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CHARACTERIZATION OF REINHARDT DOMAINS BY THEIR AUTOMORPHISM GROUPS

ALEXANDER V. ISAEV AND STEVEN G. KRANTZ

ABSTRACT. We survey results, obtained in the past three years, on characterizing bounded (and Kobayashi-hyperbolic) Reinhardt domains by their automorphism groups. Specifically, we consider the following two situations: (i) the group is non-compact, and (ii) the dimension of the group is sufficiently large. In addition, we prove two theorems on characterizing general hyperbolic complex manifolds by the dimensions of their automorphism groups.

0. Preliminaries

The problem of holomorphic equivalence of domains in complex space and, more generally, of complex manifolds, is a natural problem in complex analysis. It is well-known that two randomly chosen domains in $\mathbb{C}^n$ are most likely to be holomorphically non-equivalent (see, e.g., [18]). Our view of the equivalence problem is by way of automorphism groups. Let M be a complex manifold and $\mathrm{Aut}(M)$ the group of its holomorphic automorphisms. Note that $\mathrm{Aut}(M)$ is a topological group with the natural compact-open topology. For most domains in complex space the automorphism group consists of just one element—the identity transformation. Therefore if we restrict attention to domains with automorphism group which is in some sense "large", then we might hope to obtain a reasonable classification of such domains or manifolds.

There could be different ways of saying that the automorphism group is "large". In this paper, we either assume that the automorphism group is *non-compact*, or that its *dimension* (whenever it makes sense) is "large". Note that we can speak about $\dim \mathrm{Aut}(M)$, if the manifold M

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is Kobayashi-hyperbolic, since in this case $\mathrm{Aut}(M)$ is a Lie group (see [15]).

We deal primarily with Reinhardt domains in $\mathbb{C}^n$, that is, domains invariant under the standard action of the torus $\mathbb{T}^n$ on $\mathbb{C}^n$:

$$z_j \mapsto e^{i\phi_j} z_j, \qquad \phi_j \in \mathbb{R}, \qquad j = 1, \dots, n.$$

In Section 1 we discuss the case of bounded and hyperbolic Reinhardt domains with non-compact automorphism group and regular boundary. In Section 2 we consider arbitrary hyperbolic Reinhardt domains (and even hyperbolic complex manifolds) with automorphism group of "large" dimension and usually do not assume any regularity of the boundary.

1. Reinhardt Domains with Non-Compact Automorphism Group

In [1] (see also [14]), E. Bedford and S. Pinchuk conjectured that any bounded domain in $\mathbb{C}^n$ with C^∞-smooth boundary and non-compact automorphism group is biholomorphically equivalent to a domain of the form (1.1) below.

Fix positive integers $m_2, \dots, m_n$. For a multi-index $K = (k_2, \dots, k_n)$, define its *weight* by $\mathrm{wt}(K) := \sum_{j=2}^n \frac{k_j}{m_j}$. Then the domains in question are defined as follows:

$$\left\{ |z_1|^2 + \sum_{\mathrm{wt}(K)=\mathrm{wt}(L)=1} a_{KL} z'^K \overline{z'}^L < 1 \right\}, \tag{1.1}$$

where $z' := (z_2, \dots, z_n)$, $a_{KL} \in \mathbb{C}$ and $a_{KL} = \overline{a_{LK}}$.

The Bedford/Pinchuk conjecture holds: **(i)** for domains that have a strictly pseudoconvex boundary accumulation point, **(ii)** for convex domains of finite type, **(iii)** for weakly pseudoconvex domains of finite type such that the signature of the Levi form of the boundary has no more than one zero eigenvalue, and **(iv)** for domains in $\mathbb{C}^2$ with real-analytic boundary (see [9] for a detailed exposition of all these results).

In the case of Reinhardt domains formula (1.1) takes the form

$$\left\{ |z_1|^2 + \sum_{\mathrm{wt}(K)=1} a_K z'^K \overline{z'}^K < 1 \right\}, \tag{1.2}$$

where $a_K \in \mathbb{R}$. Note that not every domain of the form (1.1) is equivalent to a Reinhardt domain [4]. We mention here that Catlin also discussed domains of the form (1.2) as examples of domains with non-compact automorphism group (see [16]).

