J. Korean Math. Soc. **37** (2000), No. 2, pp. 225–243

EQUIVALENCE PROBLEM AND COMPLETE SYSTEM OF FINITE ORDER

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ABSTRACT. We explain the notion of complete system and how it naturally arises from the equivalence problem of G-structures. Then we construct a complete system of 3rd order for the infinitesimal CR automorphisms of CR manifolds of nondegenerate Levi form.

0. Introduction

Let M and M' be real analytic (C^ω) real hypersurfaces in $\mathbb{C}^{n+1}$ and $\mathbb{C}^{N+1}$, respectively, where $N \geq n \geq 1$. A pseudoconformal mapping of M into M' is a holomorphic mapping of a neighborhood of M that maps M into M'. Then the restriction of f is a C^ω CR embedding of M into M'. Recently many authors, [14], [22], [23], [33], [34], [35], [41] and others, studied the dependence of f on its finite jet at a point. There have been two approaches: one is using the fact that f maps the Segre varieties of M into the Segre varieties of M' and the other is prolongation of the tangential Cauchy-Riemann equations and construction of complete system of finite order of which every CR map f is a solution. In this paper we briefly review the equivalence problem of E. Cartan and define the notion of complete system as a generalization of Cartan's complete system of invariants. Then we construct a complete system of order 3 for the infinitesimal CR automorphisms of CR manifolds of nondegenerate Levi form.

Given a system of partial differential equations prolongation is a process of repeated differentiation and algebraic operations to get a

Received May 30, 1999.

1991 Mathematics Subject Classification: 32C16, 32F25, 32F40.

Key words and phrases: CR mapping, prolongation, complete system, equivalence problem , infinitesimal CR automorphism.

The author was supported by GARC-KOSEF, BSRI-POSTECH and Korea Research Foundation.

This paper was presented on February 19th, 1998 in the 2nd KSCV meeting.

new system of desired form. Prolongation seems to be a very effective and widely applicable method in the qualitative study of partial differential equations, especially for overdetermined systems. The crux of the method of prolongation is in the reduction of order by eliminating the highest order terms using the symmetry of the system. In most cases, however, one can hardly find the symmetry that can be used for this reduction of order. For the embedding equations the geometric invariants are often useful in finding the right symmetry that reduces the order by eliminating the highest order terms in the prolonged system. This is the key observation in [7] and [19]. Generically, in a finite number of steps an overdetermined system can be prolonged to a level on which all the partial derivatives of unknown functions of certain order, say m, can be solved in terms of derivatives of lower order. This level of prolongation shall be called a complete prolongation and the resulting system a complete system of order m, which we shall discuss in detail in §2.

In geometric equivalence problems where the mappings are determined by a finite number of constants it seems to the author that the defining property of the mappings can be expressed as a complete system of finite order. For instance, a biholomorphism ϕ of the unit disk D is of the form

$$\phi(z) = e^{i\theta_0}\frac{z-\alpha}{1-\bar{\alpha}z}, \quad \theta_0 \in \mathbb{R}, \quad \alpha \in D$$

and therefore ϕ is determined by three real constants θ_0, $\mathrm{Re}\alpha$ and $\mathrm{Im}\alpha$. On the other hand ϕ is an isometry of the Poincare metric of the disk. Therefore, ϕ satisfies a complete system of order 2 and thus ϕ is determined by its 1-jet at a point, that is, by three real constants, see Example 2.3. Similarly, it follows from the Chern-Moser theory [CM] that a CR equivalence between C^ω CR manifolds of nondegenerate Levi form is determined by its 2-jet at a point. In §3 we shall construct a complete system for infinitesimal CR automorphisms.

In CR geometry our primary concern associated with the complete system is the existence of integral manifolds. This is the equivalence problem when M and M' are of the same dimension and the embeddability question in different dimensional cases. Secondarily, the existence of complete system for CR mappings between C^ω real hypersurfaces implies the holomorphic extendability of CR mappings (see

[22], [23]). The existence of complete system for the infinitesimal CR automorphisms implies that the space of infinitesimal automorphisms forms a finite dimensional Lie algebra (see Theorem 3.1). In the cases $N > n$ the existence of complete system for infinitesimal deformations implies the finite dimensionality of the deformations or the infinitesimal rigidity of an embedding. In §4 we shall discuss further problems along these lines.

The author thanks D. Zaitsev for his interest and for many valuable discussions and suggestions. As he pointed out the complete differential system of order 3 for the infinitesimal automorphisms can be obtained by differentiating with respect to the 1-parameter in the 1-parameter family of CR automorphisms. However, the proof of Theorem 3.1 is more direct and simpler. As a rule we study infinitesimal change first to understand nonlinear objects.

