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INVARIANT METRICS AND COMPLETENESS

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ABSTRACT. We discuss completeness with respect to the Carathéodory distance, the Kobayashi distance and the Bergman distance, respectively.

I. Definitions

By E we denote the open unit disc in the complex plane $\mathbb{C}$, i.e., $E: =\{\lambda \in \mathbb{C}\colon |\lambda| < 1\}$. For two points $\lambda_1, \lambda_2 \in E$,

$$p(\lambda_1,\lambda_2)\colon = (1/2)\log\frac{1+m(\lambda_1,\lambda_2)}{1-m(\lambda_1,\lambda_2)}, \text{ where } m(\lambda_1,\lambda_2)\colon = \left|\frac{\lambda_1-\lambda_2}{1-\bar{\lambda}_2\lambda_1}\right|,$$

is called the *Poincaré* (or *hyperbolic*) distance on E.

The main object of our interest are *holomorphically contractible systems of functions* or *pseudodistances*:
Let $\mathcal{G}$ denote the set of all domains in all $\mathbb{C}^n$'s (or the set of all connected complex manifolds, or, even more general, the set of all complex spaces). A family $\underline{d}\colon = (d_G)_{G\in\mathcal{G}}$ of functions, respectively pseudodistances, $d_G\colon G\times G \longrightarrow \mathbb{R}_{\geq 0}$ is called *holomorphically contractible* if
1) $d_E = p$ (normalization),
2) $d_{G_2}(f(z_1), f(z_2)) \leq d_{G_1}(z_1,z_2)$, whenever $G_j \in \mathcal{G}$, $f \in \mathcal{O}(G_1, G_2)$, and $z_1, z_2 \in G_1$.

To avoid the "$\tanh^{-1}$" we sometimes use another normalization, namely m instead of p. We put $d_G =\colon p(0, d_G^*) = \tanh^{-1} d_G^*$.

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There exists a smallest and also a largest holomorphically contractible family, namely:
a) $d_G = c_G$, *the Carathéodory pseudodistance on* G, $G \in \mathcal{G}$, where

$$\begin{aligned} c_G(z,w) \colon &= \sup\{p(f(z),f(w)) \colon f \in \mathcal{O}(G;E)\} \\ &= p(0, c_G^*(z,w)), \quad z,w \in G, \end{aligned}$$

where $c_G^*(z,w) = \sup\{|f(w)| \colon f \in \mathcal{O}(G,E), f(z) = 0\}$.
b) $d_G = \widetilde{k}_G$, *the Lempert function on* G, $G \in \mathcal{G}$, where

$$\begin{aligned} \widetilde{k}_G(z,w) \colon &= \inf\{p(\lambda_1,\lambda_2) \colon \exists f \in \mathcal{O}(E,G) \colon f(\lambda_1) = z,\ f(\lambda_2) = w\} \\ &= p(0, \widetilde{k}_G^*(z,w)), \quad z,w \in G, \end{aligned}$$

where $\widetilde{k}_G^*(z,w) = \inf\{|\lambda| \colon \exists \varphi \in \mathcal{O}(E,G) \colon \varphi(0) = z,\ \varphi(\lambda) = w\}$.
Then the following is true:
Whenever $\underline{d} = (d_G)_{G\in\mathcal{G}}$ is a holomorphically contractible system of functions, then $c_G \leq d_G \leq \widetilde{k}_G$.

REMARK. a) c_G is a pseudodistance and it is continuous in both variables;
b) $\widetilde{k}_G$, in general, does not satisfy the triangle-inequality; $\widetilde{k}_G$ is, in general, not continuous but it is upper semicontinuous.

To improve the situation in b) we define

$$k_G \colon = \text{ the greatest pseudodistance below of } \widetilde{k}_G.$$

k_G is called *the Kobayashi pseudodistance on* G.

REMARK. The system $(k_G)_{G\in\mathcal{G}}$ is the largest holomorphically contractible family of pseudodistances; k_G is continuous in both variables.