THEOREM 1.1. ([5]) *Any bounded Reinhardt domain in $\mathbb{C}^n$ with non-compact automorphism group and C^∞-smooth boundary is holomorphically equivalent to a domain of the form* (1.2), *and the equivalence is given by dilations and a permutation of coordinates.*

Theorem 1.1 confirms the Bedford/Pinchuk conjecture for Reinhardt domains and is a special case of the following theorem.

THEOREM 1.2. ([11]) *Let $D \subset \mathbb{C}^n$ be a bounded Reinhardt domain with C^k-smooth boundary (where $k = 1, 2, 3, \ldots$ or $k = \infty$) and non-compact automorphism group. Then, up to dilations and permutations of coordinates, D is a domain of the form*

$$\left\{|z_1|^2 + \psi(|z_2|, \ldots, |z_n|) < 1\right\}, \tag{1.3}$$

where $\psi(x_2, \ldots, x_n)$ is a non-negative C^k-smooth function in $\mathbb{R}^{n-1}$ that is strictly positive in $\mathbb{R}^{n-1} \setminus \{0\}$ and such that $\psi(|z_2|, \ldots, |z_n|)$ is C^k-smooth in $\mathbb{C}^{n-1}$, and

$$\psi\left(t^{\frac{1}{\alpha_2}} x_2, \ldots, t^{\frac{1}{\alpha_n}} x_n\right) = t\psi(x_2, \ldots, x_n)$$

in $\mathbb{R}^{n-1}$ for all $t \geq 0$. Here $\alpha_j > 0$, $j = 2, \ldots, n$, and each α_j is either an even integer or $\alpha_j > 2k$.

For $k = \infty$, the functions ψ from (1.3) admit an explicit description as in formula (1.2). For $k < \infty$, the problem of describing such functions seems to be open. Natural examples of such functions ψ are given as follows.

EXAMPLE 1.3. ([11]) Consider the following set of $(n-1)$-tuples $s = (s_2, \ldots, s_n)$:

$$M := \Big\{ s = (s_2, \ldots, s_n) \in \mathbb{R}^{n-1} : s_j \geq 0; \text{ each } s_j \text{ is either an even integer or } s_j > 2k, \text{ and } \textstyle\sum_{j=2}^n \frac{s_j}{\alpha_j} = 1 \Big\}.$$

Let μ be an arbitrary finite measure on the set M. Then the function

$$\psi(|z_2|, \ldots, |z_n|) = \int_M |z_2|^{s_2} \ldots |z_n|^{s_n} \, d\mu \tag{1.4}$$

has all the properties stated in Theorem 1.2.

One can show, however, that not all functions ψ as in (1.3) are described by formula (1.4) ([11]).

The proofs of Theorems 1.1 and 1.2 rely on the explicit description of the automorphism groups of bounded (and even hyperbolic) Reinhardt domains given in [17], [21]. Namely, for a hyperbolic Reinhardt domain D, the results in [17] give explicit formulas for the identity component $\text{Aut}_0(D)$ of the full automorphism group $\text{Aut}(D)$ and also show that $\text{Aut}(D) = \text{Aut}_0(D) \circ \text{Aut}_{\text{alg}}(D)$, where $\text{Aut}_{\text{alg}}(D)$ is the subgroup of algebraic automorphisms in $\text{Aut}(D)$. For C^1-smoothly bounded Reinhardt domains it turns out ([5]) that $\text{Aut}_{\text{alg}}(D)$ is finite up to the standard action of the torus $\mathbb{T}^n$ on $\mathbb{C}^n$ and thus the non-compactness of $\text{Aut}(D)$ is equivalent to that of $\text{Aut}_0(D)$. Since $\text{Aut}_0(D)$ is given by explicit formulas, one can determine when precisely $\text{Aut}_0(D)$ is non-compact and use this information to prove Theorems 1.1 and 1.2.

For (possibly unbounded) hyperbolic Reinhardt domains the situation is more complicated. It can happen [10] that the subgroup $\text{Aut}_{\text{alg}}(D)$ is essentially infinite and that the non-compactness of $\text{Aut}(D)$ comes not from that of $\text{Aut}_0(D)$, but from the non-compactness of $\text{Aut}_{\text{alg}}(D)$. However, the following holds:

PROPOSITION 1.4. ([10]) *Let $D \subset \mathbb{C}^n$ be a hyperbolic Reinhardt domain with C^1-smooth boundary. Suppose that D intersects at least $n-1$ of the coordinate hyperplanes $\{z_j = 0\}$, $j = 1, \dots, n$. Then $Aut_{\text{alg}}(D)$ is finite up to the action of $\mathbb{T}^n$.*

Proposition 1.4 allows one to apply the above technique to obtain the following result.

