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# CR GEOMETRY/ANALYSIS AND DEFORMATION OF ISOLATED SINGULARITIES

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ABSTRACT. In the late 1970's, M. Kuranishi proposed to control the moduli of the germ of a normal Stein space by deformations of the CR structure on the boundary. I this paper, we will see that it is naturally accomplished by considering stably embeddable deformations of CR structures.

## 1. Introduction

Complex manifolds are investigated from various points of view; the holomorphic viewpoint (e.g. algebraic geometry and holomorphic function theory) and the differentiable viewpoint (e.g. differential geometry and partial differential equations). The subject of this paper concerns the moduli of singular varieties from the differentiable viewpoint. The moduli of singular varieties consists of two factors; the moduli of local structures and the moduli coming from the global arrangement of local pieces. As a first step, we consider the moduli of germs of singular varieties having only isolated singularities. For simplicity, throughout this paper, we assume that $V^{'}$ is an analytic subvariety (i.e., a reduced and irreducible analytic subset) of a neighborhood of $\overline{B(c)}$ in $\mathbf{C}^N$ with $\mathrm{Sing}(V^{'}) = \{0\}$ and intersecting transversely with $S_\epsilon^{2N-1} := \partial B(\epsilon)$ for all $0 < \epsilon \leq c$, where $B(c)$ denotes a ball centered at 0 with radius $c > 0$. From the differentiable viewpoint, there would be two ways to control the moduli of germs of $V^{'}$; by the moduli of its regular part and by that of the boundary. M. Kuranishi ([22]) proposed to take the latter approach and to control the moduli of the germ of a normal

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Stein $V'$ with $\dim_{\mathbf{C}} V' \geq 3$, by deformations of the CR structure on $M := V' \cap S_\epsilon^{2N-1}$. In this proposal, the normality is necessary since, without it, it is impossible to control the singularity of $V'$ by the CR structure on $M$ (cf. §4). The surface singularity case was excluded since all real three dimensional compact strongly pseudoconvex CR manifolds need not be boundaries of Stein spaces.

After M. Kuranishi, this approach was done under the assumptions $\dim_{\mathbf{C}} V' \geq 4$ and depth $\mathcal{O}_{V'} \geq 3$ by [3], [24] and [8]. Then, the following questions naturally arise (note that a normal variety is assumed to be $\dim_{\mathbf{C}} V' \geq 2$ and depth $\mathcal{O}_{V'} \geq 2$):

Is there any CR approach to treat deformations of normal isolated surface singularities?

Is there any way to weaken the dimensional condition and the depth-condition?

For the surface singularity case, [6] recently showed that stably embeddable formal deformations of the CR structure on $M$ correspond to formal deformations of $V$. We will proceed along this line in all dimensions and see that we can affirmatively answer all of the above questions in a natural manner.

This paper consists of three parts. The first part is preliminaries for the subject of this paper; compact strongly pseudoconvex CR manifolds and the moduli problem. We will review that the embeddability of a compact strongly pseudoconvex CR manifold is a critical property both in geometry and in analysis. Next, as a model case of the moduli problem, we review the moduli of compact complex manifolds. We recall that the moduli need not exists in general and, instead, the semi-universal family is taken as a center of the theory of moduli. We will review the construction of the semi-universal family of compact complex manifolds in the differentiable viewpoint, due to K. Kodaira, D. C. Spencer and M. Kuranishi. The Kodaira-Spencer's method reviewed in this part is the model of (the construction of the Kuranishi semi-universal family of) our deformation theory of CR structures.

The second part is the presentation of the problem which we consider. Before the presentation, we will review the deformation theory of normal isolated singularities. We will see that the flatness (resp. the stable embeddability) is a natural requirement in the holomorphic viewpoint (resp. in the differentiable viewpoint), in order for the correct definition of deformations of singular varieties (resp. of normal isolated singularities). In the latter half of the second part, we present the Kuranishi program for constructing the semi-universal family of deformations of normal isolated singularities by means of deformations of CR structures and analyze the meaning of the conditions $\dim_{\mathbf{C}} V' \geq 4$ and depth $\mathcal{O}_{V'} \geq 3$ in the preceding partial solution.

In the third part, we will see how to accomplish the Kuranishi program. At first, we will see that the stable embeddability is the differentiable-counterpart of the flatness condition. It was shown in [6] for the surface singularity case. Then, we will briefly see how to accomplish the Kuranishi program using the theory of stably embeddable deformations of CR structures. Details of the third part will appear in [25].

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## Part I. Preliminaries

### §1 Geometry and analysis on CR manifolds

In this section, we review geometry and analysis on a compact strongly pseudoconvex CR manifold, in particular, that the embeddability of a CR manifold is a critical property both in geometry and analysis.

