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THE BEREZIN OPERATOR ON THE INVARIANT L^p SPACE IN THE BALL

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ABSTRACT. We show various properties of the Berezin operator T and its iteration T^k on the L^p space of the invariant measure τ on the complex unit ball B by exhibiting differences between the case of $1 < p < \infty$ and $p = 1, \infty$ under the infinite iteration of T or the summation of iterations.

I. Introduction

For a positive integer n, let

$$B = \{ z \in \mathbf{C}^n \mid |z| < 1 \}$$

and ν denote the Lebesgue volume measure on $\mathbf{C}^n$ normalized so that $\nu(B) = 1$, then for each $a \in B$ we have the automorphism of B

$$\varphi_a(z) = \frac{1 - P_z - (1 - |a|^2)^{1/2} Q_z}{1 - \langle z, a \rangle}$$

where

$$P_z = \frac{\langle z, a \rangle a}{|a|^2}, \quad Q_z = z - P_z, \quad \langle z, a \rangle = \sum_{i=1}^{n} z_i \overline{a_i}.$$

For $f \in L^1(B, \nu)$, we can define the Berezin transform

$$Tf(z) = \int_B f \circ \varphi_z \, d\nu. \tag{1}$$

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From the fact that the real Jacobian of φ_z is

$$(J_R\varphi_z)(w) \;=\; \frac{(1-|z|^2)^{n+1}}{|1-\langle z,w\rangle|^{2n+2}}$$

we get another formula of Tf as

$$Tf(z) \;=\; \int_B f(w)\,\frac{(1-|z|^2)^{n+1}}{|1-\langle z,w\rangle|^{2n+2}}\,d\nu(w). \tag{2}$$

The invariant Laplacian $\tilde{\Delta}$ is defined for $f \in C^2(B)$ by

$$(\,\tilde{\Delta} f\,)(z) \;=\; \Delta(f\circ\varphi_z)(0)$$

where Δ is the ordinary Laplacian. It commutes with every $\psi \in Aut(B)$

$$(\tilde{\Delta} f)\circ\psi \;=\; \tilde{\Delta}(f\circ\psi).$$

A function f on B is called M-harmonic if $\tilde{\Delta} f \;=\; 0$.
τ is the invariant measure on B defined by

$$d\tau(z) \;=\; (1-|z|^2)^{-n-1}\,d\nu(z) \quad \text{and satisfies} \quad \int_B f\,d\tau \;=\; \int_B f\circ\psi\,d\tau$$

for every $f \in L^1(\tau)$ and $\psi \in Aut(B)$.

In the paper [3], the author exhibited properties of the Berezin operator T and its iteration T^k on $L^\infty(B)$ by using the fact that $L^\infty(B)$ is the dual of $L^1(\tau)$ on which T is a contradiction, moreover the spectrum of T on the radial subspace of $L^1(\tau)$ is found explicitly from theories of the commutative Banach algebra.
Here, we extend results of [3] so that we find properties of

$$Tf\;,\; T^k f\;,\; \sum_{k=0}^{\infty} T^k f$$

when f belongs to $L^p(\tau)$ for $1 < \;p\; < \infty$, some of which turns out to be quite different from the case of $p=1$ or $p=\infty$.

Then we compare and combine our results with those of [3] to find general properties of T on the space $L^p(\tau)$ for $1 \le \;p\; \le \;\infty$.

II. Properties and Results

We will find various properties of Tf for functions in $L^p(\tau)$ when $1 < p < \infty$, then compare to [3] which deals with $p = 1$ or $p = \infty$.
We start from the basic and easy lemma of [3] which enables us to consider the iteration of T on $L^p(\tau)$.

LEMMA 1 ([3]). *For* $1 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, *$T$ is a linear contraction on $L^p(\tau)$. Moreover, for $f \in L^p(\tau)$ and $g \in L^q(\tau)$ we have*

$$\int_B (Tf)g \, d\tau = \int_B fTg \, d\tau.$$

(Note: T is not a bounded operator on $L^1(\nu)$ but using a technique developed by [2], one can show that T is bounded on $L^p(\nu)$ for $p > 1$)
Also we frequently use the L^∞ part of the Main Theorem of [1], which states as follows.

PROPOSITION 2 ([1]). *If $f \in L^\infty(B)$ satisfies $Tf = f$, then f is M-harmonic.*

We first show that for $1 < p < \infty$, the behaviour of $f \in L^p(\tau)$ under the infinite iteration of T differs from that of $L^\infty(B)$ or $L^1(\tau)$. Proposition 3.6 of [3] showed that if $f \in L^1(\tau)$ is radial then

$$\lim_{k\to\infty} \| T^k f \|_1 = \left| \int_B f \, d\tau \right| .$$

Next proposition says that when $1 < p < \infty$, things are much simpler.

