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A REMARK ON GENERALIZED COMPLEX ELLIPSOIDS WITH SPHERICAL BOUNDARY POINTS

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Abstract. It is well-known that there is no analogue to the Riemann mapping theorem in the higher dimensional case. Therefore, it would be an interesting question to find sufficient conditions for domains to be biholomorphically equivalent to the unit ball. In this paper, we investigate this question in the case where the given domains are generalized complex ellipsoids with spherical boundary points.

1. Introduction

It is well-known that there is no analogue to the Riemann mapping theorem in $\mathbf{C}^n$ ($n > 1$), that is, there are many simply connected domains D in $\mathbf{C}^n$ that are not biholomorphically equivalent to the unit ball B^n in $\mathbf{C}^n$. Therefore, it would be an interesting question to find sufficient conditions for domains D to be biholomorphically equivalent to B^n. In connection with this, one can see several articles. For instance, Pinchuk [10] and Huang-Ji [5] obtained the following Riemann mapping type theorem. In order to state their results, let us recall some definitions. Let p be a point of ∂D, the boundary of D. Then, the point p is said to be a *spherical boundary point* of D if there is an open neighborhood U of p in $\mathbf{C}^n$ and a biholomorphic mapping $f : U \to f(U) \subset \mathbf{C}^n$ such that $f(U \cap \partial D) \subset \partial B^n$ and $f(U \cap D) \subset B^n$. Moreover, we say that ∂D is *algebraic* if it is defined by a real polynomial. Now their results may be stated as follows:

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THEOREM P-H-J. *Let D be a bounded strictly pseudoconvex domain in $\mathbf{C}^n$ with connected real analytic boundary ∂D. Then we have:*

(I) (Pinchuk [10]). *If ∂D is simply connected and D has a spherical boundary point p, then D is biholomorphically equivalent to B^n.*

(II) (Huang-Ji [5]). *If ∂D is algebraic and D has a spherical boundary point p, then D is biholomorphically equivalent to B^n.*

Note that the simply connectedness of ∂D is not assumed in (II).

In view of these results, it would be natural to ask the following question: *Given a domain D in $\mathbf{C}^n$ with a spherical boundary point, under what additional hypotheses is D biholomorphically equivalent to B^n?* Our purpose of this note is to study this question in the case when D is a *generalized complex ellipsoid*

$$E(n; n_1, \ldots, n_s; p_1, \ldots, p_s) = \left\{ (z_1, \ldots, z_s) \in \mathbf{C}^{n_1} \times \cdots \times \mathbf{C}^{n_s} \;\middle|\; \sum_{i=1}^{s} |z_i|^{2p_i} < 1 \right\}$$

in $\mathbf{C}^n = \mathbf{C}^{n_1} \times \cdots \times \mathbf{C}^{n_s}$, where $0 < p_1, \ldots, p_s \in \mathbf{R}$, $0 \leqq n_1, \ldots, n_s \in \mathbf{Z}$ with $n = n_1 + \cdots + n_s$ and $|\cdot|$ denotes the Euclidean norm on $\mathbf{C}^{n_i}$. Here, without loss of generality, we always assume that

$$(*) \qquad p_1 = 1 \quad \text{and} \quad p_i \neq 1,\ n_i > 0 \quad \text{for} \quad i = 2, \ldots, s.$$

Also, it is understood that $p_1 = 1$ does not appear if $n_1 = 0$ and this domain is the unit ball B^n if $s = 1$. Then we can prove the following theorem, which was announced at the International Conference on Several Complex Variables in Seoul, Korea, 1998:

THEOREM. *Let E be a generalized complex ellipsoid in $\mathbf{C}^n$ as above. Assume that $s \geqq 2$ and $n_2, \ldots, n_s \geqq 2$. Then E does not have a spherical boundary point. In particular, among the class of all generalized complex ellipsoids in $\mathbf{C}^n = \mathbf{C}^{n_1} \times \cdots \times \mathbf{C}^{n_s}$ with $n_1, \ldots, n_s \geqq 2$, the unit ball B^n is the only one that has a spherical boundary point.*