THEOREM 1.5. ([10]) *Let $D \subset \mathbb{C}^2$ be a hyperbolic Reinhardt domain with C^k-smooth boundary (where $k = 1, 2, 3, \dots$ or $k = \infty$) and let D intersect at least one of the coordinate complex lines $\{z_j = 0\}$, $j = 1, 2$. Assume also that $Aut(D)$ is non-compact. Then D is holomorphically equivalent to one of the following domains:*

(i) $\left\{(z_1, z_2) : |z_1|^2 + |z_2|^\alpha < 1\right\}$,
where either $\alpha < 0$, or $\alpha = 2m$ for some $m \in \mathbb{N}$, or $\alpha > 2k$;

(ii) $\{(z_1, z_2) : |z_1| < 1, (1 - |z_1|^2)^\alpha < |z_2| < R(1 - |z_1|^2)^\alpha\}$,
where $1 < R \leq \infty$ *and* $\alpha < 0$;

(iii) $\left\{(z_1, z_2) : e^{\beta|z_1|^2} < |z_2| < Re^{\beta|z_1|^2}\right\}$,
where $1 < R \leq \infty$, $\beta \in \mathbb{R}$, $\beta \neq 0$, *and, if* $R = \infty$, $\beta > 0$.

In case **(i)** *the equivalence is given by dilations and a permutation of the coordinates; in cases* **(ii)** *and* **(iii)** *the equivalence is given by a mapping of the form*

$$\begin{aligned} z_1 &\mapsto \lambda z_{\sigma(1)} z_{\sigma(2)}^a, \\ z_2 &\mapsto \mu z_{\sigma(2)}^{\pm 1}, \end{aligned}$$

where $\lambda, \mu \in \mathbb{C} \setminus \{0\}$, $a \in \mathbb{Z}$ *and* σ *is a permutation of* $\{1, 2\}$.

Despite Proposition 1.4, the result of Theorem 1.5 cannot be extended to higher dimensions. As examples in [10] show, no reasonable classification seems to exist in $\mathbb{C}^n$, $n \geq 3$, even for hyperbolic domains that contain the origin.

2. Dimensions of Automorphism Groups

In this section we will describe some results that characterize hyperbolic Reinhardt domains in terms of the dimensions of their automorphism groups—rather than characterizations by way of non-compactness, as we did above. We will start, however, with the case of general hyperbolic manifolds.

Let M be a connected complex hyperbolic manifold of complex dimension n. It is known (see [12], [15]) that $\dim \mathrm{Aut}\,(M) \leq n^2 + 2n$ and, if $\dim \mathrm{Aut}\,(M) = n^2 + 2n$, then M is holomorphically equivalent to the unit ball $B^n \subset \mathbb{C}^n$. Below we obtain a stronger version of this result.*

THEOREM 2.1. *Let* M *be a connected hyperbolic manifold of complex dimension* $n \geq 2$. *Suppose that* $\dim Aut(M) \geq n^2 + 3$. *Then* M *is biholomorphically equivalent to the unit ball* $B^n \subset \mathbb{C}^n$.

*We would like to thank W. Kaup, R. E. Greene and K.-T. Kim for helpful conversations.

First, we need the following algebraic lemma.[†].

LEMMA 2.2. *Let G be a Lie subgroup of the unitary group $U(n)$ and let G^c be its connected component of the identity. Suppose that $\dim G \geq n^2 - 2n + 3$, $n \geq 2$, $n \neq 4$. Then either $G = U(n)$, or $G^c = SU(n)$. For $n = 4$ this list has to be augmented by subgroups of $U(4)$ whose Lie algebras are isomorphic to $\mathbb{R} \oplus \mathfrak{sp}_{2,0}$.*

Proof. Since G is compact, it is completely reducible (see e.g. [22]). Thus, $\mathbb{C}^n$ splits into a sum of G-invariant pairwise orthogonal complex subspaces, $\mathbb{C}^n = V_1 \oplus \dots V_k$, such that the restriction G_j of G to every V_j is irreducible. Let $n_j := \dim_{\mathbb{C}} V_j$ (hence we have $n_1 + \dots + n_k = n$) and $U(n_j)$ be the group of unitary transformation of V_j. Clearly, $G_j \subset U(n_j)$ and therefore $\dim G_j \leq n_j^2$. Hence $\dim G \leq n_1^2 + \dots + n_k^2$. On the other hand we have $\dim G \geq n^2 - 2n + 3$, which gives that $k = 1$, i.e. G acts (complex) irreducibly on $\mathbb{C}^n$.