There is another family of functions which will be important for further discussions, namely the family of all *pluricomplex Green functions*: for $\mathcal{G} \ni G \ni a, z$ we put

$$\begin{aligned} g_G^*(a,z) \colon = \sup\{u(z) \colon u \colon G \longrightarrow [0,1),\ &\log u \in \mathcal{PSH}(G), \\ &u \leq C\|\cdot - a\| \text{ near } a\}, \end{aligned}$$

where $\mathcal{PSH}(G)$ denotes the set of all plurisubharmonic functions on G.
Then $d_G \colon = \tanh^{-1} g_G^*$ leads to a holomorphically contractible system of functions.

REMARK. g_G^* is upper semicontinuous but, in general, not continuous.
Moreover, g_G^* is, in general, not symmetric.

In order to follow the standard notations we put

$$g_G \colon = \log g_G^*.$$

Then the following inequality holds

$$\log c_G^* \leq g_G \leq \log \widetilde{k}_G^*.$$

To finish this short introduction we recall the following surprising and deep result due to L. Lempert.

THEOREM. *If $G \in \mathcal{G}$ is biholomorphically equivalent to a convex domain, then $c_G = \widetilde{k}_G$, i.e., $d_G = c_G$ for any holomorphically contractible family $\underline{d}$ of functions.*

For more details we refer to [15] and [11].

II. Completeness

Let $\underline{d}$ be a holomorphically contractible family of functions. We say that a domain $G \in \mathcal{G}$ is *d_G-hyperbolic* if $d_G(z,w) > 0$ whenever $z, w \in G$, $z \neq w$.

REMARK. 1) In the case $d_G = c_G$ or $d_G = k_G$ hyperbolicity means that the Carathéodory pseudodistance, respectively the Kobayashi pseudodistance, is in fact a distance.
2) G is c_G-hyperbolic iff $H^\infty(G)$ separates the points of G. In particular, any bounded domain G is Carathéodory hyperbolic.

Thus under the assumption of hyperbolicity w.r.t c_G and k_G, respectively, we are dealing with metric spaces (G, c_G) and (G, k_G), respectively, where c_G and k_G, respectively, are continuous on $G \times G$.

For a continuous distance $d_G \colon G \times G \longrightarrow [0, \infty)$ we say

DEFINITION. a) G is *d_G-complete* if any d_G-Cauchy sequence does converge to a point in G in the G-topology.
b) G is called *d_G-finitely-compact* if any d_G-ball $\mathbb{B}_{d_G}(a, r)$ with center $a \in G$ and finite radius r is relatively compact in G w.r.t. the G-topology.

REMARKS. 1) If G is bounded, then $\top_{c_G} = \top_{k_G} = \top(G)$, where $\top_{d_G}$, respectively $\top(G)$, means the topology induced by d_G, respectively the standard topology of G.
2) In general, $\top_{c_G} \neq \top(G)$, even if G is c_G-hyperbolic (cf. [12]).
3) If G is k_G-hyperbolic, then $\top_{k_G} = \top(G)$.
4) "c_G-finitely compact" implies "c_G-complete".
5) "k_G-complete" is equivalent to "k_G-finitely compact". Observe that k_G is an inner metric whereas, in general, c_G is not.
6) If $G \subset \mathbb{C}$ then: G is c_G-complete iff G is c_G-finitely compact.
7) OPEN PROBLEM: What about the implication "$\Leftarrow$" in 6) in the case of an arbitrary domain $G \subset \mathbb{C}^n$, $n \geq 2$ (or in case of manifolds or in the case of irreducible complex spaces)?
7') There is a highly reducible 1-dimensional complex space X for which the implication "$\Leftarrow$" in 6) is wrong (cf. [13]).
8) c_G completeness, respectively k_G-completeness, implies that G is pseudoconvex.

Because of 8) we will always assume that the domains under consideration are pseudoconvex.

III. Completeness for Reinhardt domains

It was 1984 when the following result appeared (cf. [20]).

THEOREM. *a) A bounded domain $G \subset \mathbb{C}^n$ is c_G-finitely-compact iff for any fixed point $z_0 \in G$ and any sequence $(z_\nu)_{\nu\in\mathbb{N}} \subset G$ converging to a boundary point $z^* \in \partial G$ (w.r.t. $\top(\mathbb{C}^n)$) there exists $f \in \mathcal{O}(G, E)$: $f(z_0) = 0$ and $\sup\{|f(z_\nu)| : \nu \in \mathbb{N}\} = 1$.*
b) Any bounded pseudoconvex Reinhardt domain G, $0 \in G$, is c_G-finitely-compact.

