $g^1_{de} = g^1_k + F$ for some fixed divisor F on C. Let $\Gamma_1;\Gamma_2$ be two general elements of $\mathbf{P}$. Using the lemma we find $\deg(\Gamma_1.\Gamma_2) \geq \deg(F) - [\frac{\mu^2}{4}]m - u$ (here $\Gamma_1.\Gamma_2$ is the intersection cycle of Γ_1 and Γ_2 on $\mathbf{P}^2$). Since $\deg(F) \geq ed - d + \mu$ and $\deg(\Gamma_1.\Gamma_2) = e^2$, we find $e^2 \geq ed - d + \mu - m[\frac{\mu^2}{4}] - u$. Let $g(e) = e^2 - ed + (d - \mu + m[\frac{\mu^2}{4}] + u)$. For $2 \leq e \leq d/2$, we have $g(2) \geq g(e)$, hence if $e \geq 2$ then $g(2) = 4 - 2d + (d - \mu + m[\frac{\mu^2}{4}] + u) \geq 0$, i.e. $d \leq m[\frac{\mu^2}{4}] + u + 4 - \mu$. Since we assume $d > m[\frac{\mu^2}{4}] + u + 4 - \mu$ and we already know $e < d/2$, we find $e = 1$. From this fact the proposition follows. □

(2.4) Remark. Under certain conditions the proposition can be approved. As an example, assume $3 \leq e \leq d/2$, then we find $g(3) = 9 - 3d + (d - \mu + m[\frac{\mu^2}{4}] + u) \geq 0$. So, if $d \geq 6$ and $2d > m[\frac{\mu^2}{4}] + u + 9 - \mu$ then we find $e \leq 2$. Assume g^1_k is induced by a pencil $\mathbf{P}$ of conics. This pencil has 4 fixed points, hence it induces a linear system of degree at most $2d - 4\mu$ on C. So $d - \mu \geq 2d - 4\mu$. In case $d > 3\mu$ we again conclude that $e = 1$.

One can continue using pencils $\mathbf{P}$ of plane curves of degree at least 3, however one has to take care about the possibility of fixed singularities in the pencil. In this way it becomes more involved.

(2.5) Discussion. In case $m = 0$ (i.e. curves with u singular points of multiplicity 2 of simple type) Proposition (2.3) is already obtained in [14]. In particular, under the conditions of the proposition, the normalization C of such a curve Γ has gonality $k = d-2$ and has exactly u linear systems g^1_k. Choosing such a linear system g^1_k corresponds to choosing a node and then $\dim(|K_c - 2g^1_k|)$ is the dimension of plane curves of degree $d - 5$ containing the other nodes (use canonically adjoint curves). If the nodes are in general position (this is possible– see [19]) then we conclude $\dim(|K_c - 2g^1_k|) = \frac{(d-2)(d-5)}{2} - (u-1)$, hence $\dim(|2g^1_k|) = 2$, hence g^1_k is of type I. This proves $C \in M_{g,d-2}(u)$. Note however, fixing g and k, we obtain examples in $M_{g,k}(m)$ for at most one value of m in this way.

(2.6) In the previous discussion we did not use the complete statement of the proposition. It would be enough to find examples of plane curves Γ with u nodes such that the normalisation C of Γ is $(d-2)$-gonal and each g^1_{d-2} on C is induced by a pencil of lines.

PROBLEM. Determine the exact upper bound δ_0 of δ such that for a general integral plane nodal curve Γ of degree d with δ nodes the normalization C of Γ is $(d-2)$-gonal and each g^1_{d-2} on C is induced by a pencil of lines. With respect to the gonality (without a description of all linear systems g^1_{d-2}) see [10].