#### 1.1. Strongly pseudoconvex CR structures

The following definition of a CR structure is an abstract model of the structure on a real hypersurface of a complex manifold.

Definition 1.1.1. Let $M$ be an orientable $C^\infty$-manifold of $\dim_{\mathbf{R}} M = 2n-1$ $(n \geq 2)$. A CR structure on $M$ is a complex subbundle $\overline{S} \subset \mathbf{C}TM$ of $\operatorname{rank}_{\mathbf{C}} \overline{S} = n-1$ having the following property;

(1) $S \cap \overline{S} = \{0\}$ where $S := \overline{\overline{S}}$,

(2) $[X, Y] \in C^\infty(M, S)$ holds for all $X$, $Y \in C^\infty(M, S)$.

If $M$ is a real hypersurface of a complex manifold $X$, then

$$^{\circ}T'' := T^{0,1}X_{|M} \cap \mathbf{C}TM$$

is a CR structure which is called the CR structure on $M$ induced from the complex structure of $X$.

Let $\overline{S}$ be a CR structure on $M$. If we fix a line sub-bundle $F \subset TM$ such that $F \simeq TM/Re(S \oplus \overline{S})$, then we have a type-decomposition

$$\mathbf{C}TM = \mathbf{C}F \oplus S \oplus \overline{S}. \tag{1.1.2}$$

This type-decomposition naturally induces a type-decomposition of differential forms, and then a differential complex $(A_b^{0,q},\, \overline{\partial}_b)$ is introduced analogously to the $\overline{\partial}$-complex on complex manifolds, where we denote $A_b^{0,q} := C^\infty(M, \wedge^q(\overline{S})^*)$. A holomorphic vector bundle on a CR manifold is defined analogously to the differential geometric definition of a holomorphic vector bundle over a complex manifold (cf. [34]) and the $\overline{\partial}_b$-operator is naturally defined for holomorphic vector bundle valued $(0, q)$-forms.

If we choose a local dual form $\theta$ of $F$, the *Levi-form* is defined as an hermitian form;

$$\mathcal{L}_{\overline{S},p} :\, S_p \otimes \overline{S_p} \ni (u,\, v) \longrightarrow \sqrt{-1}\theta([\tilde{u}, \overline{\tilde{v}}])(p) \in \mathbf{C}$$

with $p \in M$ and $\tilde{u}$, $\tilde{v} \in C^\infty(M, S)$ such that $\tilde{u}(p) = u$, $\tilde{v}(p) = v$. We call a CR structure *strongly pseudo-convex* if the Levi-form has a definite sign at any point of $M$. (This property is independent of the choice of $\theta$.) We remark that the CR structure on a real hypersurface defined by a strictly plurisubharmonic function is strongly pseudoconvex.

### 1.2. Geometry and analysis on a compact strongly pseudo-convex CR manifold

A CR structure $\overline{S}$ on $M$ is *embeddable* if there exists a CR embedding into some complex Euclidean space; that is, there exists an embedding

$$f :\, M \to \mathbf{C}^N$$

such that $\overline{S_p} = df_p^{-1}(T^{0,1}_{f(p)}\mathbf{C}^N)$ holds for all $p \in M$. The geometric feature of the embeddability of a compact CR manifold is in the following theorem of F. Harvey and H. Lawson.

THEOREM 1.2.1 ([16]). *Let $M$ be a compact CR manifold embedded in $\mathbf{C}^N$. Then there exists an analytic subvariety $V$ of $\mathbf{C}^N$ such that $M = \partial V$ as a current.*

Under some assumption of the Levi form (it is satisfied for a strongly pseudoconvex CR manifold), we have a strict form of the above Harvey-Lawson theorem which gives the basic correspondence between isolated singularities and compact strongly pseudoconvex CR manifolds.

(I) Geometry on a compact strongly pseudoconvex CR manifold. Let $M$ be a compact strongly pseudoconvex CR manifold.

THEOREM 1.2.2. *$M$ is embeddable if and only if $M$ is a real hypersurface of a Stein space.*