In virtue of a), a bounded domain G is c_G-finitely compact if G has "sufficiently many peak-points". In particular, strongly pseudoconvex domains or pseudoconvex domains in $\mathbb{C}^2$ with real analytic boundary are c_G- finitely compact.

OPEN PROBLEM: Are all bounded pseudoconvex domains with a smooth C^∞-boundary c_G-finitely compact (or c_G-complete)?

Then, in 1994, S. Fu proved the following more general result (cf.

[6]).

THEOREM. a) *Let $G \subset \mathbb{C}^n$ be a bounded pseudoconvex Reinhardt domain which satisfies the following condition*

$$(*) \qquad \text{if } \overline{G} \cap V_j \neq \varnothing, \text{ then } G \cap V_j \neq \varnothing,$$

where $V_j: = \{z \in \mathbb{C}^n : z_j = 0\}$.
Then G is c_G-finitely compact.
b) *Any bounded pseudoconvex Reinhardt domain is k_G-complete.*

We call condition (*) the *Fu-condition.*

That the Fu-condition is in some sense necessary is seen by looking at the following examples:

EXAMPLE. 1) Put $G: = \{(z,w) \in \mathbb{C}^2 : |w| < |z| < 1\}$. G is not c_G-complete, since $c_G((1/\nu, 0), (1/\mu, 0)) \le c_{E_*}(1/\nu, 1/\mu) = c_E(1/\nu, 1/\mu) \longrightarrow 0$ when $\nu, \mu \longrightarrow \infty$. Obviously, G does not fulfil the Fu-condition.
2) Using the biholomorphic mapping $(z,w) \longrightarrow (1/z, w)$ leads to the unbounded pseudoconvex Reinhardt domain $\widetilde{G}: = \{(z,w) \in \mathbb{C}^2 : |zw| < 1 < |z|\}$, which, of course, is not $c_{\widetilde{G}}$-complete, but fulfils the Fu-condition.

So there are two natural questions:
1) is there a good description of all c-hyperbolic (k-hyperbolic) pseudoconvex Reinhardt domains?
2) is the Fu-condition together with the boundedness necessary for being c-complete in the class of pseudoconvex Reinhardt domains?
The complete answer is due to W. Zwonek (1998) (cf. [21]).

THEOREM. *Let G be a pseudoconvex Reinhardt domain in $\mathbb{C}^n$. Then the following conditions are equivalent:*
a) *G is c_G-hyperbolic;*
b) *G is $\widetilde{k}_G$-hyperbolic;*
c) *G is Brody-hyperbolic, i.e., $\mathcal{O}(\mathbb{C}, G) = G$;*
d) *$\exists A = (A_k^j) \in \mathbb{Z}(n \times n)$, $\operatorname{rank} A = n$ and $\exists C \in \mathbb{R}^n$ such that*
α) *$G \subset G(A^1, C_1) \cap \cdots \cap G(A^n, C_n) =: G(A, C)$, where*

$$G(A^j, C_j): = \{z \in \mathbb{C}^n : z_k \neq 0, \text{ if } A_k^j < 0, \text{ and}$$
$$|z_1^{A_1^j} \cdots z_n^{A_n^j}| =: |z^{A^j}| < \exp C_j\},$$

(Even more is true: it is possible to choose A such that $|\det A| = 1$, i.e., $A^{-1} \in \mathbb{Z}(n \times n)$.)
β) $V_j \cap G = \varnothing$ or it is c-hyperbolic as a domain in $\mathbb{C}^{n-1}$;
e) G is algebraically biholomorphic to a bounded Reinhardt domain;
f) G is Kobayashi-complete.

REMARK. 1) Condition f) is a consequence of the Fu result via e). Zwonek's proof does not use the Fu-theorem; it relies on the effective formulas for invariant functions on elementary Reinhardt domains.
2) The theorem shows that all the hyperbolicity notions coincide; so, in the future, we will speak only on hyperbolic Reinhardt domains.