(2.7) CONTINUATION OF THE DISCUSSION. In case $m>0$ two problems arise: the existence of the curves and the linear systems g^1_k have to be of type I. Results concerning the existence of such integral plane curves Γ with singular points in general position are in [16]. In order to prove that the linear systems $g^1_{d-\mu}$ induced by a pencil of lines through a singular point P_0 of multiplicity μ is of type I, one needs to consider the dimension of the linear system of plane curves of degree $d-5$ having a point of multiplicity at least $\mu-3$ at $\mathbf{P}_0$; having multiplicity at least $\mu-1$ at the other points of multiplicity μ and containing the nodes. This is a problem about fat points in the plane in general position. Assume the dimension is the expected one, i.e. $\frac{(d-5)(d-2)}{2}-\frac{(\mu-2)(\mu-3)}{2}-(m-1)\frac{\mu(\mu-1)}{2}-u$, then $\dim(|2g^1_{d-\mu}|)=2(d-\mu)-[\frac{(d-1)(d-2)}{2}-m.\frac{\mu(\mu-1)}{2}-u]+1+\frac{(d-5)(d-2)}{2}-\frac{(\mu-2)(\mu-3)}{2}-(m-1).\frac{\mu(\mu-1)}{2}-u=2$. In case $\mu=3$, the problem on fat points is the problem of finding the dimension of plane curves of given degree having a singularity at a given number of points in general position. This problem is solved in [17]. In case $\mu\geq 4$ the problem is: given m general points $\mathbf{P}_0;\mathbf{P}_1;\cdots;\mathbf{P}_{m-1}$, find the dimension of the linear system of plane curves of given degree having multiplicity $\mu-1$ at $\mathbf{P}_1,\cdots,\mathbf{P}_{m-1}$ and having multiplicity $\mu-3$ at $\mathbf{P}_0$. This problem is considered in [5] for some cases.

The advantage of the case $m>0$ is that given g and k, once the difficulties are solved, we can find examples of curves belonging to $M_{g,k}(m)$ for different values of m. More concretely, let Γ be a curve with m points of multiplicity μ of simple type and u nodes and assume its normalization belongs to $M_{g,d-\mu}(m)$ (here $g=\frac{(d-1)(d-2)}{2}-m\frac{\mu(\mu-1)}{2}-u$). One can expect that Γ is the limit of a similar curve Γ_η with m' points of multiplicity μ of simple type and $u+(m-m')\frac{\mu(\mu-1)}{2}$ nodes (here

$m' \leq m$). The normalization of Γ_η has the same geometric genus, one expects it to belong to $M_{g,d-\mu}(m')$. Also one can expect that Γ specializes to curves Γ' with m points of multiplicity μ of simple type and $u' > u$ nodes and no other singularities. Within a certain range on u' one can expect that the normalization of Γ' belongs to $M_{g',d-\mu}(m)$ for $g' = g - (u' - u)$. Of course, one has to answer questions similar to the problem in (2.6) in order to proceed in this way.

3. Conjectures

(3.1) In his paper ([1]), Accola proves the following fact. Let C be a smooth curve of genus g possessing m mutually independent linear systems g^1_k. Define $k = s(m-1)+q$ with $-m+3 \leq q \leq 1$ ($s; q$ integers). Then $g \leq g(m;k) := [s^2(m^2-m) + (2sm+q-2)(q-1)]/2$.

(3.2) In case $m \geq k$, one finds $g(m;k) = k(k-1)/2$; the genus of a smooth plane curve of degree $k+1$. This already proves $M_{g,k}(m) = \emptyset$ for $m \geq k$ and $g > k(k-1)/2$. Let C be a smooth curve of genus $g = k(k-1)/2$. In [8] the following two statements are proved. The curve C has at least $k+1$ mutually independent base point free linear systems g^1_k if and only if C is isomorphic to a smooth plane curve of degree $k+1$; the curve C has exactly k mutually independent base point free linear systems g^1_k if and only if C is birationally equivalent to a plane curve of degree $k+2$ having exactly k singular points of multiplicity 2 of simple type. In particular for $g = k(k-1)/2$ and $m > k$ we also find $M_{g,k}(m) = \emptyset$.

(3.3) Assume $m \leq k$ and let C be a curve of genus $g(m,k)$ possessing m mutually independent base point free linear systems g^1_k. In [9] one finds a plane model Γ for C such that all g^1_k are visible on Γ. By this I mean that in most cases any g^1_k on C is obtained by intersecting Γ with a pencil of lines through a singular point of Γ; in some special case some g^1_k on C is obtained by intersecting Γ with a pencil of conics through 4 singular points of Γ. In particular one finds examples of curves C of genus $g(m;k)$ possessing m mutually independent base point free linear systems g^1_k by taking the normalization of a plane curve Γ of degree $\mu + k$ with m singular points of multiplicity μ of simple type for some suited value of μ.

(3.4) In case $2m \leq k+1$ such suited value can be taken to be at least 3. From the discussion in (2.7) this leads to the following conjecture.