1. Equivalence problem of G-structures

Let M be a C^∞ manifold of dimension n and G be a linear subgroup of $GL(n;\mathbb{R})$. A G-structure on M is reduction of coframe bundle of M to a subbundle with the structure group G. For instance, a Riemannian structure on M is a $SO(n)$-structure and the subbundle in this case is the orthonormal coframe bundle of M.

Now let M and $\tilde{M}$ be manifolds of dimension n with G-structure. The equivalence problem is deciding whether there exists a structure preserving mapping $f : M \to \tilde{M}$. Locally, this is a question of existence of solutions for an overdetermined system of first order partial differential equations in cases where G is a sufficiently small group.

E.Cartan's method to this problem is as follows: We fix coframes $\theta = (\theta^1, \cdots, \theta^n)^t$ of M and $\tilde{\theta} = (\tilde{\theta}^1, \cdots, \tilde{\theta}^n)^t$ of $\tilde{M}$ adapted to the G-structure, where θ and $\tilde{\theta}$ are defined over an open set U of M and an open set $\tilde{U}$ of $\tilde{M}$, respectively. Then the question is whether there exists a mapping $f : M \to \tilde{M}$ that satisfies

$$f^*\tilde{\theta}^\alpha = a^\alpha_\beta\, \theta^\beta, \tag{1.1}$$

where $a := [a^\alpha_\beta(x)]_{n\times n}$ is a G-valued function of M. In terms of local coordinates, (1.1) is a system of first order partial differential equations for $f = (f^1, \cdots, f^n)$ and system of algebraic equations for $a^\alpha_\beta(x)$. Thus we consider the product $U \times G$ and the tautological 1-form Θ, which is a vector valued 1-form defined by $\Theta = g\theta$ on $U \times G$, namely

$$\Theta_{(x,g)} = g\theta_x, \ \forall x \in U, \ \ \forall g \in G, \tag{1.2}$$

where θ_x is a column vector $(\theta_x^1, \cdots, \theta_x^n)^t$. G acts on $U \times G$ on the left by the action defined by

$$h(x,g) = (x, hg), \ \forall x \in U, \ \forall g, h \in G.$$

PROPOSITION 1.1. *A diffeomorphism $f : U \to \tilde{U}$ satisfies (1.1) if and only if there exists a diffeomorphism $F : U \times G \to \tilde{U} \times G$ satisfying*

i) $F^*\tilde{\Theta} = \Theta$

ii) *the following diagram commutes:*

$$\begin{tikzcd} U \times G \arrow[r, "F"] \arrow[d, "\pi"'] & \tilde{U} \times G \arrow[d, "\tilde{\pi}"] \\ U \arrow[r, "f"] & \tilde{U} \end{tikzcd}$$

iii) $F(x, gh) = gF(x, h)$, *for each $x \in U$, and $g, h \in G$.*

Proof. Suppose that f satisfies $f^*\tilde{\theta} = g_0\theta$, where g_0 is a G-valued function on M. Define $F : U \times G \to \tilde{U} \times G$ by $F(x,g) = (f(x), gg_0^{-1}(x))$. Then F satisfies ii) and iii). Moreover,

$$F^*\tilde{\Theta} = F^*(\tilde{g}\tilde{\theta}) = (gg_0{}^{-1})f^*\tilde{\theta} = (gg_0{}^{-1})g_0\theta = g\theta = \Theta.$$

Conversely, suppose that $F : U \times G \to \tilde{U} \times G$ satisfies i) - iii). Define $f : U \to \tilde{U}$ and $g_0 : U \to G$ by $F(x, e) = (f(x), g_0(x)^{-1})$, where e is the identity of G. Then $F(x,g) = gF(x,e) = (f(x), gg_0^{-1})$, and i) implies that

$$g\theta = F^*(\tilde{g}\tilde{\theta}) = (gg_0{}^{-1})f^*\tilde{\theta}$$

therefore, $f^*\tilde{\theta} = g_0\theta$. □

Now apply d to (1.2). We get

$$d\Theta = dg \wedge \theta + gd\theta;$$

substituting $\theta = g^{-1}\Theta$, we obtain

$$d\Theta = dgg^{-1} \wedge \Theta + gd\theta. \tag{1.3}$$

We need the following

HYPOTHESIS: There exists unique 1-forms ω^i_j, $i, j = 1, \cdots, n$, *such that*

$$d\theta^i = -\omega^i_j \wedge \theta^j \tag{1.4}$$

and

$$[\omega^i_j(x)] \in \mathcal{G}, \ \textit{for each } x \in U,$$

where $\mathcal{G}$ *is the Lie algebra of* G.