To complete this part of the discussion we mention the following result.

THEOREM. *Let G be a pseudoconvex hyperbolic Reinhardt domain in $\mathbb{C}^n$. Then:*
G is algebraically equivalent to an unbounded pseudoconvex Reinhardt domain iff G is algebraically equivalent to a bounded pseudoconvex Reinhardt domain which does not fulfil the Fu-condition.

Now, since we know all the hyperbolic pseudoconvex Reinhardt domains we can turn to investigate Carathéodory completeness for hyperbolic pseudoconvex Reinhardt domains.

Let us first discuss two examples that look very similar.

EXAMPLE. Put

$$G_k: = \{z \in \mathbb{C}^2 \colon 1/2|z_1|^k < |z_2| < 2|z_1|^k, |z_1| < 2\},\ k = 2 \text{ or } k = \sqrt{2}.$$

In the case $k = 2$, using the embedding $\lambda \longrightarrow (\lambda, \lambda^2)$, $\lambda \in E_*$, we are led to

$$c_{G_2}((1/\nu, (1/\nu)^2), (1/\mu, (1/\mu)^2)) \leq c_{E_*}(1/\nu, 1/\mu) = c_E(1/\nu, 1/\mu) \longrightarrow 0, \quad \text{when } \nu, \mu \longrightarrow \infty,$$

i.e., G_2 is not c_{G_2}-complete.

Although $G_{\sqrt{2}}$ looks geometrically very similar to G_2 there is no way to embed a punctured disc to conclude that it is not Carathéodory complete.

Nevertheless, the following theorem (due to W. Zwonek (1998)) holds (cf. [22]).

THEOREM. *Let G be a hyperbolic pseudoconvex Reinhardt domain in $\mathbb{C}^n$. Then the following conditions are equivalent:*

a) *G is c_G-finitely compact;*
b) *G is c_G-complete;*
c) *there is no boundary sequence $(z_\nu)_{\nu\in\mathbb{N}} \subset G$ with $\sum_{\nu=1}^{\infty} g_G^*\,(z_\nu, z_{\nu+1}) < \infty$;*
d) *G is bounded and G fulfils the Fu-condition.*

Moreover, the following characterization of hyperconvex Reinhardt domains (due to M. Carlehed, U. Cegrell, and F. Wikström (cf. [3]), and P. Pflug, W. Zwonek (1998) (cf. [22]) is true.

COROLLARY. *Let G be a pseudoconvex Reinhardt domain in $\mathbb{C}^n$. Then the following properties are equivalent:*

a) *G is hyperconvex in the sense that there is $u \in \mathcal{PSH}(G) \cap \mathcal{C}(G)$, $u < 0$, such that $\{z \in G\colon u(z) < -\varepsilon\} \subset\subset G$ for any $\varepsilon > 0$;*
b) *G is bounded and fulfils the Fu-condition.*

Hence the description of Carathéodory completeness in the class of pseudoconvex Reinhardt domains is completely settled. Observe that both notions of "Carathéodory complete" coincide in that class of domains.

REMARK. If G is a pseudoconvex bounded Reinhardt domain in $\mathbb{C}^2$ one can prove that for any $z^0 \in \partial G \cap (\mathbb{C}_*)^2$ and any $a \in G$ the following is true: $c_G(a, z) \underset{z\to z^0}{\longrightarrow} \infty$, i.e., $\partial G \cap (\mathbb{C}_*)^2$ is c_G-infinitely far away from points in G. In contrast to the two–dimensional case there exists a bounded pseudoconvex Reinhardt domain $G \subset \mathbb{C}^3$, a point $a \in G$ and a boundary sequence $(z_\nu)_{\nu\in\mathbb{N}} \subset G$ with $\lim_{\nu\to\infty} z_\nu \in (\partial G \cap \mathbb{C}_*^3)$ such that the sequence $(c_G(a, z_\nu))_{\nu\in\mathbb{N}}$ is bounded (cf. [23]).