Let $\mathfrak{g} \subset \mathfrak{u}_n$ be the Lie algebra of G and $\mathfrak{g}^{\mathbb{C}} \subset \mathfrak{gl}_n$ its complexification. It then follows that $\mathfrak{g}^{\mathbb{C}}$ acts irreducibly and faithfully on $\mathbb{C}^n$. Therefore by a theorem of É. Cartan (see e.g. [7]), $\mathfrak{g}^{\mathbb{C}}$ is either semisimple or is the direct sum of a semisimple ideal $\mathfrak{h}$ and $\mathbb{C}$, where $\mathbb{C}$ acts on $\mathbb{C}^n$ by multiplication. Clearly the action of the ideal $\mathfrak{h}$ on $\mathbb{C}^n$ is irreducible and faithful.

Suppose first that $\mathfrak{g}^{\mathbb{C}}$ is semisimple, and let $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_m$ be its decomposition into the direct sum of simple ideals. It then follows (see e.g. [7]) that the representation of $\mathfrak{g}^{\mathbb{C}}$ is the tensor product of some irreducible faithful representations of $\mathfrak{g}_j$. Let n_j denote the dimension of the representation of $\mathfrak{g}_j$, $j = 1, \dots, m$. Then $n = n_1 \cdot \dots \cdot n_m$ and $\dim_{\mathbb{C}} \mathfrak{g}_j \leq n_j^2 - 1$, $n_j \geq 2$ for $j = 1, \dots, m$. It is now not difficult to prove the following claim. □

CLAIM. *If $n = n_1 \cdot \dots \cdot n_m$, $m \geq 2$, $n_j \geq 2$ for $j = 1, \dots, m$, then $\sum_{j=1}^m n_j^2 \leq n^2 - 2n$.*

It follows from the claim that $m = 1$, i.e., $\mathfrak{g}^{\mathbb{C}}$ is simple. The minimal dimensions of irreducible faithful representations of complex simple Lie algebras are well-known (see e.g. [22]). In the table below V denotes representations of minimal dimension.

[†]We are grateful to A. Stolin for help with the proof of this lemma

$\mathfrak{g}$	$\dim V$	$\dim \mathfrak{g}$
$\mathfrak{sl}_k$ $k \geq 2$	k	$k^2 - 1$
$\mathfrak{o}_k$ $k \geq 7$	k	$\frac{k(k-1)}{2}$
$\mathfrak{sp}_{2k}$ $k \geq 2$	$2k$	$2k^2 + k$
$\mathfrak{e}_6$	27	78
$\mathfrak{e}_7$	56	133
$\mathfrak{e}_8$	248	248
$\mathfrak{f}_4$	26	52
$\mathfrak{g}_2$	7	14

Since $\dim_{\mathbb{C}} \mathfrak{g}^{\mathbb{C}} \geq n^2 - 2n + 3$, it follows that $\mathfrak{g}^{\mathbb{C}} \simeq \mathfrak{sl}_n$. Since $\mathfrak{g}$ is a compact algebra, we get $\mathfrak{g} = \mathfrak{su}_n$ (see [22]) and therefore $G^c = SU(n)$ (note here that if $\mathfrak{g}$ is a subalgebra in $\mathfrak{u}_n$ and $\mathfrak{g}$ is isomorphic to $\mathfrak{su}_n$, then $\mathfrak{g}$ coincides with $\mathfrak{su}_n$, i.e., it consists exactly of matrices with zero trace).