CONJECTURE A. Assume $2m \leq k+1$ and $g(m+1;k) < g \leq g(m;k)$. Then $M_{g,k}(m')$ is not empty if and only if $1 \leq m' \leq m$.

This conjecture is clearly related to the problems alluded to at the end of (2.7).

(3.5) Next, let $m_0 = [(k+3)/2]$ and $m_0 \leq m \leq k$ (so we are in the case $2m > k+1$). Then $g(m;k) = \frac{k(k-1)}{2} + (k-m)$. From the plane models mentioned in (3.3) it follows that C is birationally equivalent to a plane curve Γ of degree $k+2$ having exactly m singularities of multiplicity 2 of simple type as its only singularities. From the discussion in (2.5) one finds that $M_{g(m;k),k}(m)$ is not empty. This does not indicate a statement as in Conjecture A (this is the difference between the discussions in (2.5) and (2.7)). For $m_0 \leq m \leq k$, define

$$s(m;k) = \max\left(\left\{m'' \in \mathbf{Z} : g(m;k) \leq \frac{(k+2)(k+1)}{2} - 3m''\right\}\right)$$

PROPOSITION 3.6. *Let $k \geq 7$ and let $1 \leq m' \leq s(m;k)$ Then $M_{g(m;k),k}(m')$ is not empty.*

Proof. Assume Γ is plane curve of degree $k+3$ having m' triple points of simple type and u double points of simple type as their only singularities. Let C be the normalization of Γ and assume C has genus $g(m;k)$. Since $g(m;k) \geq \frac{k(k-1)}{2}$ we find $3m'+u \leq \frac{(k+2)(k+1)}{2} - \frac{k(k-1)}{2} = 2k+1$. Since $k^2-6k+5 \geq 0$ we find $(k+1)^2 \geq 4(2k+1)-8$ and so $(k+1)^2 \geq 4(3m'+u)-8$. This is the first condition in Proposition (2.3).

Because of [12] we can assume $m' \geq 3$. Then $3m'+u \leq 2k+1$ implies the inequality $2m'+u < 2k$. Hence $2(k+3) > 2m'+u+9-3$. Since $k \geq 7$ we find that the conditions mentioned in Remark (2.4) hold. So C is k-gonal and each g^1_k is induced by a pencil of lines through a triple point. Next we use Theorem 5 in [16]. It implies the existence of such curve Γ having its singularities at $m'+u$ general points on $\mathbf{P}^2$ if $\frac{(k+3)^2+6(k+3)-1}{4} - [\frac{k+3}{2}] > 6m'+3u$. But $6m'+3u = 3(3m'+u)-3m' \leq 3(2k+1)-9$ (because $3m'+u \leq 2k+1$ and $m' \geq 3$).

So the inequality holds if $6k-6<\frac{(k+3)^2+6(k+3)-1}{4}-[\frac{k+3}{2}]$. Of course, this inequality holds. Then using the results of [17], the proposition follows as explained in (2.7). □

REMARK. We only used $m_0 \leq m$ to obtain $3m'+u \leq 2k+1$. This proposition can be extended to more genus cases. Then we enter in the range $2m \leq k+1$ and the existence result is not optimal.

REMARK 3.7. An easy computation shows that it is not possible to find plane curves Γ of degree $k+\mu$ for some $\mu \geq 4$ with $m' > s(m,k)$ singular points of multiplicity μ such that the normalization C has genus $g(m;k)$ in case $2m > k+1$. This suggests the following.

CONJECTURE B. If $k \geq 7$; $2m > k+1$; $m \leq k$ and $s(m;k) < m' < m$ then $M_{g(m;k),k}(m')$ is empty.

4. Low gonality

(4.1) In case $k \in \{2,3\}$ it is well know that, for $g \geq 2k-1$, one has $M_{g,k}(m)$ is not empty if and only if $m=1$.

(4.2) The case of gonality $k=4$ is intensively studied in [6]. One finds for $g \geq 7$ that $M_{g,4}(m)$ is not empty if and only if one of the following conditions hold: $m=1$ or $m=2$ and $g \in \{8,9\}$ or $m=3$ and $g=7$. There exist 4-gonal curves of genus 7 having exactly 2 linear systems g^1_4, however one of them is not of type I and actually such curve is the limit of 4-gonal curves belonging to $M_{7,4}(3)$.