In the special case where all p_i's are integers, our theorem is an immediate consequence of Dini and Selvaggi Primicerio [2], [3]. However, if some of p_i's are not integers, then ∂E is not smooth and E is not geometrically convex, in general. Hence, their technique is not directly applicable to our case. Finally, it should be remarked that the assumption $n_2, \dots, n_s \geqq 2$ in our theorem cannot be dropped as one may see in the examples such as $E(\alpha) = \{(z, w) \in \mathbf{C}^{n-1} \times \mathbf{C} \mid |z|^2 + |w|^{2\alpha} < 1\}$ with arbitrary $0 < \alpha \neq 1$. Indeed, every point $p_o = (z_o, w_o) \in \partial E(\alpha)$ with $w_o \neq 0$ is a spherical boundary point, but obviously we have $E(\alpha) \neq B^n$ in this case.

In Section 2 below, we give a proof of our theorem and, in Section 3, we make a conjecture and pose some questions, that arise from the characterization problem of generalized complex ellipsoids from the viewpoint of biholomorphic automorphism groups.

2. Proof of Theorem

We will proceed along the same line as in the proof of [6; Lemma]. Although there are some overlaps with that paper [6], we carry out the proof in detail for the sake of completeness and self-containedness.

Before undertaking the proof, we need to introduce the following notation: For a point $z = (z_1, \dots, z_s) \in \mathbf{C}^{n_1} \times \cdots \times \mathbf{C}^{n_s} = \mathbf{C}^n$ and for the generalized complex ellipsoid E appearing in the theorem, we set

$$z = (z_1, \dots, z_s) = (u_1, \dots, u_n), \quad \partial^* E = \{z \in \partial E \mid |z_2| \cdots |z_s| \neq 0\}$$

and denote by $\mathrm{Aut}_o(E)$ the identity component of the Lie group $\mathrm{Aut}(E)$ consisting of all biholomorphic automorphisms of E. Then, by using the facts in the previous paper [7; Section 1], the following assertions are easily proved:

(2.1) $\partial^* E$ is a connected, strictly pseudoconvex, real analytic hypersurface in $\mathbf{C}^n$ consisting of all strictly pseudoconvex boundary points of E. Moreover, it is simply connected, since $n_2, \dots, n_s \geqq 2$ in our case [4; p. 346].

(2.2) $\mathrm{Aut}_o(E)$ can be regarded as a subgroup of $\mathrm{Aut}(B^{n_1} \times \mathbf{C}^{n_2} \times \cdots \times \mathbf{C}^{n_s})$.

(2.3) $\partial^* E$ is a subset of $B^{n_1} \times \mathbf{C}^{n_2} \times \cdots \times \mathbf{C}^{n_s}$ invariant under $\mathrm{Aut}_o(E)$ and $\mathrm{Aut}_o(E)$ acts on $\partial^* E$ as a real analytic CR-automorphism group of $\partial^* E$.

With this preparation, we shall start the proof of our theorem. The proof is by contradiction, so we assume that E has a spherical boundary point p. Notice that p is necessarily a strictly pseudoconvex boundary point of E; hence $p \in \partial^* E$. Therefore, there is an open neighborhood U of p in $\mathbf{C}^n$ and a biholomorphic mapping $f : U \to f(U) \subset \mathbf{C}^n$ such that

$$U \cap \partial E = U \cap \partial^* E, \quad f(U \cap \partial^* E) \subset \partial B^n \text{ and } f(U \cap E) \subset B^n.$$

Since $\partial^* E$ is a connected and simply connected, strictly pseudoconvex, real analytic hypersurface in $\mathbf{C}^n$ by (2.1), it follows from a result of Pinchuk [10], [11; p. 193] that:
(2.4) f extends to a locally biholomorphic mapping F defined on some connected open neighborhood V of $\partial^* E$ in $\mathbf{C}^n$ such that $F(\partial^* E) \subset \partial B^n$ and $F(V \cap E) \subset B^n$.