IV. Completeness for pseudoconvex circular domains

Recall that a domain $G \subset \mathbb{C}^n$ is called *balanced* (or *complete circular*) if $\overline{E}G \subset G$. Let G be a balanced domain. Then there exists an upper semicontinuous function $h = h_G\colon \mathbb{C}^n \longrightarrow [0, \infty)$, $u(\lambda z) = |\lambda| u(z)$, $\lambda \in \mathbb{C}$, $z \in \mathbb{C}^n$, such that

$$G = G_h = \{z \in \mathbb{C}^n\colon h(z) < 1\}.$$

The function h is uniquely defined; it is called the *Minkowski function* of G.

We mention that G is pseudoconvex iff $\log h \in \mathcal{PSH}(\mathbb{C}^n)$.

The following fact, due to T. Barth, is well known: if $G = G_h$ is a bounded pseudoconvex balanced domain which is k_G-complete, then its Minkowski function $h = h_G$ is necessarily continuous. So the following natural question appeared: is any bounded pseudoconvex balanced domain $G = G_h$ with continuous h complete in one of the three possible interpretations?

The answer found by M. Jarnicki & P. Pflug in 1991 is negative (cf. [8]).

THEOREM. *For any $n \geq 3$ there exists a bounded pseudoconvex balanced domain $G = G_h \subset \mathbb{C}^n$ with continuous h which is not k_G-complete.*

We mention that the main example is in $\mathbb{C}^3$; for the general case take $G \times E^{n-3}$.

OPEN PROBLEMS: 1) Is there also a counterexample in $\mathbb{C}^2$?
2) How to characterize completeness in the class of all bounded pseudoconvex balanced domains?
3) What about completeness for bounded pseudoconvex circular domains?

V. Bergman completeness

First let us repeat the necessary definitions. Let $G \subset \mathbb{C}^n$ be a bounded domain. By $L_h^2(G) := \mathcal{O}(G) \cap L^2(G)$ we denote the Hilbert space of all holomorphic square-integrable functions on G. Recall that for $a \in G$ the evaluation map

$$L_h^2(G) \ni f \longrightarrow f(a) \in \mathbb{C}$$

is a continuous linear functional. Thus there exists $K_G(\cdot, a) \in L_h^2(G)$ with $f(a) = (f, K_G(\cdot, a))_{L^2(G)}$; here $(\cdot,\cdot)_{L^2(G)}$ denotes the scalar product in the space $L^2(G)$.

K_G is called the *Bergman function* on G; it is antiholomorphic in the second variable and holomorphic in the first one; moreover $K_G(z,w) = \overline{K_G(w,z)}$.

By $\beta_G\colon G \times \mathbb{C}^n \longrightarrow [0, \infty)$,

$$\beta_G(z;X)\colon = \left[\sum_{j,k=1}^{n} \frac{\partial^2 \log K_G(z,z)}{\partial z_j \partial \bar{z}_k} X_j \bar{X}_k \right]^{1/2},$$

we denote the *Bergman metric* on G. The associated distance, *the Bergman distance*, is denoted by b_G. Recall that $c_G \leq b_G$.
Observe that the family $(b_G)_G \in \mathcal{G}_b$, $\mathcal{G}_b$ the set of all bounded domains in all $\mathbb{C}^n$'s, is not holomorphically contractible. Nevertheless, it is invariant under biholomorphic mappings.

We start summarizing some of the known results on b-completeness.

1) Since b_G is inner there is only one notion of "Bergman complete".
2) If $H^\infty(G)$ is dense in $L_h^2(G)$ and if $\lim\limits_{z \to \partial G} K_G(z,z) = \infty$, then G is b_G-complete.
3) If G is a bounded pseudoconvex domain satisfying an "outer cone condition" at $z_0 \in \partial G$, then $\lim\limits_{z \to z_0} K_G(z,z) = \infty$ (cf. [19]).
4) Any bounded pseudoconvex domain with C^1-boundary is b_G-complete (cf. [17]).
5) For a bounded hyperconvex domain we have that $\lim\limits_{z \to \partial G} K_G(z,z) = \infty$ (cf. [18]).
6) There are bounded pseudoconvex domains, not hyperconvex, such that the Bergman kernel $K_G(z,z)$ tends to ∞ when $z \to \partial G$; for example, take the Hartogs triangle $G\colon = \{(z,w) \in \mathbb{C}^2\colon |z| < |w| < 1\}$. Observe that G is not b_G-complete. Other examples are the so called Zalcman domains in $\mathbb{C}$ (cf. [18]).
7) Any bounded pseudoconvex balanced domain $G = G_h$ with continuous Minkowski function h is b_G-complete (cf. [9]).