This Lie algebra valued 1-form $\omega = [\omega^i_j]$ is called a torsion-free connection (see [9]). Substitute $d\theta = -\omega \wedge \theta$ and $\theta = g^{-1}\Theta$ in (1.3), to get

$$d\Theta = dgg^{-1} \wedge \Theta - g\omega \wedge g^{-1}\Theta = (dgg^{-1} - g\omega g^{-1}) \wedge \Theta.$$

Let

$$\Omega = -(dgg^{-1} - g\omega g^{-1}), \tag{1.5}$$

then Ω is a $\mathcal{G}$-valued 1-form on $U \times G$ and we have

$$d\Theta = -\Omega \wedge \Theta. \tag{1.6}$$

Now it is easy to show

PROPOSITION 1.2. *Let* Θ^i *and* Ω^i_j, $i, j = 1, \cdots, n$, *be the* 1*-forms defined by* (1.2) *and* (1.5) *on* $U \times G$. *Then* Θ^i, Ω^i_j *spans the cotangent space at each point of* $U \times G$. *Furthermore, if* $\tilde{\Theta}^i, \tilde{\Omega}^i_j$ *are the corresponding* 1*-forms on* $\tilde{U} \times G$ *and*

$$F : U \times G \to \tilde{U} \times G$$

is the mapping as in Proposition 1.1, *then*

$$F^*\tilde{\Omega}^i_j = \Omega^i_j. \tag{1.7}$$

The set $\{\Theta^i, \Omega^i_j\}$ is called a complete set of invariants for the equivalence problem. Ω is called a torsion-free connection form on $U \times G$. Note that ω is a 1-form on the base manifold U and that the restriction of Ω on each fibre is the Maurer-Cartan form of G.

2. Complete systems

Let f be a smooth (C^∞) mapping of an open subset X of $\mathbb{R}^n$ into an open subset U of $\mathbb{R}^m$. In this section we use superscripts for each components of vectors , thus $x = (x^1, \cdots, x^n)$ and $u = (u^1, \cdots, u^m)$ are the standard coordinates of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and $f(x) = (f^1(x), \cdots, f^m(x))$.

Let U_k be the space of all the different k-th order partial derivatives of the component of f at a point x. Set $U^{(q)} = U \times U_1 \times \cdots \times U_q$ be the Cartesian product space whose coordinates represent all the derivatives of a mapping $u = f(x)$ of all orders from 0 to q. A point in $U^{(q)}$ will be denoted by $u^{(q)}$.

The space $J^q(X, U) = X \times U^{(q)}$ is called the q-th order jet space of the space $X \times U$. If $f : X \to U$ is smooth, let $(j^q f)(x) = (x, f(x), \partial^\alpha f(x) : |\alpha| \leq q)$, then $j^q f$ is a smooth section of $J^q(X, U)$ called the q-graph of f.

Consider a system of partial differential equations of order q ($q \geq 1$) for unknown functions $u = (u^1, \cdots, u^m)$ of independent variables $x = (x^1, \cdots, x^n)$,

$$\Delta_\lambda(x, u^{(q)}) = 0, \qquad \lambda = 1, \cdots, l, \tag{2.1}$$

where $\Delta_\lambda(x, u^{(q)})$ are smooth functions in their arguments. Then $\Delta = (\Delta_1, \cdots, \Delta_l)$ is a smooth map from $X \times U^{(q)}$ into $\mathbb{R}^l$, so that the given system of partial differential equations describes the subset $\mathcal{S}_\Delta$ of zeros of Δ_λ in $X \times U^{(q)}$, called the solution subvariety of (2.1). Thus, a smooth solution of (2.1) is a smooth map $f : X \to U$ whose q-graph is contained in $\mathcal{S}_\Delta$.

A differential function $P(x, u^{(q)})$ of order q defined on $X \times U^{(q)}$ is a smooth function of x, u, and derivatives of u up to order q. The total derivatives of $P(x, u^{(q)})$ with respect to x^i is the unique smooth function defined by

$$D_i P(x, u^{(q+1)}) := \frac{\partial P}{\partial x^i} + \sum_{a=1}^{m} \sum_{J} u^a_{J,i} \frac{\partial P}{\partial u^a_J},$$

where $J = (j_1, \cdots, j_n)$ is a multi-index such that $|J| \leq q$ and $J, i = (j_1, \cdots, j_i + 1, \cdots, j_n)$. For each nonnegative integer r, the rth-prolongation $\Delta^{(r)}$ of the system (2.1) is the system consisting of all the total derivatives of (2.1) of order up to r. Let $(\Delta^{(r)})$ be the ideal generated by $\Delta^{(r)}$ of the ring of differential functions on $X \times U^{(q+r)}$. If $\widetilde{\Delta} \in (\Delta^{(r)})$ for some r, the equation

$$\widetilde{\Delta}(x, u^{(q+r)}) = 0 \tag{2.2}$$

is called a prolongation of (2.1). Note that any smooth solution of (2.1) must satisfy (2.2). If k is the order of the highest derivative involved in $\widetilde{\Delta}$, we call (2.2) a prolongation of order k.

We now define the complete system.