Once it is shown that this F induces a biholomorphic equivalence between E and B^n, it follows from a result of Naruki [9] that $p_2, \ldots, p_s = 1$ and so $E = B^n$ as sets. This contradicts our assumption $p_2, \ldots, p_s \neq 1$. Thus our proof is now reduced to showing the following:
(**) *F extends to a biholomorphic mapping from E onto B^n.*
We will prove this assertion by several steps.

1) *F extends to a holomorphic mapping $\widetilde{F}$ from E into B^n.* To prove this, take an arbitrary r with $0 < r < 1$ and put $K_r = \{z \in \partial E \mid |z_i| \geqq r \ (2 \leqq i \leqq s)\}$. Notice that there exists a small $\delta > 0$ such that $K_r \neq \emptyset$ for $0 < r \leqq 2\delta$. Since $K_r \subset \partial^* E \subset V$ and K_r is compact in V for $0 < r < \delta$, one can choose a small $\varepsilon = \varepsilon(r) > 0$ in such a way that

$$U_{r,\varepsilon} := \left\{ z \in E \;\middle|\; 1 - \varepsilon < \sum_{i=1}^{s} |z_i|^{2p_i} < 1, \quad |z_i| > r \ (2 \leqq i \leqq s) \right\} \subset V.$$

Clearly, $U_{r,\varepsilon}$ is a bounded Reinhardt domain in $\mathbf{C}^n$. Moreover, since $n_2, \ldots, n_s \geqq 2$, we have

$$U_{r,\varepsilon} \cap \{u = (u_1, \ldots, u_n) \in \mathbf{C}^n \mid u_j = 0\} \neq \emptyset \quad \text{for} \quad j = 1, \ldots, n.$$

Hence, by a well-known fact [8; p. 15] every component function F_j of F has a holomorphic extension F_j^r to the smallest complete Reinhardt domain $\widehat{U}_{r,\varepsilon}$ in $\mathbf{C}^n$ containing $U_{r,\varepsilon}$. On the other hand, by simple (but a little bit complicated) computations one can check that $\widehat{U}_{r,\varepsilon}$ is defined by the following inequalities:

$$\sum_{i=1}^{s} |z_i|^{2p_i} < 1 \quad \text{and} \quad \sum_{i \notin I} |z_i|^{2p_i} + \sum_{i \in I} r^{2p_i} < 1$$

for all subsets $I = \{i_1, \dots, i_k\}$ of the set $\{2, \dots, s\}$ with $2 \leqq i_1 < \cdots < i_k \leqq s$, $1 \leqq k \leqq s-1$. Hence, putting $E_r = \widehat{U}_{r,\varepsilon}$ for simplicity, we have seen that $F = (F_1, \dots, F_n)$ has a holomorphic extension $F^r := (F_1^r, \dots, F_n^r)$ to $E_r \cup V$. Note that $E_r \subset E_s$ for $0 < s < r < \delta$, $\bigcup_{0<r<\delta} E_r = E$ and that the holomorphic extensions F^r are uniquely determined by the values of F on a small neighborhood of a given point of $K_{2\delta} \subset \bigcap_{0<r<\delta} K_r$. Then, by standard argument, one can define a holomorphic extension $\widetilde{F} : E \cup V \to \mathbf{C}^n$ of $F : V \to \mathbf{C}^n$.