Recently, it was shown that this result remains true even without assuming that h is continuous (cf. [14]). In fact, for any bounded balanced pseudoconvex domain G the Bergman kernel $K_G(z,z)$ tends to infinite if $z \longrightarrow \partial G$. Moreover, this result gives new examples of bounded pseudoconvex domains, not hyperconvex, but Bergman complete.
Recall that any bounded pseudoconvex balanced domain is an L_h^2-domain of holomorphy (cf. [10]).

Observe that the domains in 4) and 7) (when h is continuous) are hyperconvex. So there was the question whether any hyperconvex bounded domain G is b_G-complete?

The answer due to Z. Błocki & P. Pflug and to G. Herbort (cf. [2], [7]) is the following:

THEOREM. *Any bounded hyperconvex domain $G \subset \mathbb{C}^n$ is Bergman complete.*

Sketch of the proof. Assuming that G is not Bergman complete leads to the existence of a boundary sequence $(a_\nu)_\nu \subset G$ and a sequence of real numbers $(\Theta_\nu)_\nu$ such that

$$(\exp i\Theta_\nu \frac{K_G(\cdot, a_\nu)}{\sqrt{K_G(a_\nu, a_\nu)}})_\nu$$

is a Cauchy-sequence in $L_h^2(G)$. Thus there is a function $f \in L_h^2(G)$, $\|f\|_{L^2(G)} = 1$, such that

$$\exp i\Theta_\nu \frac{K_G(\cdot, a_\nu)}{\sqrt{K_G(a_\nu, a_\nu)}} \overset{L^2(G)}{\longrightarrow} f.$$

In particular, we obtain that

$$\lim_{\nu\to\infty} \frac{|f(a_\nu)|}{\sqrt{K_G(a_\nu, a_\nu)}} = 1.$$

On the other side using $\bar\partial$-methods B.-Y. Chen (cf. [4]) and G. Herbort (cf. [7]) have shown that there are functions $\widetilde{f}_\nu \in L^2(G)$, $\widetilde{f}_\nu(a_\nu) = 0$, such that

$$\|f - \widetilde{f}_\nu\|_{L^2(G)} \le C_{G,n}\|f\|_{L^2(G_{a_\nu})},$$

where $G_{a_\nu}\colon = \{z \in G\colon g_G(a_\nu, z) \le -1\}$.

Therefore substituting above f by $f - \widetilde{f}_\nu + \widetilde{f}_\nu$ gives the following inequality

$$\frac{|f(a_\nu)|}{\sqrt{K_G(a_\nu, a_\nu)}} \le \|f - \widetilde{f}_\nu\|_{L^2(G)} \le C_{G,n}\|f\|_{L^2(G_{a_\nu})}.$$

What remains is to see that $\mathrm{vol}(G_{a_\nu}) \underset{\nu\to\infty}{\longrightarrow} 0$. But this follows from the fact that

$$\int_G (-g_G(a_\nu, z))^n d\lambda(z) \underset{\nu\to\infty}{\longrightarrow} 0,$$

which is a consequence of properties of the Monge-Ampère operator. □

REMARK. According to [3], for a bounded hyperconvex domain G the following is true: if $(a_\nu)_{\nu\in\mathbb{N}} \subset G$ is a boundary sequence in G, then there exists a pluripolar set $E \subset G$ such that $\overline{\lim}_{\nu\to\infty} g(a_\nu, z) = 0$, $z \in G \setminus E$. Using a result of Hörmander it follows that $\mathrm{vol}(G_{a_\nu}) \longrightarrow 0$. We mention that there are pseudoconvex Reinhardt domains, not hyperconvex, but nevertheless satisfying the above property.