*Proof.* Suppose that $f_0 := (f_1, \ldots, f_N) : M \to \mathbf{C}^N$ is a CR embedding. For $p \in M$, there exist $f_{i_1}, \ldots, f_{i_n}$ such that $x \to (f_{i_1}(x), \ldots, f_{i_n}(x))$ ($\dim_{\mathbf{R}} M = 2n-1$) is a CR embedding of a neighborhood of $p$ as a real hypersurface of an open domain of $\mathbf{C}^n$. Let

$$F_i := (f_{i_1}, \ldots, f_{i_n}) : U_i \to D_i \subset \mathbf{C}^n$$

be such an embedding, $M_i := F_i(U_i)$ and $D_i^+$ (resp. $D_i^-$) a domain of $\mathbf{C}^n$ such that $F_i(U_i)$ is the convex boundary of $D_i^+$ (resp. the concave boundary of $D_i^-$) and $D_i = D_i^+ \cup D_i^- \cup M_i$ holds. Let $F_i : U_i \to D_i \subset \mathbf{C}^n$, $F_j : U_j \to D_j \subset \mathbf{C}^n$, be two embeddings as above. Then the coordinate transformation

$$F_{ij} : f_j(U_i \cap U_j) \to f_i(U_i \cap U_j)$$

on $U_i \cap U_j$ is uniquely extended to a biholomorphic transformation

$$\tilde{F}_{ij} : D_j^- \supset D_{ij}^- \to D_{ji}^- \subset D_i^-$$

by the Lewy extension theorem. By the uniqueness of the extension, we can patch $D_i^-$'s together and obtain a strip of a complex manifold $W$ whose convex boundary is $M$. Next, by [31] ($\dim_{\mathbf{R}} M \geq 5$) or [36] ($\dim_{\mathbf{R}} M \geq 3$), we can uniquely complete the holes of $W$ in the concave side to obtain a normal Stein space $V$ such that $\partial V = M$. Finally, by [28] ($\dim_{\mathbf{R}} M \geq 5$) or [10] ($\dim_{\mathbf{R}} M \geq 3$), we can enlarge $V$

to a normal Stein space $V'$ such that $V \subset\subset V'$ holds. Therefore $M$ is a real hypersurface of a normal Stein space $V'$.

On the contrary, we suppose that $M$ is a real hypersurface of a Stein space $V$. Since a Stein space is holomorphically embedded in some complex Euclidean space, the restriction of that holomorphic embedding gives a CR embedding of $M$. □

A remarkable fact in geometry is the following theorem due to L. Boutet de Monvel.

THEOREM 1.2.3 ([7]). *Any compact strongly pseudoconvex CR manifold of* $\dim_{\mathbf{R}} M \geq 5$ *is embeddable.*

(II) Analysis on a compact strongly pseudoconvex CR manifold. Let $M$ be a compact strongly pseudoconvex CR manifold as above.

THEOREM 1.2.4 ([20]). *$M$ is embeddable if and only if the range of $\overline{\partial}_b$ in the $L^2$-space is closed.*

Analytical counterpart of Theorem 1.2.3 is the following.

THEOREM 1.2.5 ([11]). *The range of $\overline{\partial}_b$ is closed for any compact strongly pseudoconvex CR manifold of* $\dim_{\mathbf{R}} M \geq 5$.

### 1.3. $\overline{\partial}_b$-analysis on a compact strongly pseudoconvex embeddable CR manifold

In order to speak of the harmonic analysis, $M$ need to be embeddable (cf. Theorem 1.2.3). Let $M$ be a compact strongly pseudoconvex CR manifold embedded in $\mathbf{C}^N$ and denote by ${}^{\circ}T''$ the CR structure induced from the complex structure of $\mathbf{C}^N$. For the deformation theories of CR structures, we need the vector bundle valued $\overline{\partial}_b$-analysis. Since the tangential Cauchy-Riemann operator $\overline{\partial}_b$ contains derivations only in ${}^{\circ}T''$, we employ the Folland-Stein norm $\|\ \|_{(k)}$ as a function norm (cf. [12] for the Folland-Stein norm).

(I) The case of $\dim_{\mathbf{R}} M \geq 5$:

The following Hodge-Kodaira-type decomposition is proved by directly showing a priori estimate.

THEOREM 1.3.1 ([11], [12]). *Let* $\mathbf{E}$ *be a holomorphic vector bundle on $M$. Then, for $1 \leq q \leq n-2$, we have*

(1) *$u = \rho_b u + \Box_b N_b u$ holds for $u \in A_b^{0,q}(\mathbf{E})$, where $\rho_b$ denotes the orthogonal projection onto the harmonic space $\mathbf{H}_b^{0,q}(\mathbf{E}) := \{u \in A_b^{0,q}(\mathbf{E}) \mid \Box_b u = 0\}$,*

(2) *$||N_b u||_{(k+2)} \leq C||u||_{(k)}$ holds.*

(II) The case of $\dim_{\mathbf{R}} M = 3$:

The Hodge-Kodaira-type decomposition is obtained by utilizing the $\bar{\partial}$-analysis on the inside strongly pseudoconvex domain and the estimate is proved by the technique of [5].