Now we wish to show that $\widetilde{F}(E) \subset B^n$. We first claim that $\widetilde{F}(E) \subset \overline{B}^n$, the closure of B^n in $\mathbf{C}^n$. Indeed, assume the contrary. Then there exists a point $z^o = (z_1^o, \dots, z_s^o) \in E$ such that $\widetilde{F}(z^o) \notin \overline{B}^n$. By taking a nearby point if necessary, one may assume that $|z_2^o| \cdots |z_s^o| > 0$. Set $H(z^o) = \{(z_1^o, \dots, z_{s-1}^o)\} \times \mathbf{C}^{n_s}$ and $E(z^o) = E \cap H(z^o)$. Then $E(z^o)$ can be regarded as an open ball in $\mathbf{C}^{n_s}$ containing the point z^o and the set $U(z^o) := H(z^o) \cap (E \cup V)$ is an open neighborhood of the closure $\overline{E(z^o)}$ in $\mathbf{C}^{n_s}$. Consider now the continuous plurisubharmonic function $\psi : z_s \mapsto -1 + |\widetilde{F}(z_1^o, \dots, z_{s-1}^o, z_s)|^2$ defined on $U(z^o)$. It follows then from (2.4) that $\psi(\partial E(z^o)) = 0$ and $\psi(z_s) < 0$ on $E(z^o) \cap V$. This, combined with the maximum principle for plurisubharmonic functions, guarantees that $\psi(z^o) < 0$. However, this means $\widetilde{F}(z^o) \in B^n$, a contradiction; and hence $\widetilde{F}(E) \subset \overline{B}^n$. Therefore, considering the non-constant, continuous plurisubharmonic function $\widetilde{\psi} : z \mapsto -1 + |\widetilde{F}(z)|^2$ defined on E, we have now that $0 \geqq \sup_{u \in E} \widetilde{\psi}(u) > \widetilde{\psi}(z)$ for every point $z \in E$, i.e., $\widetilde{F}(E) \subset B^n$.

2) *There exists a locally injective, real analytic homomorphism* $\Phi : \mathrm{Aut}_o(E) \to \mathrm{Aut}(B^n)$ *such that* $\Phi(\sigma) \circ \widetilde{F} = \widetilde{F} \circ \sigma$ *on* E *for all* $\sigma \in \mathrm{Aut}_o(E)$. Indeed, take an arbitrary $\sigma \in \mathrm{Aut}_o(E)$. By virtue of (2.2), (2.3) and (2.4), one can choose an open neighborhood W of the spherical boundary point $p \in \partial^* E$ so small that $W \cup \sigma(W) \subset V$ and $\widetilde{F}$ is injective on W and on $\sigma(W)$. Let us consider the biholomorphic mapping $\Psi := \widetilde{F} \circ \sigma \circ (\widetilde{F}|W)^{-1} : \widetilde{F}(W) \to \widetilde{F}(\sigma(W))$. Then Ψ gives a homeomorphism from $\widetilde{F}(W) \cap \overline{B}^n$ onto $\widetilde{F}(\sigma(W)) \cap \overline{B}^n$ such that $\Psi(\widetilde{F}(W) \cap \partial B^n) \subset \partial B^n$. Hence, by an extension theorem due to Alexander [1] or Rudin [12; p. 311] we obtain an element $\widetilde{\Psi} \in \mathrm{Aut}(B^n)$ such that $\widetilde{\Psi}(z) = \Psi(z)$ for all $z \in \widetilde{F}(W \cap E)$. Note that $W \cap E$ and

$\widetilde{F}(W \cap E)$ are non-empty open subsets of E and B^n, respectively. Then, by the principle of analytic continuation, we have that $\widetilde{\Psi} \circ \widetilde{F} = \widetilde{F} \circ \sigma$ on E and $\widetilde{\Psi}$ is uniquely determined by σ. Accordingly, one can define a mapping

$$\Phi : \mathrm{Aut}_o(E) \to \mathrm{Aut}(B^n)$$

by setting $\Phi(\sigma) = \widetilde{\Psi}$ so that $\Phi(\sigma) \circ \widetilde{F} = \widetilde{F} \circ \sigma$ on E for all $\sigma \in \mathrm{Aut}_o(E)$.