PROPOSITION 3. *For $1 < p < \infty$, if $f \in L^p(\tau)$ then*

$$\lim_{k\to\infty} \| T^k f \|_p = 0.$$

Proof. Since T is a positive linear contraction on $L^p(\tau)$, by the standard approximation, it is enough to prove the proposition when f is a characteristic function χ_K for come compact subset K of B.
First, we'll show that

$$(3) \qquad \lim_{k\to\infty} \| T^k \chi_K \|_\infty = 0.$$

Pick $0 < r < 1$ such that $K \subset rB$. Then define $u : [0,1] \rightarrow \mathbf{R}$ by

$$u(t) = -1 \quad \text{for } 0 \leq t \leq r$$
$$u(t) = \frac{t-1}{1-r} \quad \text{for } r \leq t \leq 1$$

then $v(z) = u(|z|)$ is subharmonic in B, which implies that $v \circ \varphi_a$ is subharmonic for each $a \in B$. Thus from the definition (1) of Tv and the submean value property, we get $Tv \geq v$. Since T is a positive operator, $\{T^k v\}$ is increasing and uniformly bounded on B.
Hence $\lim T^k v = g$ exists and satisfies $Tg = g$.
Since g is bounded, Proposition 2 forces g to be M-harmonic. So $g \equiv 0$ since $g = 0$ on ∂B.
By Dini's theorem, therefore, $\{ T^k v \}$ converges uniformly to zero and this makes $\{ T^k \chi_K \}$ also converge uniformly to zero because

$$T^k v \leq -T^k \chi_K \leq 0 .$$

This proves (3).

Next let $p = 1 + \alpha$ for some $\alpha > 0$, and then for a given $\epsilon > 0$, define

$$A_k = \{ z \in B \mid T^k \chi_K > \epsilon \}.$$

then from (3), A_k is empty for all k sufficiently large.
Moreover, since

$$\| T^k \chi_k \|_\infty \leq 1$$

we have

$$\begin{aligned} \| T^k \chi_K \|_p^p &= \int_B | T^k \chi_K |^p \, d\tau \\ &= \int_{A_k} (T^k \chi_K)\,(T^k \chi_K)^\alpha \, d\tau + \int_{B/A_k} (T^k \chi_K)\,(T^k \chi_K)^\alpha \, d\tau \\ &\leq \tau(A_k) + \tau(K)\epsilon^\alpha \end{aligned}$$

Therefore, we complete the proof by taking $k \rightarrow \infty$. □

Even though $T^k f$ generally does not converge to zero in norm when $f \in L^1(\tau)$. Next proposition tells that it converges pointwise to zero in B.

PROPOSITION 3. If $f \in L^1(\tau)$, then

$$\sum_{k=0}^{\infty} \mid T^k f(z) \mid \quad < \quad \infty \qquad \text{for every } z \in B$$

Proof. First, let's observe that for $u(z) = |z|^2 - 1$, we get $Tu > u$. To show this we have to calculate directly using 1.4.9 and 1.4.10 of [4].

$$\begin{aligned}
& Tu(z) \\
= & -(1-|z|^2)^{n+1} \int_B \frac{1-|w|^2}{|1-\langle z,w\rangle|^{2n+2}}\, d\nu(w) \\
= & -(1-|z|^2)^{n+1} 2n \int_0^1 (1-r^2) \int_S \left| \sum_{k=0}^{\infty} \frac{(k+n)!}{k!n!} (\langle z, r\xi\rangle)^k \right| d\sigma(\xi) r^{2n+1} dr
\end{aligned}$$

(Here σ is the rotation-invariant probability measure on $S = \partial B$)

$$\begin{aligned}
= & -(1-|z|^2)^{n+1} 2n \int_0^1 (1-r^2) \frac{1}{n!} \sum_{k=0}^{\infty} \frac{(k+n)!(k+n)}{k!n} |z|^{2k} r^{2n+2k-1} dr \\
= & -(1-|z|^2)^{n+1} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!n!(k+n+1)} |z|^{2k} \\
> & -(1-|z|^2)^{n+1} \sum_{k=0}^{\infty} \frac{(k+n-1)!}{k!(n-1)!(k+n+1)} |z|^{2k} \\
= & |z|^2 - 1 \quad = \quad u(z)
\end{aligned}$$