DEFINITION 2.1. We say that (2.1) admits a complete prolongation to a system of order k if there exist prologations of (2.1) of order k

$$\widetilde{\Delta}_\nu(x, u^{(k)}) = 0, \quad \nu = 1, \cdots, N \tag{2.3}$$

which can be solved for all the k-th order partial derivatives as smooth functions of lower order derivatives of u, namely, for each $a = 1, \cdots, m$ and for each multi-index J with $|J| = k$,

$$u^a_J = H^a_J(x, u^{(p)} : p < k) \tag{2.4}$$

for some function H^a_J which is smooth in its arguments.

Every ordinary differential equation of order n

$$y^{(n)} = F(x, y, y', y'', \dots, y^{(n-1)})$$

is obviously a complete system of order n. Then the existence and uniquensess of solutions for given initial condition follow from the fundamental theorem of ordinary differential equation. Moreover, if F is analytic in its arguments then the solutions are analytic. If a given system of partial differential equations admits prolongation to a complete system the solutions have the same properties, see Proposition 2.5.

EXAMPLE 2.2. Consider the following system for one unknown function $u(x, y)$ of two independent variables.

$$\begin{cases} u_x + uu_y &= a(x, y) \\ u_{yy} + u^2 &= b(x, y). \end{cases} \tag{2.5}$$

We shall show that (2.5) admits a prolongation to a complete system of order 2. Differentiate the first equation of (2.5) with respect to x and y , respectively. then we have

$$
\begin{aligned}
u_{xx} + u_x u_y + u u_{xy} &= a_x \\
u_{xy} + u_y^2 + u u_{yy} &= a_y \\
u_{yy} + u^2 &= b.
\end{aligned}
\tag{2.6}
$$

(2.6) is a quasi-linear system of second order. The second order terms are

$$
\begin{pmatrix} 1 & u & 0 \\ 0 & 1 & u \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{pmatrix}.
$$

Since the matrix of the coefficients is invertible we can solve for all the second order derivatives of u in terms of lower order terms to get a complete system of order 2. Thus a C^2 solution u of (2.5) is uniquely determined by $u(0), u_x(0)$, and $u_y(0)$ and u is C^ω.

Coming back to the equivalence problem let G be a Lie-subgroup of $GL(n;\mathbb{R})$. Suppose that a manifold E of dimension n has a G-structure and $\pi : Y \to E$ is the associated principal bundle. The equivalence problem is finding canonically a system of differential 1-forms

$$
\omega^1, \cdots, \omega^N, \quad \text{where} \quad N = n + \dim G,
\tag{2.7}
$$

so that a mapping $f : E \to \tilde{E}$ preserves the G-structure if and only if there exists a mapping $F; Y \to \tilde{Y}$, which is a lift of f,that is, $\tilde{\pi} \circ F = f \circ \pi$, and such that

$$
F^* \tilde{\omega}^i = \omega^i, \quad i = 1, \cdots N,
\tag{2.8}
$$

where $\tilde{E}$ is a manifold of dimension n with a G-structure and $\tilde{\pi} : \tilde{Y} \to \tilde{E}$ is the associated principal bundle and $\tilde{\omega}^i$ are the corresponding 1-forms on $\tilde{Y}$. (2.7) is called a complete system of invariants of the G-structure and (2.8) is a complete system of order 1 for F in the sense of Definition 2.1. It turns out that (2.8) is equivalent to a complete system of order 2 for f . In the following we present a direct construction of a complete system for Riemannian isometries.

EXAMPLE 2.3. Let (M, g) and $(\tilde{M}, \tilde{g})$ be smooth Riemannian manifolds with Riemannian metric g and $\tilde{g}$, respectively. A C^1 map $u : M \to \tilde{M}$ is an isometry if

$$u^* \tilde{g} = g. \tag{2.9}$$

In term of local coordinates (2.9) can be written as

$$u_i^\alpha u_j^\beta \tilde{g}_{\alpha\beta}(u) = g_{ij}(x), \quad \text{(summation convention)}, \tag{2.10}$$

for each $i, j = 1, \ldots, n$. By applying ∂_k to (2.10) we have

$$(u_{ik}^\alpha u_j^\beta + u_i^\alpha u_{jk}^\beta)\tilde{g}_{\alpha\beta}(u) + u_i^\alpha u_j^\beta \frac{\partial \tilde{g}_{\alpha\beta}}{\partial u_\gamma}(u) u_k^\gamma = \frac{\partial g_{ij}}{\partial x_k}(x), \tag{2.11}$$

for each $i, j, k = 1, \ldots, n$. We may assume that $u(0) = 0, g_{ij}(0) = \delta_{ij}, \tilde{g}_{\alpha\beta}(0) = \delta_{\alpha\beta}$, and $u_j^\alpha(0) = \delta_j^\alpha$. Then at the reference point 0 (2.11) is