It is easy to check that Φ is a group homomorphism. Once it is shown that Φ is continuous at the identity element id_E of $\mathrm{Aut}_o(E)$, it follows that Φ is real analytic on $\mathrm{Aut}_o(E)$ (cf. [4; p. 117]). Since the topology of $\mathrm{Aut}_o(E)$ satisfies the second axiom of countability, we have only to show that Φ is sequentially continuous at id_E. For this let us take an arbitrary sequence $\{\sigma_\nu\}$ in $\mathrm{Aut}_o(E)$ which converges to id_E and assume that $\{\Phi(\sigma_\nu)\}$ does not converge to the identity element id_{B^n} of $\mathrm{Aut}(B^n)$. Passing to a subsequence, we may assume that there is an open neighborhood O of id_{B^n} in $\mathrm{Aut}(B^n)$ such that $\Phi(\sigma_\nu) \notin O$ for all ν. Pick an arbitrary point $x \in E$. Then $\lim_{\nu\to\infty} \Phi(\sigma_\nu)(\widetilde{F}(x)) = \lim_{\nu\to\infty} \widetilde{F}(\sigma_\nu(x)) = \widetilde{F}(x) \in B^n$, which implies that $\{\Phi(\sigma_\nu)(\widetilde{F}(x))\}$ lies in a compact subset of B^n. Hence, after taking a subsequence if necessary, we may assume that $\{\Phi(\sigma_\nu)\}$ converges to some element $g \in \mathrm{Aut}(B^n)$ (cf. [8; p. 82]). Since $g \notin O$, we see that $g \neq \mathrm{id}_{B^n}$. On the other hand, we have $g(\widetilde{F}(z)) = \lim_{\nu\to\infty} \Phi(\sigma_\nu)(\widetilde{F}(z)) = \lim_{\nu\to\infty} \widetilde{F}(\sigma_\nu(z)) = \widetilde{F}(z)$ for all $z \in W \cap E$; consequently, $g = \mathrm{id}_{B^n}$ by analytic continuation. This is a contradiction. Therefore, Φ is continuous at id_E, as desired.

Finally we claim that Φ is locally injective. It suffices to prove that Φ is injective in some neighborhood O of id_E. To this end, let us select a small open neighborhood W of the point $p \in \partial^* E$ in $\mathbf{C}^n$ and non-empty open subsets W_1, W_2 of $W \cap E$ with the properties: $\widetilde{F}$ is injective on W, and W_1 is a relatively compact subset of W_2. We claim that $O = \{\sigma \in \mathrm{Aut}_o(E) \mid \sigma(\overline{W_1}) \subset W_2\}$ is what is required. Indeed, it is clear that O is an open neighborhood of id_E in $\mathrm{Aut}_o(E)$. Moreover, assume that $\Phi(\sigma_1) = \Phi(\sigma_2)$ for $\sigma_1, \sigma_2 \in O$. It follows that $\widetilde{F}(\sigma_1(z)) = \Phi(\sigma_1)(\widetilde{F}(z)) = \Phi(\sigma_2)(\widetilde{F}(z)) = \widetilde{F}(\sigma_2(z))$ for all $z \in E$. Since $\widetilde{F}$ is injective on $W_2 \subset W$ and since $\sigma_1(z), \sigma_2(z) \in W_2$ for all $z \in W_1$, this says that $\sigma_1 = \sigma_2$ on W_1; and hence $\sigma_1 = \sigma_2$ on E by analytic continuation. Therefore, we have shown that Φ is locally injective on $\mathrm{Aut}_o(E)$, completing the proof of 2).