Suppose now that $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h} \oplus \mathbb{C}$, where $\mathfrak{h}$ is a semisimple ideal in $\mathfrak{g}^{\mathbb{C}}$. Then, repeating the above argument for $\mathfrak{h}$ and taking into account that $\dim \mathfrak{h} \geq n^2 - 2n + 2$, we conclude that $\mathfrak{h} \simeq \mathfrak{sl}_n$ for $n \neq 4$ and, for $n = 4$, either $\mathfrak{h} \simeq \mathfrak{sl}_4$, or $\mathfrak{h} \simeq \mathfrak{sp}_4$. Therefore, for $n \neq 4$, $\mathfrak{g}^{\mathbb{C}} = \mathfrak{gl}_n$ and hence $\mathfrak{g} = \mathfrak{u}_n$, which implies that $G = U(n)$. For $n = 4$ we get that either $\mathfrak{g} = \mathfrak{u}_4$ (in which case $G = U(4)$), or $\mathfrak{g} \simeq \mathbb{R} \oplus \mathfrak{sp}_{2,0}$.

The lemma is proved. □

Proof of Theorem 2.1. Let $p \in M$ and let I_p denote the isotropy group of p in $\mathrm{Aut}(M)$. Since the complex dimension of M is n, the real dimension of any orbit of the action of $\mathrm{Aut}(M)$ on M does not exceed $2n$, and therefore we have $\dim I_p \geq n^2 - 2n + 3$.

Consider the mapping $\alpha_p : I_p \to GL(T_p(M), \mathbb{C})$:

$$\alpha_p(f) := df(p), \qquad f \in I_p.$$

The mapping α_p is a continuous group homomorphism (see, e.g., Lemma 1.1 of [8]) and thus is a Lie group homomorphism (see [23]). Since I_p is compact, there is a positive-definite Hermitian form h_p on $T_p(M)$ such that $\alpha_p(I_p) \subset U_{h_p}(n)$, where $U_{h_p}(n)$ is the group of complex linear transformations of $T_p(M)$ preserving the form h_p. We choose a basis in $T_p(M)$ such that h_p in this basis is given by the identity matrix.

By [3] and [13], the mapping α_p is one-to-one. Further, since $\dim I_p \geq n^2 - 2n + 3$ and I_p is compact (see [15]), $\alpha(I_p)$ is a compact subgroup of $U_{h_p}(n)$ of dimension at least $n^2 - 2n + 3$. We are now going to use Lemma 2.2.

Assume first that $n \neq 4$. Then we have that either $\alpha_p(I_p) = U_{h_p}(n)$, or $\alpha_p(I_p)^c = SU_{h_p}(n)$ (the latter denotes the subgroup of $U_{h_p}(n)$ consisting of matrices with determinant 1). The groups $U_{h_p}(n)$ and $SU_{h_p}(n)$ act transitively on the unit sphere in $T_p(M)$ and thus act transitively on directions in $T_p(M)$ (see [8] and [2] for terminology). Since M is non-compact (because the dimension of $\mathrm{Aut}(M)$ is positive—see [15]), the main result of [8] and its generalization in [2] applies. Thus M is biholomorphically equivalent to B^n (and therefore the possibility $\alpha_p(I_p)^c = SU_{h_p}(n)$ is in fact not realizable).

Suppose now that $n = 4$. If we have that either $\alpha_p(I_p) = U_{h_p}(4)$, or else $\alpha_p(I_p)^c = SU_{h_p}(4)$ for some $p \in M$, then by the above argument M is equivalent to B^4. Suppose now that the Lie algebra of $\alpha_p(I_p)$ is isomorphic to $\mathbb{R} \oplus \mathfrak{sp}_{2,0}$ for every $p \in M$. Then $\dim \alpha_p(I_p) = 11$ for any p. Since $\dim \mathrm{Aut}(M) \geq 19$, we have that in fact $\dim \mathrm{Aut}(M) = 19$, and thus M is homogeneous. Therefore, by [19], M is biholomorphically equivalent to a Siegel domain $D \subset \mathbb{C}^4$ of the first or second kind. Further, we note that any representation $\phi : \mathfrak{sp}_{2,0} \to \mathfrak{gl}_4$ is conjugate to the standard embedding of $\mathfrak{sp}_{2,0}$ into $\mathfrak{gl}_4$ by an element from $GL(4, \mathbb{C})$ (to see this, one can extend ϕ to a 4-dimensional representation of the complex Lie algebra $\mathfrak{sp}_4$ and notice that such a representation is unique up to conjugation by elements of $GL(4, \mathbb{C})$). Therefore $\phi(\mathfrak{sp}_{2,0})$ contains an element X such that $\exp(X) = -\mathrm{id}$, and thus $\alpha_p(I_p)$ contains $-\mathrm{id}$ for any p. Hence the domain D is in fact symmetric. It now follows from the explicit classification of symmetric Siegel domains (see [20]) that in fact there is no symmetric Siegel domain in $\mathbb{C}^4$ with automorphism group of dimension equal to 19.