$$u_{jk}^i + u_{ik}^j = -\tilde{g}_{ij,k}(0) + g_{ij,k}(0). \tag{2.12}$$

By permuting the indices $\{i, j, k\}$ in (2.12) we get

$$u_{ki}^j + u_{ji}^k = -\tilde{g}_{jk,i}(0) + g_{jk,i}(0) \tag{2.13}$$

and

$$u_{ij}^k + u_{kj}^i = -\tilde{g}_{ki,j}(0) + g_{ki,j}(0). \tag{2.14}$$

Then (2.12) + (2.14) − (2.13) yields

$$2u_{jk}^i = -\tilde{g}_{ij,k}(0) + \tilde{g}_{jk,i}(0) - \tilde{g}_{ki,j}(0) + g_{ij,k}(0) - g_{jk,i}(0) + g_{ki,j}(0).$$

Therefore, on a neighborhood of $(0, u(0), u_i(0))$ in the space of first jets of u we have

$$u_{jk}^i = H_{jk}^i(x, u^{(1)}), \tag{2.15}$$

which is a complete system of order 2.

Now we observe that solving the given system of partial differential equations (2.1) is equivalent to finding an integral manifold of the corresponding exterior differential system

$$du_I^a - \sum_{i=1}^{n} u_{I,i}^a dx^i = 0$$

for all multi-index I with $|I| < q$ and $a = 1, \cdots, m$, with an independence condition $dx_1 \wedge \cdots \wedge dx_n \neq 0$ on $\mathcal{S}_\Delta$ (see [2]). If a solution of (2.1) satisfies a complete system of order k then we have the following Pfaffian system on $J^{k-1}(X, U)$:

$$\text{(2.16)} \qquad \begin{cases} du^a - \sum_{j=1}^{n} u_j^a dx^j = 0, & \\ \vdots & \\ du_I^a - \sum_{j=1}^{n} u_{I,j}^a dx^j = 0, & |I| = k-2, \\ du_I^a - \sum_{i=1}^{n} H_{I,i}^a dx^i = 0, & |I| = k-1. \end{cases}$$

with an independence condition $dx^1 \wedge \cdots \wedge dx^n \neq 0$, where $H_{I,i}^a$ are as in (2.4). Thus a solution $u = f(x)$ of (2.1) of class C^k satisfies a complete system of order k if and only if

$$(x) \mapsto (x, f(x), \partial_J f(x) : |J| \leq k-1)$$

is an integral manifold of the Pfaffian system (2.16). In particular, we have

PROPOSITION 2.5. *Let f be a solution of (2.1) of class C^k. Suppose that f satisfies a complete system (2.4), then f is C^∞. If each H_J^a is real analytic then f is real analytic. Furthermore, a solution is uniquely determined by its $(k-1)$-jet at a point.*

3. Complete system for the infinitesimal CR automorphisms

In this section we construct a complete system for the infinitesimal CR automorphisms of a CR manifolds of nondegenerate Levi forms. Let M be a differentiable manifold of dimension $2n+1$. A CR structure of hypersurface type on M is a subbundle $\mathcal{V}$ of the complexified tangent bundle $T_{\mathbb{C}}M$ having the following properties:
i) each fiber is of complex dimension n,
ii) $\mathcal{V} \cap \bar{\mathcal{V}} = \{0\}$,
iii) $[\mathcal{V}, \mathcal{V}] \subset \mathcal{V}$ (integrability).
Given a CR structure $\mathcal{V}$ the Levi form $\mathcal{L}$ is defined by

$$\mathcal{L}(L_1, L_2) := \sqrt{-1}[L_1, \bar{L_2}], \quad \mathrm{mod} \quad (\mathcal{V} + \bar{\mathcal{V}}).$$

$\mathcal{L}$ is a hermitian form on $\mathcal{V}$ with values in $T_{\mathbb{C}}M/(\mathcal{V} + \bar{\mathcal{V}})$. M is said to be strictly pseudoconvex if $\mathcal{L}$ is definite. A real hypersurface in a complex manifold has natural CR structure induced from the complex structure of the ambient space. A complex valued function f is called a CR function if f is annihilated by $\bar{\mathcal{V}}$. Let $\{L_1, \cdots, L_n\}$ be a set of complex vector fields that generates $\mathcal{V}$. Then f is a CR function if and only if
(3.1)

$$\bar{L}_i f = 0, \quad i = 1, \cdots, n \quad \text{(tangential Cauchy-Riemann equations)}.$$