Before proceeding further, we need some preparation. First, notice that B^n is homogeneous and each element $g \in \mathrm{Aut}(B^n)$ extends to a biholomorphic mapping defined in an open neighborhood of $\overline{B}^n$. Thus, shrinking the neighborhood V of $\partial^* E$ and replacing $\widetilde{F}$ by a suitable mapping of the form $g \circ \widetilde{F}$ with some $g \in \mathrm{Aut}(B^n)$, if necessary, we may assume that the holomorphic mapping $\widetilde{F} : E \cup V \to \mathbf{C}^n$ satisfies an additional condition $\widetilde{F}(o) = o$, where o stands for the origin of $\mathbf{C}^n$. Next, let us consider the toral subgroups T_E and T_{B^n} of $\mathrm{Aut}_o(E)$ and $\mathrm{Aut}(B^n)$, respectively, induced by the rotations R_θ on $\mathbf{C}^n$ as follows:

$$R_\theta \ : \ (u_1, \dots, u_n) \longmapsto \big((\exp\sqrt{-1}\theta_1)u_1, \dots, (\exp\sqrt{-1}\theta_n)u_n\big)$$

for $\theta = (\theta_1, \dots, \theta_n) \in \mathbf{R}^n$. Then $\Phi(T_E)(o) = \Phi(T_E)(\widetilde{F}(o)) = \widetilde{F}(T_E(o)) = \widetilde{F}(o) = o$, which says that $\Phi(T_E)$ is contained in the unitary group $U(n)$ of degree n (the isotropy subgroup of $\mathrm{Aut}(B^n)$ at the origin o). Since $\Phi(T_E)$ as well as T_{B^n} is now a maximal torus in $U(n)$ by 2), it is well-known that they are conjugate to each other in $U(n)$, that is, there exists an element $\tau \in U(n)$ such that $\tau \cdot \Phi(T_E) \cdot \tau^{-1} = T_{B^n}$. Thus, considering $\tau \circ \widetilde{F}$, $\tau \circ \Phi \circ \tau^{-1}$ instead of $\widetilde{F}$, Φ if necessary, we may further assume that $\Phi(T_E) = T_{B^n}$. Under these assumptions, we claim the following:

3) $\widetilde{F} = \big(\widetilde{F}_1, \dots, \widetilde{F}_n\big) : E \to B^n$ *can be written in the form*

$$\widetilde{F}_j(u_1, \dots, u_n) \ = \ A_j(u_1)^{a_{j1}} \cdots (u_n)^{a_{jn}} \quad (1 \leqq j \leqq n),$$

where $\mathbf{C} \ni A_j \neq 0$, $\mathbf{Z} \ni a_{jk} \geqq 0$ *and* $a_{j1} + \cdots + a_{jn} \geqq 1$ *for all* j, k. Indeed, since the restriction $\Phi | T_E$ gives rise to a local isomorphism between T_E and T_{B^n} and since $\Phi(\sigma)(\widetilde{F}(u)) = \widetilde{F}(\sigma(u))$ on E for all $\sigma \in T_E$, there exists a non-singular real $n \times n$ matrix $A = (a_{jk})$ such that

$$(2.5) \quad \big(\exp\sqrt{-1}(a_{j1}\theta_1 + \cdots + a_{jn}\theta_n)\big)\widetilde{F}_j(u) \ = \ \widetilde{F}_j\big(R_\theta(u)\big) \quad \text{on} \quad E$$

for every $j = 1, \dots, n$ and all $\theta = (\theta_1, \dots, \theta_n) \in \mathbf{R}^n$. On the other hand, being a holomorphic function on the holomorphically convex Reinhardt domain E with center o, each $\widetilde{F}_j$ can be expanded in E in a convergent power series $\widetilde{F}_j(u) = \sum_{|\nu|=0}^{\infty} A_\nu^j u^\nu$. Substituting this in (2.5) and comparing the coefficients of u^ν, we have

$$A_\nu^j \exp\sqrt{-1}(a_{j1}\theta_1 + \cdots + a_{jn}\theta_n) \ = \ A_\nu^j \exp\sqrt{-1}(\nu_1\theta_1 + \cdots + \nu_n\theta_n)$$

for all ν and all θ. This implies that $A_\nu^j = 0$ if $\nu \neq (a_{j1}, \ldots, a_{jn})$ for $j = 1, \ldots, n$. Combining this with the fact that $\widetilde{F}_j$ is not constant by (2.4), we conclude that each $\widetilde{F}_j$ has the form required in 3).