Recall that any bounded balanced domain is Bergman complete but not necessarily hyperconvex. Other such examples are the following so-called Zalcman domains discussed by T. Ohsawa (cf. [18] and [4]). We put

$$D_N := E \setminus \overline{\cup_{k=1}^{\infty} \mathbb{B}(a_k, r_k)},$$

where $a_k := (1/2)^k$ and $r_k := (1/2)^{kN(k)}$ with $N = (N(k))_k \subset \{2, 3, 4, \dots\}$.
Applying the Wiener criterion we obtain that

$$D_N \text{ is hyperconvex iff } \sum_{k=1}^{\infty} 1/(N(k)) = \infty.$$

For example, D_N is not hyperconvex if $N(k) := k^2 + 1$. On the other side if $kN(k)/(2^{2k})$ tends to zero, then $\lim\limits_{z\to\partial D_N} K_{D_N}(z, z) = \infty$ (cf. [4] and [18]).

We could conclude that D_N is Bergman complete if we knew that $H^\infty(D_N)$ is dense in $L_h^2(D_N)$. But this information follows from a result due to P. Lindberg (1977) (cf. [16]).

In order to be able to formulate that result we need the following notion of a capacity: Let $K \subset \Omega \subset \mathbb{C}$ be an arbitrary compact subset of the open set $\Omega\colon = 2E$. We put

$$C_2(K,\Omega)\colon = \inf\{\|f\|_2^2\colon f\in L^2(\Omega), f\geq 0,\\ \int_\Omega \frac{f(\zeta)}{|\zeta - z|}d\zeta \geq 1 \text{ for all } z\in K\}.$$

In the case that $F\subset\Omega$ is arbitrary we set

$$C_2(F,\Omega)\colon = \sup\{C_2(K,\Omega)\colon K\subset F, K \text{ compact}\}.$$

Now we give a special case of the theorem of P. Lindberg.

THEOREM. *Let G be a bounded domain in $\mathbb{C}$. Assume that there exists a set $F\subset\partial G$ with $C_2(F)=0$ such that for any $z\in\partial G\setminus F$ the following inequality holds:*

$$\overline{\lim_{r\to\infty}}\,\gamma(\overline{\mathbb{B}(z,r)}\setminus G)/r > 0,$$

where γ denotes the analytic capacity.
Then $H^\infty(G)$ is dense in $L_h^2(G)$.

Observe that in the case of Zalcman domains $F=\{0\}$, i.e., $C_2(F)=0$. Thus there are a lot of Zalcman domains, not hyperconvex, but Bergman complete.

REMARKS. a) In fact, instead of using the above result of P. Lindberg, it suffices to apply the following weaker one: let $f\in L_h^2(G)$, $G\subset\mathbb{C}$ a bounded domain, and let $F\subset\partial G$ with $C_2(F)=0$. Then, if $\varepsilon>0$, there exists an open set $W\supset F$ and a function $g\in L_h^2(G\cup W)$ such that $\|f-g\|_{L^2(G)}<\varepsilon$. Taking a boundary point $a\in\partial G$ as the set F this result says that the $L_h^2(G)$-functions, that are bounded near a, are dense in $L_h^2(G)$.

Recently, also Chen (cf. [5]) using complete Kaehler metrics showed (compare Lindberg's result quoted in a)) that for any bounded domain $G\subset\mathbb{C}$ and any point $z^0\in\partial G$ the holomorphic functions that are bounded in a neighborhood of z^0 are dense in $L_h^2(G)$. Therefore any bounded plane domain G that satisfies $K_G(z,z)\underset{z\to\partial G}{\longrightarrow}\infty$ is Bergman complete.

We mention that this result can also be proven by using the Berndtsson solution of the $\overline{\partial}$–problem (cf. [1]) instead of using complete Kaehler metrics.
b) In [14] it is shown that there are fat plane domains G of Zalcman type (*fat* means that $\operatorname{int}\overline{G} = G$) with:

(i) $K_G(z, z)$ does not tend to ∞ if $z \longrightarrow \partial G$;
(ii) G is not Bergman complete.

c) Those Zalcman domains, that are Bergman complete, but not hyperconvex, do allow a boundary sequence $(a_j)_{j\in\mathbb{N}}$ and $\varepsilon > 0$ such that $\operatorname{vol}(G_{a_j}) > \varepsilon$, $j \in \mathbb{N}$, i.e., the potential theory condition on the volumes of the sublevel sets of the Green function is strictly stronger than the Bergman completeness.