(4.3) CURVES OF GONALITY 5. From results mentioned in (1.5) and from Accola's bound (see (3.1)) it follows that, if $g \geq 17$, then $M_{g,5}(m)$ is not empty if and only if $m=1$ and if $13 \leq g \leq 16$ then $M_{g,5}(m)$ is not empty if and only if $m \in \{1,2\}$. Next, $M_{12,5}(m)$ is not empty if and only if $m \in \{1,2,3\}$. Here the value $m=3$ is obtained using plane curves of degree 8 with 3 triple points. In [13] it is proved that $M_{11,5}(m)$ is not empty if and only if $m \in \{1,2,4\}$. The fact that it is not possible to find elements in $M_{11,5}(3)$ can be explained as follows. If Γ is a plane curve of degree 8 with 3 triple points and 1 double point of simple type then the normalization C of Γ has genus 11. Also the pencil of conics through the four singular points induces a g^1_5 on C. In [13] it is also proved that $M_{10,5}(m)$ is not empty if and only if

$m \in \{1,2,5\}$. It is possible to give a short proof of this claim using the methods used to study gonality 6 further on, we leave it to the reader to make this short proof. In a forthcoming paper it will be proved that $M_{9,5}(m)$ is not empty if and only if $m \in \{1,2,3,6\}$. This is beyond the restriction $g \geq k(k-1)/2$ used in the conjectures in Section 3.

(4.4) CURVES OF GONALITY 6. If $g \geq 26$, then $M_{g,6}(m)$ is not empty if and only if $m = 1$; if $20 \leq g \leq 25$ then $M_{g,k}(m)$ is not empty if and only if $m \in \{1,2\}$; if $g \in \{18,19\}$ then $M_{g,6}(m)$ is not empty if and only if $m \in \{1,2,3\}$. In this last statement examples with $m = 3$ can be found using plane curves of degree 9 with 3 triple points and also 1 double point for the case $g = 18$. $M_{17,6}(m)$ is not empty if and only if $m \in \{1,2,3,4\}$. The value $m = 3$ can be obtained using plane curves of degree 9 with 3 triple points and 2 double points, the value $m = 4$ can be obtained using plane curves of degree 8 having 1 double points.

(4.4.1) CLAIM. $M_{16,6}(m)$ is not empty if and only if $m \in \{1,2,3,5\}$.

Proof. The value $m = 3$ can be obtained from plane curves of degree 9 having 3 triple points and 3 double points, the value $m = 5$ can be obtained from plane curves of degree 8 having 5 double points. It should be noted that, if Γ is a plane curve of degree 9 with 4 triple points of simple type as its only singularities, then the pencil of conics through those 4 triple points also induces a g^1_6 on the normalization C of Γ. This is the reason for $k \geq 7$ in the statement of Conjecture B in (3.8).

Now we prove $M_{16,6}(4)$ is empty. Let C be a smooth 6-gonal curve of genus 16 and assume C has at least 4 mutually independent linear systems g^1_6 of type I. Take 4 such linear systems g_1, g_2, g_3 and g_4. Then $\dim(|g_1+g_2+g_3+g_4|) \geq 10$ ([1]), hence $\dim(|K_c-(g_1+g_2+g_3+g_4)|) \geq 1$ while $\deg(K_c - (g_1+g_2+g_3+g_4)) = 6$. Since C is 6-gonal we find $|K_c - (g_1+g_2+g_3+g_4)|$ is a base point free g^1_6 on C. If C belongs to $M_{16,6}(4)$ then this g^1_6 is one of the linear systems g_1, g_2, g_3 or g_4; say it is g_4. Then $|K_c-(g_1+g_2+g_3)| = |2g_4|$. Since $\dim(|g_1+g_2+g_3|) \geq 6$ ([1]) it follows that $\dim(|2g_4|) \geq 3$, a contradiction to g_4 being of type I. □

(4.4.2) CLAIM. $M_{15,6}(m)$ is not empty if and only if $m \in \{1,2,3,5,6\}$.