Moreover u is a uniform limit of a sequence of functions in $C_c(B)$ and if $v \in C_c(B)$, then we can show exactly the same way as the proof of (3) of Proposition 3 that

$$\lim_{k\to\infty} \| T^k v \|_\infty = 0.$$

Hence we get

$$\lim_{k\to\infty} \| T^k u \|_\infty = 0.$$

Thus if we define $g = Tu - u$, then $g > 0$ and $\|g\|_\infty \leq 2$, moreover

$$\sum_{k=0}^{m} T^k g$$

converges uniformly to $-u$ as $m \to \infty$.
Combining this with Lemma 1, we get

$$\begin{aligned} \int_B \left(\sum_{k=0}^{\infty} T^k|f| \right) g\, d\tau &= \int_B |f| \sum_{k=0}^{\infty} T^k\, g\, d\tau \\ &= \int_B |f|\, (-u)\, d\tau \\ &\leq \| f \|_1 \| u \|_\infty \quad < \quad \infty. \end{aligned}$$

Since $g > 0$, the proof is complete. □

[3] showed that when $p = 1$ or $p = \infty$, the space $(I - T)L^p(\tau)$ is not dense in $L^p(\tau)$ by showing that every $f \in \overline{(I - T)L^p(\tau)}$ satisfies

$$\lim_{k\to\infty} \| T^k f \|_p = 0 \quad \text{for} \quad p = 1\ ,\ \infty.$$

However next proposition shows that for $1 < p < \infty$, $(I - T)L^p(\tau)$ is dense in $L^p(\tau)$.

PROPOSITION 5. *If* $1 < p < \infty$, *then* $\overline{(I - T)L^p(\tau)} = L^p(\tau)$.

Proof. Let L be a bounded linear functional on $L^p(\tau)$, then there is a $g \in L^q(\tau)$ $(\frac{1}{p} + \frac{1}{q} = 1)$ such that

$$L(f) = \int_B f\, g\, d\tau \quad \text{for all} \quad f \in L^p(\tau).$$

Assume that $L(h) = 0$ for all $h \in (I - T)L^p(\tau)$. Then

$$\int_B (f - Tf)g\, d\tau = 0 \quad \text{for every} \quad f \in L^p(\tau)$$

which means, by Lemma 1

$$\int_B f(g - Tg)\, d\tau = 0 \quad \text{for every} \quad f \in L^p(\tau).$$

Thus $g = Tg$.
However, for every $z \in B$

$$\begin{aligned}
& \mid g(z) \mid \\
= & \mid Tg(z) \mid \\
\leq & \int_B |g(w)| \frac{(1-|z|^2)^{n+1}}{\mid 1-\langle z,w\rangle \mid^{2n+2}} \, d\nu(w) \\
= & (1-|z|^2)^{n+1} \int_B |g(w)| \frac{(1-|w|^2)^{n+1}}{\mid 1-\langle z,w\rangle \mid^{2n+2}} \, d\tau(w) \\
\leq & (1-|z|^2)^{n+1} \Big(\int_B |g|^q d\tau \Big)^{\frac{1}{q}} \Big(\int_B \frac{(1-|w|^2)^{pn+p}(1-|w|^2)^{-n-1}}{|1-\langle z,w\rangle|^{2np+2p}} d\nu(w) \Big)^{\frac{1}{p}} \\
\leq & (1-|z|^2)^{n+1} \parallel g \parallel_q c(1-|z|^2)^{-n-1} \\
& \text{for some } c>0 \quad \text{by 1.4.10 of [4]} \\
= & c \parallel g \parallel_q .
\end{aligned}$$

Thus g is bounded and $Tg = g$, which means that g is M-harmonic (Proposition 2). In view of 4.2.3 of [4], the radialization of such g is a constant but nonzero constant can't belong to $L^q(\tau)$, which forces g to be the constant zero.
Therefore $L(f) = 0$ for all $f \in L^p(\tau)$.
From Hahn-Banach theorem, we conclude that $(I-T)L^p(\tau)$ is dense in $L^p(\tau)$. This ends the proof. □

If we combine the Proposition 5 with the Theorem 3.7, corollary 2.3 of [3] we get the following corollary.

COROLLARY 6. (*a*) *For* $1 \leq p < \infty$,

$$\overline{(I-T)L^p(\tau)} = \{ f \in L^p(\tau) \mid \{ T^k f \} \text{ converges} \}$$

(*b*) *For* $1 \leq p \leq \infty$,

$$\overline{(I-T)L^p(\tau)} = \{ f \in L^p(\tau) \mid \{ T^k f \} \text{ converges to } 0 \}.$$

We have characterize the closure of $(I-T)L^p(\tau)$ *for* $1 \leq p \leq \infty$ *by using the iteration* T^k *on* $L^p(\tau)$. *Moreover we can also characterize the space* $(I-T)L^p(\tau)$ *using the iteration* T^k *on* $L^p(\tau)$ *when* $1 < p \leq \infty$.