A system of CR functions $(f_1, \cdots, f_{n+1})$ with $df_1 \wedge \cdots \wedge df_{n+1} \neq 0$ is a CR immersion into $\mathbb{C}^{n+1}$.
Let $(N, \mathcal{V}')$ be a CR manifold of dimension $2N+1$, $N \geq n$, with the CR structure bundle $\mathcal{V}'$. A mapping $F : M \to M'$ is called a CR mapping if F preserves the CR structure, that is,

$$F_* \mathcal{V} \subset \mathcal{V}'. \tag{3.2}$$

A real vector field X on a CR manifold $(M, \mathcal{V})$ is an infinitesimal CR automorphism if the flow maps ϕ_t of X are the local CR diffeomorphisms for each t with $|t| \leq \epsilon$. A smooth vector field X is an infinitesimal CR automorphism if and only if the Lie derivative of a section L of $\mathcal{V}$ with respect to X is again a section of $\mathcal{V}$, that is, $[X, L] \in \mathcal{V}$. We set

$$[X, L_i] = \alpha_i^j L_j, \quad \text{(summation convention)}, \tag{3.3}$$

for some functions α_i^j for each $i = 1, \ldots, n$. We have

THEOREM 3.1. *Let M^{2n+1} be a C^ω CR manifold of nondegenerate Levi form. Then the defining equation (3.3) of the infinitesimal CR automorphisms admits prolongation to a complete system of order 3. Therefore, a C^3 infinitesimal CR automorphism is in fact C^ω. Moreover, the set of infinitesimal CR automorphisms of M forms a finite dimensional Lie algebra.*

Proof. In this proof we use summation convention: repeated indices mean the summation over 1 through n. (3.3) is linear in X. Suppose that X_1 and X_2 are solutions of (3.3). Then

$$[[X_1, X_2], L] = -[[X_2, L], X_1] - [[L, X_1], X_2], \tag{3.4}$$

by Jacobi identity. Each X_j, $j = 1, 2$, is an infinitesimal CR automorphism and hence $Z_j := [X_j, L]$ is a section of $\mathcal{V}$. Then the right hand side of (3.4) is $-[Z_2, X_1] + [Z_1, X_2]$, which is again a section of $\mathcal{V}$. This shows that infinitesimal CR automorphisms are closed under the bracket operation and hence form a Lie algebra. To prove the other assertions of the theorem it suffices to show that (3.3) admits a prolongation to a complete system of order 3. Choose a C^ω nonvanishing real vector field T of M which is transversal to the CR structure bundle $\mathcal{V}$ and set

$$[L_i, \bar{L}_j] = \sqrt{-1}\rho_{i\bar{j}}T, \quad \mathrm{mod} \quad (\mathcal{V} + \bar{\mathcal{V}}). \tag{3.5}$$

The hermitian matrix $(\rho_{i\bar{j}})_{i,j=1,\dots,n}$ is the Levi form. We may assume that at the reference point $(\rho_{i\bar{j}})$ is a diagonal matrix with diagonal elements ± 1, for otherwise we make a suitable linear change of basis $L_1, \dots, L_n$. Set

$$X = f^\lambda L_\lambda + \bar{f}^\lambda \bar{L}_\lambda + gT, \quad g \text{ is real} \tag{3.6}$$

$$[L_i, \bar{L}_j] = \sqrt{-1}\rho_{ij}T + a_{ij}^\lambda L_\lambda + b_{ij}^\lambda \bar{L}_\lambda \tag{3.7}$$

$$[L_i, L_j] = c_{ij}^\lambda L_\lambda \tag{3.8}$$

and

$$[T, L_i] = A_i^\lambda L_\lambda + B_i^\lambda \bar{L}_\lambda + E_i T. \tag{3.9}$$

By substituting (3.6) for X in (3.3) and by equating the corresponding componets we have

$$\alpha_i^\lambda = -L_i f^\lambda + f^\mu c_{\mu i}^\lambda - \bar{f}^\mu a_{i\mu}^\lambda + gA_i^\lambda, \tag{3.10}$$

and

$$-L_i \bar{f}^\lambda - \bar{f}^\mu b_{i\mu}^\lambda + gB_i^\lambda = 0$$

or equivalently

$$\bar{L}_i f^\lambda = -f^\mu \bar{b}_{i\mu}^\lambda + g\bar{B}_i^\lambda \tag{3.11}$$

and

$$L_i g = -\bar{f}^\mu \rho_{i\mu} + gE_i. \tag{3.12}$$

On the other hand by Jacobi identity

$$\begin{aligned}
[X,[L_i,\bar{L}_j]] &= -[L_i,[\bar{L}_j,X]] - [\bar{L}_j,[X,L_i]] \\
&= [L_i, \quad -\bar{\alpha}_j^\lambda \bar{L}_\lambda] - [\bar{L}_j, \quad \alpha_i^\lambda L_\lambda] \\
&= (\bar{\alpha}_j^\lambda \rho_{i\lambda} + \alpha_i^\lambda \rho_{\lambda j})T \quad + \quad L \text{ and } \bar{L} \text{ components}
\end{aligned} \tag{3.13}$$