OPEN PROBLEM: Characterize those sequences N for which D_N is Bergman complete.

To conclude this survey we return to the discussion of bounded pseudoconvex Reinhardt domains.
Let G be a bounded pseudoconvex Reinhardt domain. As usual, we denote by $\log G$ its logarithmic image. Recall it is always a convex domain. Without loss of generality, we will always assume that the point $(1, \ldots, 1) \in G$, i.e., $0 \in \log D$. We put

$$S(G) \colon = \{v \in \mathbb{R}^n \colon \mathbb{R}_{\geq 0} v \subset \log D\}.$$

$S(G)$ is a closed convex cone in $[-\infty, 0]^n$. Moreover, we set

$$\widetilde{S}(G) \colon = \{v \in S(G) \colon \exp([-\infty, 0]v) \subset G\} \text{ and } S'(G) \colon = S(G) \backslash \widetilde{S}(G).$$

EXAMPLE. Using the notions of the second example in section III we observe that $S'(G_2) = \mathbb{R}_{>0}(-1, -2)$, i.e., $S'(G_2)$ contains the non trivial rational vector $v \colon = (-1, -2)$. Choose the matrix

$$A \colon = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}; \text{ put } B \colon = A^{-1} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} B^1 \\ B^2 \end{pmatrix}.$$

Then

$$\Phi_B \colon \mathbb{C}_*^2 \longrightarrow \mathbb{C}_*^2, \ (z_1, z_2) \longrightarrow (z^{B^1}, z^{B^2}) = (z_1^3 z_2^{-1}, z_1^{-2} z_2),$$

gives a biholomorphic mapping with

$$\widetilde{G}_2 \colon = \Phi_B(G_2) = \{(z_1, z_2) \in \mathbb{C}_*^2 \colon 1/2 < |z_2| < 2, |z_1 z_2| < 2\}.$$

It is easily seen that $\widetilde{G}_2$ is not Bergman complete; hence G_2 is not Bergman complete.
On the other side we see that $S'(G_{\sqrt{2}}) = \mathbb{R}_{>0}(-1, -\sqrt{2})$, i.e., $S'(G_{\sqrt{2}})$ does not contain any non trivial rational vector. Recall that $G_{\sqrt{2}}$ is not hyperconvex.

It turns out that there is a complete characterization of Bergman complete bounded pseudoconvex Reinhardt domains in terms of the set $S'(G)$ due to W. Zwonek (cf. [23] and [24]).

THEOREM. *Let G be a bounded pseudoconvex Reinhardt domain in $\mathbb{C}^n$. Then:*
G is Bergman complete iff $S'(G) \cap \mathbb{Q}^n = \varnothing$.

EXAMPLE. In particular, this theorem implies that
a) $G_{\sqrt{2}}$ is Bergman complete, but not hyperconvex;
b) the example of G. Herbort (cf. [7])

$$G \colon = \{(z, w) \in \mathbb{C}^2 \colon |w|^2 < \exp(-1/|z|^2), |z| < 1\}$$

is Bergman complete, but not hyperconvex; one simply has to observe that $S(G) = \{0\}$.

We mention that for bounded pseudoconvex Reinhardt domains in $\mathbb{C}^2$ the following properties are equivalent (cf. [23]):
(i) G is Bergman complete;
(ii) for any $\varepsilon > 0$: $\mathrm{vol}(\{g_G < -\varepsilon\}) \longrightarrow 0$ if $z \longrightarrow \partial G$;
(iii) for any $w \in G \cap (\mathbb{C}_*)^2$ we have $g_G(z, w) \longrightarrow 0$ if $z \longrightarrow \partial G$.

Hence, for this class of domains Bergman completeness can be characterized via pluripotential properties.

Recall that any bounded balanced pseudoconvex is Bergman complete. Thus it remains to solve the following question.

OPEN PROBLEM: Characterize all bounded circular pseudoconvex domains that are Bergman complete.

Observe that a solution of this problem would contain the given one for Reinhardt domains.

By analogy to the Reinhardt case one may ask:

OPEN PROBLEM: (posed by H. Upmeier) Is there a characterization of Bergman complete matrix Reinhardt domains?

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