PROPOSITION 7. *For* $1 < p \leq \infty$

$$(I-T)L^p(\tau) = \left\{ f \in L^p(\tau) \,\middle|\, \limsup_{m\to\infty} \| \sum_0^m T^k f \|_p < \infty \right\}.$$

Proof. Let $f = g - Tg$ for some $g \in L^p(\tau)$ then

$$\left\| \sum_0^m T^k f \right\|_p = \| g - T^{m+1} g \|_p \leq 2\|g\|_p.$$

Hence

$$(I-T)L^p \subset \left\{ f \in L^p \mid \limsup_m \left\| \sum_0^m T^k f \right\|_p < \infty \right\}.$$

on the other hand, for $1 < p \leq \infty$, pick $f \in L^p(\tau)$ such that

$$\limsup_m \left\| \sum_0^m T^k f \right\|_p = M < \infty.$$

Define

$$f_k = \sum_{j=0}^k T^j f$$

then $f_k - Tf_k = f - T^{k+1} f$.
hence if we define

$$F_m = \frac{1}{m+1} \sum_{k=0}^m f_k$$

then $\| F_m \|_P \leq M$ and

$$\begin{aligned} (I-T)F_m &= \frac{1}{m+1} \sum_{k=0}^m (I-T) f_k \\ &= \frac{1}{m+1} \sum_{k=0}^m (f - T^{k+1} f) \\ &= f - \frac{1}{m+1} \sum_{k=0}^m T^{k+1} f. \end{aligned}$$

Hence

$$\| (I-T)F_m - f \|_p \leq \frac{1}{m+1} M \to 0 \quad \text{as} \quad m \to \infty.$$

Since $\{F_m\}$ is norm bounded , it has a subspace $\{F_{m_j}\}$ that converges weak $*$ to some $g \in L^p(\tau)$ and the operator $(I-T)$ is self-adjoint (Lemma 1) in $L^q(\tau)$ which makes $(I-T)F_{m_j}$ converge to $(I-T)g$ weak $*$ in $L^p(\tau)$. Since $(I-T)F_m$ norm converges to f , f is the unique weak $*$ limit of $(I-T)F_m$.
Therefore

$$f \;=\; (I-T)g \;\in\; (I-T)L^p(\tau)$$

This completes the proof. □

[3] showed that if $f \in L^1(\tau)$ satisfies $Tf = \mu f$ for some $\mu \in \mathbf{C}$ with $|\mu| = 1$, then $f \equiv 0$.
Next proposition is an extension of it.

PROPOSITION 8. Let $1 \leq p \leq 2$. If $f \in L^p(\tau)$ satisfies $Tf = \mu f$ for some $\mu \in \mathbf{C}$, then $f \equiv 0$.

Proof. It is obvious when μ is zero , so assume that $\mu \neq 0$. Observe that when $f \in L^p(\tau)$ satisfies $Tf = \mu f$ then
(i) When $p = 1$, for every $z \in B$

$$\begin{aligned} |\,\mu f(z)\,| \;=\; |\,Tf(z)\,| \;&\leq\; \sup_{z\in B} \left(\frac{(1-|z|^2)\,(1-|w|^2)}{|1-\langle z,w\rangle^2} \right)^{n+1} \|f\|_1 \\ &=\; \|f\|_1 \end{aligned}$$

(ii) When $1 < p < \infty$

$$\begin{aligned} |\,\mu f(z)\,| \;&\leq\; (1-|z|^2)^{n+1} \int_B |f(w)| \left(\frac{1-|w|^2}{|1-\langle z,w\rangle|^2} \right)^{n+1} d\tau(w) \\ &\leq\; c\|f\|_p \quad \text{(for some } c \text{ just as the proof of Proposition 5)} \end{aligned}$$

Thus all such f belongs to $L^\infty(B)$.
For $1 \leq p \leq \infty$ and $\mu \in \mathbf{C}$, we define

$$M_{p,\mu} \;=\; \{\; f \in L^p(\tau) \mid Tf = \mu f \;\}.$$

First we will characterize the space $M_{\infty,\mu}$. The spectrum of T on $L^\infty(B)$ is found in [3] (Theorem 2.1), which is

$$E \;=\; \left\{ \frac{\Gamma(z+1)\Gamma(n+1-z)}{\Gamma(n+1)} \;\middle|\; 0 \;\leq\; Re\; z \;\leq\; n \right\}.$$