In the left hand side of (3.13) substitute (3.6) for X and (3.7) for $[L_i, \bar{L}_j]$. By equating the T-components and by (3.12) we obtain

$$\bar{\alpha}_j^\lambda \rho_{i\lambda} + \alpha_i^\lambda \rho_{\lambda j} = -\sqrt{-1}\rho_{ij} Tg + \langle 1, f, \bar{f}, g\rangle, \tag{3.14}$$

where $\langle\ \rangle$ denotes an element of the module over the ring of analytic functions of M generated by the arguments inside $\langle\ \rangle$, that is, a linear combination of the arguments with analytic coefficients. In (3.14) substitute (3.10) for α_i^λ then we have

$$\rho_{ij} Tg = (L_i f^\lambda)\rho_{\lambda j} + (\bar{L}_j \bar{f}^\lambda)\rho_{i\lambda} + \langle 1, f, \bar{f}, g\rangle. \tag{3.15}$$

Since the matrix of coefficients $(\rho_{\lambda j})_{\lambda, j=1,\dots,n}$ is invertible we can solve (3.15) for $L_i f^\lambda$ to obtain

$$L_i f^\lambda = \langle 1, f, g, \bar{f}, \bar{L}\bar{f}, Tg\rangle. \tag{3.16}$$

Now we shall express all the 3rd order partial derivatives $L^\alpha \bar{L}^\beta T^t f$ and $L^\alpha \bar{L}^\beta T^t g$ as analytic functions of lower order derivatives, where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ are multi-indices and $|\alpha|+|\beta|+t = 3$. If $|\beta| \neq 0$ then by (3.11) $L^\alpha \bar{L}^\beta T^t f$ reduces to the second order. If $|\alpha|+|\beta| \neq 0$ then by (3.12) and its complex conjugate $L^\alpha \bar{L}^\beta T^t g$ reduces to the second order. Thus it suffices to express $L^\alpha T^t f$ with $|\alpha|+|\beta| = 3$ and $T^3 g$ in terms of derivatives of f and g of order ≤ 2. By applying L_j to (3.16) we obtain

$$L_j L_i f = \langle 1, f, g, , L_j f, L_j g, L_j \bar{L} \bar{f}, \bar{L} \bar{L} \bar{f}, L_j T g \rangle.$$

In the right hand side $L_j \bar{L}_i \bar{f} = ([L_j, \bar{L}_i] + \bar{L}_i L_j) \bar{f} = \langle \bar{L} \bar{f}, T \bar{f} \rangle$ by (3.7) and (3.11), $L_i g = \langle g, \bar{f} \rangle$ by (3.12) and $Lf = \langle 1, f, g, \bar{f}, \bar{L} \bar{f}, Tg \rangle$ by (3.16) and

$$\begin{aligned} L_j T g &= (-[T, L_j] + T L_j) g \\ &= \langle g, Tg, \bar{f}, T\bar{f} \rangle \end{aligned}$$

by (3.12). Hence we have

$$L_j L_i f = \langle 1, f, g, Tg, \bar{f}, \bar{L}\bar{f}, T\bar{f} \rangle. \tag{3.17}$$

Apply L_k to (3.17). Then in the same way as above we have

$$L_k L_j L_i f = \langle 1, f, g, Tg, \bar{f}, \bar{L}\bar{f}, T\bar{f} \rangle, \quad \forall i, j, k = 1, \dots, n. \tag{3.18}$$

Apply $\bar{L}_l$ to (3.18). Then we have

$$\begin{aligned} \bar{L}_l L_k L_j L_i f = L_k L_j L_i \bar{L}_l f &- \{\rho_{kl} T L_j L_i + \rho_{jl} T L_k L_i + \rho_{il} T L_j L_k\} f \\ &+ \langle f, Lf, Tf, L^2 f, TLf, \rangle, \end{aligned}$$

where the last two terms occur when $\bar{L}_l$ commutes with L's. In the first term of the RHS $\bar{L}_l f = \langle f, g \rangle$ by (3.11) hence by (3.18) and (3.12) the first term of the RHS is $\langle 1, f, g, Tg, \bar{f}, \bar{L}\bar{f}, T\bar{f} \rangle$. Since (ρ_{kl}) is diagonal and invertible at the reference point we can solve for $TL_j L_i$ for each $i, j = 1, \dots, n$. Thus we have

$$TL_j L_i f = \langle 1, f, g, Lf, Tf, Tg, TLf \rangle + \langle \bar{f}, \bar{L}\bar{f}, T\bar{f}, \bar{L}^2 \bar{f}, T\bar{L}\bar{f} \rangle. \tag{3.19}$$

Apply $\bar{L}_k$ to (3.19). Then

$$T^2L_if = \langle 1, f, g, Lf, Tf, Tg, L^2f, TLf, T^2f\rangle + \langle \bar{f}, \bar{L}\bar{f}, T\bar{f}, \bar{L}^2\bar{f}, T\bar{L}\bar{f}, \bar{L}^3\bar{f}, T\bar{L}^2\bar{f}\rangle. \tag{3.20}$$