4) *$\widetilde{F} : E \to B^n$ is a proper holomorphic mapping from E onto B^n with $\widetilde{F}^{-1}(o) = \{o\}$.* Thanks to the fact 3), in order to prove this, one may regard $\widetilde{F}$ as a holomorphic mapping defined on the whole space $\mathbf{C}^n$. Then we have $\widetilde{F}(\partial E) \subset \partial B^n$, since $\widetilde{F}(\partial^* E) \subset \partial B^n$ and $\partial^* E$ is dense in ∂E. This implies that $\widetilde{F} : E \to B^n$ is a proper holomorphic mapping and $\widetilde{F}^{-1}(o)$ is to be a finite set of points (cf. [12; p. 300]). Moreover, we claim that $\widetilde{F}^{-1}(o) = \{o\}$. Indeed, assume that $\widetilde{F}^{-1}(o)$ contains a point $u^o \neq o$. Then $\widetilde{F}(T_E(u^o)) = \Phi(T_E)(\widetilde{F}(u^o)) = T_{B^n}(o) = o$, which says that $T_E(u^o) \subset \widetilde{F}^{-1}(o)$. But, this is impossible, since the orbit $T_E(u^o)$ is a positive dimensional real torus imbedded in E and $\widetilde{F}^{-1}(o)$ is a finite subset of E. Therefore we conclude that $\widetilde{F}^{-1}(o) = \{o\}$, as desired.

5) *$\widetilde{F} : E \to B^n$ is, in fact, a biholomorphic mapping.* Keeping the fact 3) in mind, we define the complex analytic subvariety V_j of E by

$$V_j \ = \ \left\{ u \in E \;\middle|\; (u_1)^{a_{j1}} \cdots (u_n)^{a_{jn}} \ = \ 0 \right\} \quad \text{for} \quad j = 1, \ldots, n.$$

It is clear that each V_j is a non-empty open subset of the union of finitely many coordinate hyperplanes in $\mathbf{C}^n$. Moreover, by virtue of 4) we know that $\dim(V_1 \cap \cdots \cap V_n) = \dim \widetilde{F}^{-1}(o) = 0$. Now we assert the following:

(2.6) For each k, $1 \leqq k \leqq n$, there exists a unique $j = j(k)$ such that $a_{jk} \neq 0$ and $a_{lk} = 0$ for all $l \neq j$.

Indeed, assume first $a_{jk} = 0$ for all $j = 1, \ldots, n$. This means that $\widetilde{F}$ is independent on the variable u_k by 3), which is impossible by (2.4). Assume next that $a_{jk} \neq 0$, $a_{lk} \neq 0$ for some j, l with $j \neq l$. Then $V_j \cap V_l$ contains the complex analytic subset $E \cap \{u \in \mathbf{C}^n \,|\, u_k = 0\}$ of E of dimension $n - 1$; and hence $\dim(V_1 \cap \cdots \cap V_n) \geqq 1$. This is a contradiction; proving our assertion (2.6). Recall here that, for every $j = 1, \dots, n$, we have $a_{jk} \neq 0$ for some k by 3). Then, this combined with the assertion (2.6) guarantees that $\widetilde{F}$ has an expression

$$\widetilde{F}(u) = \left(A_1(u_{\sigma(1)})^{a_{1\sigma(1)}}, \dots, A_n(u_{\sigma(n)})^{a_{n\sigma(n)}}\right), \quad u \in \mathbf{C}^n$$

where σ is a permutation of $\{1, \dots, n\}$. Note that, for every $j = 1, \dots, n$, there exists a point $u^o \in \partial^* E$ whose j-th component $u_j^o = 0$, since $n_2, \dots, n_s \geqq 2$ by our assumption. Then, the fact (2.4) that the holomorphic Jacobian of $\widetilde{F}$ does not vanish at such a point u^o yields at once that $a_{j\sigma(j)} = 1$ for every $j = 1, \dots, n$. Therefore, $\widetilde{F} : E \to B^n$ is a biholomorphic mapping, completing the proof of our assertion $(**)$. We have thus proved the theorem.