In view of this if $\mu \notin E$, then $M_{\infty,\mu} = \{\;0\;\}$. Furthermore every $\mu \in E$ is a point spectrum (an eigenvalue) of T on $L^\infty(B)$. Indeed if $\mu \in E$

and β with $0 \leq Re\,\beta \leq n$ satisfies

$$\frac{\Gamma(\beta+1)\,\Gamma(n+1-\beta)}{\Gamma(n+1)} = \mu$$

then we can see that the function

$$g_\alpha(z) = \int_S \left(\frac{1-|z|^2}{|1-\langle z,\xi\rangle|^2} \right)^\beta d\sigma(\xi)$$

satisfies $Tg = \mu g$.
However the Main Theorm of [1] has the precise characterization of the space $M_{1,1}$, which is;

$\tilde{\Delta}$ is a bounded linear operator on the Banach space $M_{1,1}$ and the finite set E_1 defined by

$$E_1 = \{ \lambda \in \mathbf{C} \mid \lambda = -4\beta(n-\beta),\ \frac{\Gamma(\beta+1)\Gamma(n+1-\beta)}{\Gamma(n+1)} = 1, \\ -1 < Re\,\beta < n+1 \}$$

is the set of all eigenvalues of $\tilde{\Delta}$ on $M_{1,1}$.
Indeed, if $E_1 = \{ \lambda_1, \cdots, \lambda_N \}$ then

$$M_{1,1} = X_{\lambda_1} \oplus \cdots \oplus X_{\lambda_N}$$

where

$$X_{\lambda_i} = \{ f \in L^1(B) \mid \tilde{\Delta} f = \lambda_i f \}.$$

Now by the identically the same proof as that of the Main Theorem of [1], we can also get the following characterization of the space $M_{\infty,\mu}$ such as;

For each $\mu \in E$, $\tilde{\Delta}$ is a bounded linear operator on the Banach Space $M_{\infty,\mu}$ and

$$E_\mu = \{ \lambda \in \mathbf{C} \mid \lambda = -4\beta(n-\beta),\ \frac{\Gamma(\beta+1)\Gamma(n+1-\beta)}{\Gamma(n+1)} = \mu, \\ 0 \leq Re\,\beta \leq n \}$$

is the set of all eigenvalues of $\tilde{\Delta}$ on $M_{\infty,\mu}$ and is always finite.
Let

$$E_\mu = \{ \lambda_1, \cdots, \lambda_m \}$$

and

$$Q(z) = \prod_{i=1}^{m}(z - \lambda_i)$$

then $Q(\tilde{\Delta}) = 0$ *on* $M_{\infty,\mu}$.
Thus by Lemma 4.1 *of* [1], *we have*

$$M_{\infty,\mu} = X_{\lambda_1} \oplus \cdots \oplus X_{\lambda_m}$$

where

$$X_{\lambda_i} = \{ f \in L^\infty(B) \mid \tilde{\Delta} f = \lambda_i f \}.$$

Since we've showed that

$$M_{p.\mu} = M_{\infty,\mu} \cap L^p(\tau)$$

to prove the proposition it is enough to show that
if $1 \le p \le 2, f \in L^p(\tau)$ and $\tilde{\Delta} f = \lambda f$ for some $\lambda \in \mathbf{C}$, then f is identically zero.
The radial functions in X_λ are known to be the constant multiple of

$$g_\alpha(z) = \int_S \frac{(1-|z|^2)^{n\alpha}}{|1-\langle z,\xi\rangle|^{2n\alpha}}\, d\sigma(\xi)$$

where $\lambda = -4n^2\alpha(1-\alpha)$ and the function g_α satisfies $g_\alpha = g_{1-\alpha}$ (see 4.2.3 of [4]).
Also note that the radialization carries $X_\lambda \cap L^p(\tau)$ into itself and if $\alpha = s + it(s,t \in \mathbf{R})$, then $g_\alpha \in L^p(\tau)$ if and only if $g_s \in L^p(\tau)$.
However by direct calculation using 1.4.10 of [4], we get that when $1 \le p \le 2, g_s$ belongs to $L^p(\tau)$ at no $s \in \mathbf{R}$, which completes the proof of the proposition. □

(Note : When $p > 2$, the function $g_{\frac{1}{2}}$ belongs to $L^p(\tau)$ and satisfies $Tg_{\frac{1}{2}} = \mu g_{\frac{1}{2}}$ where

$$\mu = \frac{\Gamma(\frac{n+1}{2})}{\gamma(n+1)}.)$$

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