The theorem is proved. □

If $\dim \mathrm{Aut}(M) \leq n^2 + 2$, then Theorem 2.1 does not hold. Indeed, for $M = B^{n-1} \times \Delta$, where Δ is the unit disc in $\mathbb{C}$, the dimension of the automorphism group is $n^2 + 2$. Below we show that this is indeed the only possibility up to biholomorphic equivalence.

THEOREM 2.3. *Let M be a connected hyperbolic manifold of dimension $n \geq 2$. If $\dim \mathrm{Aut}(M) = n^2 + 2$, then M is biholomorphically equivalent to $B^{n-1} \times \Delta$.*

First, the proof of Lemma 2.2 shows that the following holds.

LEMMA 2.4. *Let $U_h(n)$ be the group of linear transformations of a complex n-dimensional space V that preserve a positive-definite Hermitian form h on V, and let G be a Lie subgroup of $U_h(n)$ with $\dim G \geq n^2 - 2n + 2$, $n \geq 2$, $n \neq 4$. Then either $G = U_h(n)$, or $G^c = SU_h(n)$, or V splits into a sum of 1- and $(n-1)$-dimensional h-orthogonal complex subspaces V^1 and V^2 such that $G = U_{h^1}(1) \times U_{h^2}(n-1)$, where h^j is the restriction of h to V^j. For $n = 4$, G can also be any subgroup of $U_h(4)$ with Lie algebra isomorphic to either $\mathfrak{sp}_{2,0}$ or $\mathbb{R} \oplus \mathfrak{sp}_{2,0}$.*

Proof of Theorem 2.3. [‡] We will use the notation from the proof of Theorem 2.1 above. Let $p \in M$ and I_p be the isotropy group of p in $\mathrm{Aut}(M)$. Then we have $\dim I_p \geq n^2-2n+2$ and thus $\alpha_p(I_p)$ is a subgroup of U_{h_p} of dimension at least $n^2 - 2n + 2$. We now use Lemma 2.4. If, for some $p \in M$, we have that either $\alpha_p(I_p) = U_{h_p}(n)$ or $\alpha_p(I_p)^c = SU_{h_p}(n)$, then $\alpha_p(I_p)$ acts transitively on directions in $T_p(M)$. Hence, as in the proof of Theorem 2.1, M is biholomorphically equivalent to B^n; this is impossible since $\dim \mathrm{Aut}(M) = n^2 + 2$.

Further suppose that, for any point $p \in M$, $T_p(M)$ splits into the sum of 1- and $(n-1)$-dimensional h_p-orthogonal complex subspaces V_p^1 and V_p^2 such that $\alpha_p(I_p) = U_{h_p^1}(1) \times U_{h_p^2}(n-1)$. In particular, $\dim I_p = n^2-2n+2$ for all $p \in M$ and therefore M is homogeneous. Then, by [19], M is biholomorphically equivalent to a homogeneous Siegel domain D of the first or second kind. Since $\alpha_p(I_p)$ contains the transformation $-\mathrm{id}$ for all $p \in M$, the domain D is in fact symmetric. The theorem for $n \neq 4$ now follows from the explicit classification of symmetric Siegel domains (see [20]).

Suppose now that $n = 4$ and that, for some point $p \in M$, the Lie algebra of $\alpha_p(I_p)$ is isomorphic to either $\mathfrak{sp}_{2,0}$ or to $\mathbb{R} \oplus \mathfrak{sp}_{2,0}$. In the proof of Theorem 2.1 we noted that any embedding of $\mathfrak{sp}_{2,0}$ into $\mathfrak{gl}_4$ is conjugate by an element of $GL(4, \mathbb{C})$ to the standard one. Therefore $\alpha_p(I_p)$ contains a subgroup conjugate by an element of $GL(4, \mathbb{C})$ to $Sp_{2,0}$. Since $Sp_{2,0}$ acts transitively on the sphere of dimension 7, we get that $\alpha_p(I_p)$ acts transitively on directions in $T_p(M)$ and therefore, as in the proof of Theorem 2.1, M is biholomorphically equivalent to the unit ball which is impossible. □

[‡]We are grateful to K. Nakajima for valuable suggestions concerning the proof of this theorem.