Proof. The value $m = 3$ can be obtained from plane curves of degree 9 having 3 triple points and 4 double points, the value $m = 6$ can be obtained from plane curves of degree 8 having 6 double points. The example with $m = 5$ comes from plane curves Γ of degree 9 with 4 triple points and 1 double point, all of them of simple type. It can be proved using the arguments of (2.4) that each g^1_6 on the normalization C is given by a pencil of plane curves of degree at most 2. Also, there exist such plane curves with the singular points in general position (one can apply Theorem 5 from [16]). Take a g^1_6 on C obtained from the pencil of lines through some triple point of Γ. Then, using canonically adjoint curves, $|K_c - 2g^1_6|$ is obtained from the linear system of plane curves of degree 4 having a singularity at the other triple points of Γ and containing the double point of Γ. Next, take the g^1_6 on C obtained from the pencil of conics in the plane containing the triple points of Γ. Now $|K_c - 2g^1_6|$ is obtained from the linear system of conics containing the double point of Γ. In both cases we find $\dim(|K_c - 2g^1_6|) = 4$, hence $\dim(|2g^1_6|) = 2$, i.e., g^1_6 is of type I. This proves $C \in M_{15,6}(5)$. We still need to prove that $M_{15,6}(4)$ is empty.

Assume C is a smooth 6-gonal curve of genus 15 and assume C has at least 4 mutually independent linear systems g^1_6 of type I. Fix two of them, call them g_1 and g_2. Then take two more such g^1_6, call them g' and g''. From [1] we know that $\dim(|g_1 + g_2|) \geq 3$; $\dim(|g_1 + g_2 + g'|) \geq 6$ and $\dim(|g_1 + g_2 + g' + g''|) \geq 10$. Moreover equality in the last inequality implies equality in the other inequalities too. In case $\dim(|g_1 + g_2 + g' + g''|) > 10$ then $\dim(|K_c - (g_1 + g_2 + g' + g'')|) > 0$, while $\deg(K_c - (g_1 + g_2 + g' + g'')) = 4$. This contradicts C having gonality 6. Hence $\dim(|g_1 + g_2 + g'|) = 6$ and $\dim(|g_1 + g_2|) = 3$. Now consider the cup-product map

$$\mu : H^0(g_1 + g_2) \otimes H^0(g') \to H^0(g_1 + g_2 + g')$$

(here we write $H^0(g_1 + g_2)$ to indicate the space of global sections of the invertible sheaf associated to $g_1 + g_2$, and so on). We conclude that $\dim(\ker(\mu)) \geq 1$. From the base point free pencil trick it follows that $|g_1 + g_2 - g'|$ is not empty. Let $E' \in |g_1 + g_2 - g'|$ and let $\phi : C \to \mathbf{P}^3$ be the morphism associated to $|g_1 + g_2|$. Using $g_1 + g_2 \subset |g_1 + g_2|$

one finds that $\Gamma = \phi(C)$ is birationally equivalent to C (g_1 and g_2 are independent) and Γ is contained in a smooth quadric $Q \subset \mathbf{P}^3$. Since $|g_1 + g_2 - E'| = g'$ there is a line $L' \subset \mathbf{P}^3$ such that the pencil of planes containing L' induces $g' + E'$ on C. Since $g' \notin \{g_1, g_2\}$ one finds $L' \not\subset Q$ hence $L' \cap Q = \{s'_1, s'_2\}$ (here $s'_1 \in Q; s'_2$ could be an infinitely near point of s'_1 on Q). Let m'_i be the multiplicity of Γ at s'_i; we can assume $m'_1 \geq m'_2$. Projection on $\mathbf{P}^2$ with center s'_1 gives rise to a plane model Γ' of C of degree $12 - m'_1$ and g' induced by a pencil of lines in $\mathbf{P}^2$ through the image s' of s'_2 under the projection.