In the right hand side of (3.20) substitute for $\bar{L}^3\bar{f}$ and for $T\bar{L}^2\bar{f}$ the complex conjugate of (3.18) and (3.19), respectively, then we have

$$T^2L_if = \langle 1, f, g, Lf, Tf, Tg, L^2f, TLf, T^2f\rangle + \langle \bar{f}, \bar{L}\bar{f}, T\bar{f}, \bar{L}^2\bar{f}, T\bar{L}\bar{f}\rangle. \tag{3.21}$$

Apply $\bar{L}_i$ to (3.21). Then we have

$$T^3f = \langle 1, f, g, Lf, Tf, L^2f, TLf, T^2f\rangle + \langle \bar{f}, \bar{L}\bar{f}, \bar{L}^2\bar{f}, T\bar{L}\bar{f}\rangle. \tag{3.22}$$

Finally, by applying T^2 to (3.15) we have

$$T^3g = \langle \text{terms of order} \leq 2\rangle + \langle T^2Lf, T^2\bar{L}\bar{f}\rangle.$$

In the right hand side we substitute (3.21) for T^2Lf and the complex conjugate of (3.21) for $T^2\bar{L}\bar{f}$, to obtain

$$T^3g = \langle 1, f, g, Lf, Tf, Tg, L^2f, TLf, T^2f\rangle + \langle \bar{f}, \bar{L}\bar{f}, T\bar{f}, \bar{L}^2\bar{f}, T\bar{L}\bar{f}, T^2\bar{f}\rangle. \tag{3.23}$$

Thus we obtained a complete system of order 3 for $(f, \bar{f}, g)$. □

4. Complete system, rigidity of embeddings and regularity of rigid embeddings

Hayashimoto ([24]) constructed complete systems of finite order for the cases of CR manifolds of degenerate Levi forms of finite type. He used the method of prolongation and made essential use of the fact that the CR functions $f = (f_1, \ldots, f_{n+1})$ are related by $r \circ f = 0$, where r is a C^ω defining function of the target manifold. In pseudo-hermitian embedding (see [39] for definition) , however, there is no such relation among $f_1, \ldots, f_{n+1}$. Instead, the mappings' being psudo-hermitian gives relation among derivatives of f and Kim [33] obtained a complete system for this case also under a very complicated but generic assumption. In general, if a family of mappings of a manifold M into M' satisfy a complete system (2.4) of order k then a mapping of the family is determined by its $(k-1)$-jet at a point. Conversely, if a mapping of a certain family is determined by its finite jet at a point our question is whether the system of differential equations that defines the family admits prolongation to a complete system. The first result in this direction is [34], where Kim proved using the Segre variety the finiteness of CR immersions of a C^ω CR manifold M into another C^ω CR manifold M' with degenerate Levi forms of finite type and then constructed a complete system for the CR immersions of M into M'. It seems that if the Segre variety argument shows the finite determination of CR mappings then this finiteness actually comes from a complete system. Our further question is whether a rigid local embedding necessarily satisfies a complete system, namely

PROBLEM 4.1. *Let M be a C^ω CR manifold of hypersurface type of dimension $2n+1$, $n \geq 1$. A CR immersion f of M into a sphere S^{2N+1}, $N \geq n$, is said to be rigid if for any CR immersion g of M into the sphere there exists a CR automorphism ϕ of the sphere such that $g = \phi \circ f$. If M admits a rigid immersion into the sphere does there exist a complete system of finite order for the CR immersions?*

PROBLEM 4.2. *Let (M, g) be a Riemannian manifold of dimension n. A smooth map $u = (u^1, \ldots, u^N) : M \to \mathbb{R}^N$ is an isometric immersion if*

$$\langle du, du \rangle = g. \tag{4.1}$$

In terms of local coordinates $x = (x_1, \ldots, x_n)$ (4.1) is

$$\sum_{\alpha=1}^{N} \frac{u^\alpha}{\partial x_i} \frac{u^\alpha}{\partial x_j} = g_{ij}(x)$$

for each $i, j = 1, \ldots, n$, where $g_{ij}(x) = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle$. A solution $f = (f^1, \ldots, f^N)$ of (4.1) is said to be rigid if for any solution g of (4.1) there exists an isometry ϕ of $\mathbb{R}^N$ such that $f = \phi \circ g$. If M admits a rigid isometric immersion then does (4.1) admit a prolongation to a *complete system of finite order? If M is C^ω then is a rigid isometric immersion f necessarily C^ω?*

There is a conformal version of the above questions where the embedding equation is

$$\langle du, du \rangle = \lambda g, \tag{4.2}$$

for some positive valued function λ and the other analogies are obvious.

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