3. A conjecture

We finish this note by a conjecture and related questions. Let us now consider two generalized complex ellipsoids

$$E := E(n; n_1, \dots, n_s; p_1, \dots, p_s) \ \text{ and } \ \widetilde{E} := E(n; m_1, \dots, m_t; q_1, \dots, q_t)$$

in $\mathbf{C}^n$. Then, under the same assumption $(*)$ as in the introduction, we have the following:

Conjecture. Let $x \in \partial E$, $\tilde{x} \in \partial \widetilde{E}$ and U, $\widetilde{U}$ open neighborhoods of $x, \tilde{x}$ in $\mathbf{C}^n$, respectively. Assume that

(1) n_i, $m_j \geqq 2$ for i, $j \geqq 2$, if s, $t \geqq 2$;
(2) $f : U \to \widetilde{U}$ is a biholomorphic mapping such that $f(U \cap \partial E) \subset \partial \widetilde{E}$ and $f(U \cap E) \subset \widetilde{E}$.

Then f extends to a biholomorphic mapping F from E onto $\widetilde{E}$.

Once it is shown that this is true, then one would obtain the same characterization theorem as in [6; Theorem 2] for any generalized complex ellipsoids in $\mathbf{C}^n$.

By the proof of our theorem, the conjecture is true in the special case where $\widetilde{E}$ is the unit ball B^n; and, in fact, we have proved that $E = B^n = \widetilde{E}$ and F is an automorphism of B^n in this case. In the situation when all p_i's and q_j's are positive integers, Dini and Selvaggi Primicerio's result [2] confirms this conjecture. Moreover, it is shown in [6; Theorem 1] that this is also true for the class of generalized complex ellipsoids $E(k, \alpha) := E(n; k, n-k; 1, \alpha)$ with $0 < k \leqq n-2$, $0 < \alpha \in \mathbf{R}$. In its proof, we used efficiently the following fact on the structure of $E(k, \alpha)$ with $\alpha \neq 1$: Let $\partial^* E(k, \alpha)$ be the set of all strictly pseudoconvex boundary points of $E(k, \alpha)$. Then we have

(3.1) $\partial^* E(k, \alpha)$ is a connected and simply connected, strictly pseudoconvex real analytic hypersurface in $\mathbf{C}^n$ without umbilical points in the sense of CR-geometry;

(3.2) $\mathrm{Aut}(E(k, \alpha))$ acts transitively on $\partial^* E(k, \alpha)$ as a CR-automorphism group of $\partial^* E(k, \alpha)$.

Hence, Webster's CR-invariant Riemannian metric g can be defined on the whole space $\partial^* E(k, \alpha)$ and $(\partial^* E(k, \alpha), g)$ is a complete real analytic Riemannian manifold. This played an essential role in the proof of [6; Theorem 1].

In view of this fact, we would like to pose the following questions: For a given generalized complex ellipsoid E satisfying the assumption $(*)$ in the introduction, we denote by $\partial^* E$ and $U(\partial^* E)$ the sets of strictly pseudoconvex boundary points of E and of umbilical points of $\partial^* E$, respectively. Assume that $s \geqq 2$ and $n_2, \ldots, n_s \geqq 2$. Then

(Q.1) *is $U(\partial^* E)$ always the empty set?*;

(Q.2) *if $U(\partial^* E)$ is empty, is it true that Webster's metric on $\partial^* E$ is complete?*

At least, we know that $U(\partial^* E)$ cannot contain a non-empty open subset of $\partial^* E$. Indeed, assume that there exists a non-empty open subset O of $\partial^* E$ contained in $U(\partial^* E)$. Then O must be locally CR-equivalent to the unit sphere ∂B^n (cf. [13; p. 213]). This combined with our theorem yields that $E = B^n$ as sets, a contradiction.

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