This point has multiplicity $6 - m'_1$ on Γ'; this has to be equal to m'_2, hence $m'_1 + m'_2 = 6$ and $m'_1 \geq 3$. First assume $m'_1 \geq 4$. Then any other g^1_6 (mutually independent with g_1 g_2 and g')-call it g''-would give rise to s''_1; s''_2 and multiplicity of Γ at s''_1 would be at least 3. Since C has gonality 6 it follows that $s''_1 = s'_1$ (otherwise the projection of s''_1 would be a point of multiplicity at least 3 on Γ' while $\deg(\Gamma') \leq 8$, hence the pencil of lines through that singular point of Γ' would give rise to a linear system g^1_t on C with $t < 6$) and so the multiplicity at s''_2 is $6 - m'_1$. Also the projection s'' of s''_2 is a point of multiplicity $6 - m'_1$ on Γ and the pencil of lines through s'' gives rise to the linear system g'' on C. Hence, in the case $m'_1 \geq 4$ each g^1_6 on C is induced by a pencil of lines through a point of multiplicity $6 - m'_1$ of Γ'. In case $m'_1 = 5$ then C has infinitely many linear systems g^1_6, in case $m'_1 = 4$ then C is birationally equivalent to a plane curve of degree 8 with double points. If some double point is not ordinary, then the associated g^1_6 is not of type I. So, in order for each linear system g^1_6 on C to be of type I it is necessary that all double points of Γ' are simple. (This can be checked using canonically adjoint curves. It can also be explained as follows. If not all double points of Γ' are simple then nevertheless Γ' is the limit in a family of plane nodal curves of degree 8 with 6 ordinary nodes. The double point of Γ' that is not simple is the limit of two different nodes in this family. Hence the associated g^1_6 on C is the limit of two different g^1_6 in a family of curves.) Then C has 6 linear systems g^1_6. (We do not need the assumption that all g^1_6 on C are of type I, since C has already 4 linear systems g^1_6 of type I, Γ' has already 4 double points of simple type, hence Γ' has at least 5 double points.) In case $m'_1 = 3$ then either $s'_1 \in \{s''_1, s''_2\}$ or $s'_1 \notin \{s''_1, s''_2\}$. In the first case there is a point s'' on Γ' of multiplicity 3 such that g'' is induced by a pencil of lines through s'', in the second case g'' is induced by the pencil of conics through 4 points of multiplicity 3 of Γ' (the projections of s_1'', s_2'' and the projections of the 2 lines on Q containing s_1'; some of those points can be infinitely near). In both cases Γ' has 4 points of multiplicity 3. Each pencil of lines through such a point induces a g_6^1 on C.

In case no three of the triple points are on a line then the pencil of conics through the triple points induces a g_6^1 on C. So, either C has at least 5 linear systems g_6^1 or at least one of those singular points is infinitesimally near to another one or three of the triple points are on a line. Assume one of the triple points of Γ' is infinitely near. Let s be the point on Γ', let g_6^1 be the linear system on C obtained from the pencil of lines through s. Let $b : X \to \mathbf{P}^2$ be the blowing-up of $\mathbf{P}^2$ at s, let E be the exceptional divisor, let $\tilde{\Gamma}'$ be the strict transform of Γ' and let $\tilde{s} = \tilde{\Gamma}' \cap E$, a triple point of $\tilde{\Gamma}'$. Then $\tilde{\Gamma}'$ has 3 triple points and 1 double point as its only singularities (still, some of them can be infinitely near). The canonical linear system $|K_c|$ is described by elements in $|6L-2E|$ (here L is the inverse image of a line in $\mathbf{P}^2$) having a double point at the three triple points of $\tilde{\Gamma}'$ and containing the double point of $\tilde{\Gamma}'$. Let D be an element of $|6L-2E|$ containing two general divisors of g_6^1. Those divisors are obtained from intersections with general elements of $|L-E|$, hence D contains those elements of $|L-E|$. So $|K_c-2g_6^1|$ is described by elements of $|4L|$ such that its sum with the proper transforms of two general lines through s defines an element of $|6L-2E|$ having a double point at $\tilde{s}$ and also at the other triple points of $\tilde{\Gamma}'$ and containing the double point of $\tilde{\Gamma}'$. Now $\tilde{s}$ corresponds to a tangent direction of $\mathbf{P}^2$ at s and so the conditions coming from $\tilde{s}$ on $|4L|$ is that the curve contains s and that tangent direction. Those are 2 conditions on $|4L|$. So, altogether we obtain at most $2+2\cdot 3+1=9$ conditions on $|4L|$, hence $\dim(|K_c-2g_6^1|) \geq \dim(|4L|)-9=5$, hence $\dim(|2g_6^1|) \geq 3$, so g_6^1 is not of type I. So Γ' has 4 different triple points. The pencil of conics through them induces one more g_6^1 on C unless 3 of those 4 triple points are on a line. Assume this is the case and let s be the other triple point. Let g_6^1 be the linear system on C obtained from the pencil of lines through s. Using canonically adjoint curves one finds that $|K_c-2g_6^1|$ is obtained by the linear system of plane curves of degree 4 having a double point at the other 3 triple points and containing the double point. Such a curve needs to contain the line containing the other triple points. So $\dim(|K_c - 2g^1_6|)$ is the dimension of the space of plane curves of degree 3 containing the other triple points and the double point of Γ'. Hence $\dim(|K_c - 2g^1_6|) \geq \dim(|3L|) - 4 = 5$ and so $\dim(|2g^1_6|) > 2$. Again we obtain that g^1_6 is not of type I. This finishes the proof. □

(4.4.3) REMARK. If Conjectures A and B hold, then in the range $g \geq \frac{k(k-1)}{2}$ the case $g = 15$; $k = 6$ would be the only case for which there exists $m_0 > 1$ such that $M_{g,k}(m_0) = \emptyset$ and for two different values $m'; m'' > m_0$ one has $M_{g,k}(m')$ and $M_{g,k}(m'')$ are not empty.