REMARK. The argument in the last paragraph in the proof of Theorem 2.3 could also be used in the proof of Theorem 2.1 for the case $n = 4$ without referring to the classification theory of symmetric domains. For hyperbolic Reinhardt domains, Theorems 2.1 and 2.3 were obtained by a different argument in [6].

We now turn to the case of hyperbolic Reinhardt domains. An important observation that follows from [17] is that, for such a domain $D \subset \mathbb{C}^n$, $\dim \mathrm{Aut}(D)$ has the same parity as n. Therefore the next largest possible value that $\dim \mathrm{Aut}(D)$ can take is n^2.

THEOREM 2.5. ([6]) *Let $D \subset \mathbb{C}^n$ be a hyperbolic Reinhardt domain such that $\dim Aut(D) = n^2$. Then D is holomorphically equivalent to one of the following domains:*

(i) $\{z \in \mathbb{C}^n : r < |z| < R\}$, $0 \le r < R < \infty$;

(ii) Δ^3 *(here $n = 3$);*

(iii) $B^2 \times B^2$ *(here $n = 4$);*

(iv) $\{(z', z_n) \in \mathbb{C}^n : |z'|^2 + |z_n|^\beta < 1\}$, $\beta \in \mathbb{R}$, $\beta \neq 0, 2$;

(v) $\{(z', z_n) \in \mathbb{C}^n : |z'| < 1, r(1 - |z'|^2)^\alpha < |z_n| < R(1 - |z'|^2)^\alpha\}$, $\alpha \in \mathbb{R}$, $0 < r < R \le \infty$;

(vi) $\{(z', z_n) \in \mathbb{C}^n : re^{\beta|z'|^2} < |z_n| < Re^{\beta|z'|^2}\}$, *where* $0 < r < R \le \infty$, $\beta \in \mathbb{R}$, $\beta \neq 0$ *and, if* $R = \infty$, *then* $\beta > 0$.

Thus we have been able to classify all hyperbolic Reinhardt domains that have automorphism groups of dimension n^2 and greater. It is certainly not possible to obtain explicit classification results similar to the ones above for any value of $\dim \mathrm{Aut}(D)$ between n and $n^2 - 2$. Nevertheless, we have the following result.

Assume for simplicity that all domains are C^1-smoothly bounded and denote by $\mathcal{R}^1(n)$ the collection of all such Reinhardt domains in $\mathbb{C}^n$. Let $S(n)$ be the set of all dimensions of the automorphism groups of domains from $\mathcal{R}^1(n)$. A number $N \in S(n)$ is called a *non-compact dimension* if any Reinhardt domain $D \in \mathcal{R}^1(n)$ with $\dim \mathrm{Aut}(D) = N$ has non-compact automorphism group. Otherwise a number $N \in S(n)$ is called a *compact dimension*. Let $h(n)$ and $c(n)$ denote the numbers of non-compact and compact dimensions respectively.

THEOREM 2.6. ([6]) *We have*

(i) $\liminf_{n\to\infty} \frac{h(n)}{n} \geq 1$;

(ii) $\lim_{n\to\infty} \frac{c(n)}{n^2} \to \frac{1}{2}$.

The second statement in Theorem 2.6 means that asymptotically compact dimensions fill the whole of $S(n)$. The first statement gives that the probability of selecting a non-compact dimension from $S(n)$ is asymptotically of order $1/n$; this is positive information, for it shows that the number of domains with non-compact automorphism group is substantial among all Reinhardt domains.

3. Closing Remarks

In this short paper we have attempted to give the reader some idea of questions that are currently being explored in the subject of automorphism groups of domains in complex space. Clearly each new result begs further questions, and we intend to explore these in future work.

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Alexander V. Isaev
Centre for Mathematics and Its Applications
The Australian National University
Canberra, ACT 0200, AUSTRALIA
E-mail: Alexander.Isaev@anu.edu.au

Steven G. Krantz
Department of Mathematics
Campus Box 1146
Washington University in St. Louis
One Brookings Drive
St. Louis, Missouri 63130, USA
E-mail: sk@math.wustl.edu