(4.5) PROPOSITION.

a) *Let* $g = \frac{k(k-1)}{2}$
  ai) *If* $k \geq 7$ *then* $M_{g,k}(k-1)$ *is empty.*
  aii) *If* $k \geq 8$ *then* $M_{g,k}(k-2)$ *is empty.*
b) *Let* $g = \frac{k(k-1)}{2} + 1$
  bi) *If* $k \geq 7$ *then* $M_{g,k}(k-2)$ *is empty.*
  bii) *If* $k \geq 10$ *then* $M_{g,k}(k-3)$ *is empty.*
c) *Let* $g = \frac{k(k-1)}{2} + 2$
  ci) *If* $k \geq 9$ *then* $M_{g,k}(k-3)$ *is empty.*

Proof. a) Assume C is a smooth k-gonal curve of genus $g = \frac{k(k-1)}{2}$; $k \geq 7$ and C has at least $k-2$ mutually independent g^1_k of type I. Fix two of them, call them g_1 and g_2. Let $g_3, \cdots, g_{k-2}$ be $k-4$ more of them. From [1] it follows that $\dim(|g_1 + \cdots + g_{k-2}|) \geq \frac{(k-1)(k-2)}{2}$. If the inequality would be strictly, then $\dim(|K_c - (g_1 + \cdots + g_{k-2})|) > 0$; while $\deg(K_c - (g_1 + \cdots + g_{k-2})) = k-2$. This would contradict C having gonality k. So we find equality $\dim(|g_1 + \cdots + g_{k-2}|) = \frac{(k-1)(k-2)}{2}$. As explained in the proof of claim 4.4.2 this implies that $\dim|g_1 + g_2|) = 3$ and $|g_1 + g_2 - g_3|$ is not empty. Using the morphism $\phi : C \to \mathbf{P}^3$ associated to $|g_1 + g_2|$ one finds $\phi(C) = \Gamma$ is birationally equivalent to C and Γ is contained in a smooth quadric Q. Also one finds s'_3 and s''_3 on Q (s''_3 can be infinitesimally near to s'_3) with m'_3 and m''_3 the multiplicity of Γ at s'_3 and s''_3; $m'_3 + m''_3 = k$; $m'_3 \geq m''_3$ and the projection on $\mathbf{P}^2$ with center s'_3 is a plane curve of degree $2k - m'_3$ having singular points s_1, s_2, s_3 of multiplicities $k - m'_3$ such that the pencil of lines through s_i induces g_i on C. In case $m'_3 > k/2$ it follows that each of $g_3, \cdots, g_{k-2}$ is induced by a pencil of lines, hence Γ has at least $k-2$ singular points of multiplicity $k - m'_3$. In case $k \geq 8$ this implies $m'_3 = k-2$. Hence Γ is a plane curve of degree $k+2$ with no singular point of multiplicity more than 2. Repeating the arguments used in Claim (4.4.2) the singular points have to be of simple type and this implies the statement of the proposition. In case $k = 7$ and C has at least $k-1$ mutually independent g^1_k of type I, again it implies $m'_3 = k-2$ and hence the statement of the proposition. In case $m'_3 = k/2$ then first of all $k \geq 8$. Because of the genus, Γ has at most 3 singular points of multiplicity $k/2$. On the other hand, one should find at least 5 linear systems g^1_k using pencils of lines or pencils of conics through singular points of multiplicity $k/2$. Hence we need at least 4 points of multiplicity $k/2$, a contradiction.

b) Assume $C \in M_{g;k}(m)$ for some $m \geq k-2$ and $g = \frac{k(k-1)}{2} + 1$. Fix $k-2$ linear systems g^1_k, call them $g_1, \cdots, g_{k-2}$. From [1] we know $\dim(|g_1+\cdots+g_{k-2}|) \geq \frac{(k-1)(k-2)}{2}$, hence $\dim(|K_c-(g_1+\cdots+g_{k-2})|) \geq 1$ while $\deg(K_c-(g_1+\cdots+g_{k-2})) = k$. Hence $|K_c-(g_1+\cdots+g_{k-2})|$ is a g^1_k. If e.g. $g_1 = |K_c-(g_1+\cdots+g_{k-2})|$, then $\dim(|K_c-2g_1|) = \dim(|g_2+\cdots+g_{k-2}|) \geq \frac{(k-2)(k-3)}{2}$ and we find $\dim(|2g_1|) \geq 3$, hence g_1 is not of type I. This proves $m > k-2$. Now assume $k \geq 10$; $C \in M_{g,k}(k-3)$ and $g = \frac{k(k-1)}{2} + 1$. Fix two of the linear systems g^1_k; call them g_1 and g_2. Let g_3 be another one and let $g_4; \cdots; g_{k-3}$ be the others. In case $\dim(|g_1+g_2+g_3|) = 6$ for any choice of g_3, then $|g_1+g_2-g_3|$ is not empty for any choice of g_3 and as before one obtains a contradiction because of the non-existence of a suited plane model for C. Now assume $\dim(|g_1+g_2+g_3|) > 6$. Using [1], one finds $\dim(|g_1+g_2+\cdots+g_{k-4}+2g_{k-3}|) > \frac{(k-1)(k-2)}{2}$, hence $\dim(|K_c-(g_1+g_2+\cdots+g_{k-4}+2g_{k-3})|) \geq 1$. Since $\deg(K_c-(g_1+g_2+\cdots+g_{k-4}+2g_{k-3})) = k$ this would imply $|K_c-(g_1+g_2+\cdots+g_{k-4}+2g_{k-3})| \in \{g_1, \cdots, g_{k-3}\}$. This condition holds also if one changes g_{k-3} and g_i for any $4 \leq i \leq k-4$. On the other hand the linear system $g^1_k = |K_c-(g_1+g_2+\cdots+g_{k-4}+2g_{k-3})|$ changes. Since $k \geq 10$ we can assume $|K_c-(g_1+g_2+\cdots+g_{k-4}+2g_{k-3})| \in \{g_4, \cdots, g_{k-3}\}$. In case it is not g_{k-3} - say it is g_{k-4} - then $2g_{k-4} = |K_c-(g_1+\cdots+g_{k-5}+2g_{k-3})|$. Since $\dim(|g_1+g_2+g_3|) > 6$, one has $\dim(|g_1+\cdots+g_{k-5}+2g_{k-3}|) \geq \frac{(k-2)(k-3)}{2}$ hence $\dim(|2g_{k-4}|) \geq 3$, a contradiction to g_{k-4} being of type I. In case it is g_{k-3} one finds $|2g_{k-3}| = |K_c - (g_1 + \cdots + g_{k-3})|$ and since $\dim(|g_1 + \cdots + g_{k-3}|) > \frac{(k-2)(k-3)}{2}$ in this case one concludes again a contradiction to g_{k-3} being of type I.

c) Finally assume $C \in M_{g,k}(k-3)$ with $g = \frac{k(k-1)}{2} + 2$ and $k \geq 9$. Fix two linear systems g^1_k on C, call them g_1 and g_2. If for any other g^1_k one has $\dim(|g_1 + g_2 + g|) = 6$, then the non-existence of a suited plane model for C implies the statement of the proposition (here we use $k \geq 9$). So assume for some g^1_k - call it g_3 - one has $\dim(|g_1 + g_2 + g_3|) > 6$. Let $g_4, \cdots, g_{k-3}$ be the other linear systems g^1_k. Then (again using [1]) $\dim(|2g_1 + g_2 + \cdots + g_{k-3}|) \geq \frac{(k-1)(k-2)}{2}$ hence $\dim(|K_c - (2g_1 + g_2 + \cdots + g_{k-3})|) \geq 2$, while $\deg(K_c - (2g_1 + g_2 + \cdots + g_{k-3})) = k+2$, hence C would have a g^2_{k+2}. Then C cannot belong to $M_{g,k}(k-3)$. □

(4.7) REMARK. This proposition implies Conjecture B in the range $7 \leq k \leq 10$. For $k = 11$ the only statement that still is not proved is: $M_{58,11}(7)$ is empty.

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