THEOREM 1.3.2 ([20], [5], [26]). *Let $X'$ be a two-dimensional complex manifold and $M = \partial\Omega$ for some strongly pseudoconvex bounded domain $\Omega \subset X'$ and $\mathbf{E} = E_{|M}$ for a holomorphic vector bundle $E$ on $X'$. Then,*

(1) *$u = \rho_b u + Q_b \bar{\partial}_b u$ holds for $u \in A_b^0(\mathbf{E})$, where $\rho_b$ denotes the orthogonal projection onto the space $\mathbf{H}_b^0(\mathbf{E}) := \{u \in A_b^0(\mathbf{E}) \mid \bar{\partial}_b u = 0\}$,*

(2) *$u = \rho_b u + \bar{\partial}_b Q_b u$ holds for $u \in A_b^{0,1}(\mathbf{E})$, where $\rho_b$ denotes the orthogonal projection onto the space $\mathbf{H}_b^{0,1}(\mathbf{E}) := \{u \in A_b^{0,1}(\mathbf{E}) \mid \bar{\partial}_b^* u = 0\}$,*

(3) *$||Q_b u||_{(k+1)} \leq C||u||_{(k)}$ and $||\rho_b u||_{(k)} \leq C||u||_{(k)}$ hold.*

### 1.4. Perturbations of CR structure

Let $M$ be a strongly pseudoconvex CR manifold with the reference CR structure ${}^\circ T''$. We fix a type-decomposition

$$\mathbf{C}TM = \mathbf{C}F \oplus {}^\circ T' \oplus {}^\circ T''$$

and denote

$$T' := \mathbf{C}F \oplus {}^\circ T'$$

where ${}^\circ T' := \overline{{}^\circ T''}$.

THEOREM 1.4.1 ([2]). *A small perturbation of ${}^\circ T''$ is given by*

$${}^\phi T'' := \{\bar{u} - \phi(\bar{u}) \mid \bar{u} \in C^\infty(M, {}^\circ T'')\}$$

*where $\phi \in A_b^{0,1}(T')$ satisfying the integrability condition*

$$\bar{\partial}_b \phi - \frac{1}{2}[\phi, \phi] - R_3(\phi) = 0$$

*with denoting by $R_3(\phi)$ the third order term.*

PROPOSITION 1.4.2. *$f \in C^\infty(M)$ is a CR fuction with respect to the CR structure ${}^\phi T''$ if and only if it satisfies the perturbed Cauchy-Riemann equation*

$$\overline{\partial}_b^\phi f := \overline{\partial}_b f - \phi \cdot f = 0.$$

## §2 Moduli and the semi-universal family

As a model case of the moduli problem, we review the moduli of compact complex manifolds. For simplicity, we restrict ourselves to the case of non-singular moduli (or families with non-singular parameter spaces). We note that everything in this section can be generalized to the case of singular moduli (or families with singular parameter spaces).

### 2.1. Moduli of compact complex manifolds

Moduli is a natural parameterization of the objects we consider. It goes back to Riemann. Compact Riemann surfaces have natural holomorphic parameters of dimension 0, 1 and $3g-3$ according to its genus 0, 1 and $g \geq 2$ respectively. It is a dimension of the space

$$\{\text{all compact Riemann surfaces of genus } g\}/\{\text{biholomorphic equivalence}\}$$

which is called the moduli space of compact Riemann surfaces of genus $g$. In their fundamental paper of the deformation theory, K. Kodaira and D. C. Spencer (cf. [18] or [17]) showed that, in the higher dimensional case, such moduli space need not exist even locally and the semi-universal family (a complete and effective family, in their terminology) should be taken as a center of the theory of moduli.

According to K. Kodaira and D. C. Spencer, a family of deformations of a compact complex manifold $X$, parametrized by a neighborhood of $D \subset \mathbf{C}^d$, is a proper surjective holomorphic map

$$\pi : \mathcal{X} \to D \tag{2.1.1}$$

with $\pi^{-1}(0) = X$ and such that $d\pi_x : T_x^{1,0}\mathcal{X} \to T_{\pi(x)}^{1,0}D$ is surjective for all $x \in \mathcal{X}$. Among all families of deformations of $X$, the universal family is defined as a family from which any other family of deformations of $X$ is obtained by a uniquely determined parameters-change. Then, the parameter space of the universal family (if it exists) provides the local structure of the moduli space around (the point of the moduli space corresponding to) $X$. As was shown in [18], the universal family need not exist and its substitute is the semi-universal family: A family is called *semi-universal* if any other family of deformations of $X$ is obtained from that family by a parameters-change which is uniquely determined up to the first order term. Therefore, the existence and the construction of the semi-universal family is a principal problem of deformation theories.

In the modern deformation theory due to A. Grothendieck and M. Schlessinger (cf. [32] and [29]), the formal construction of the semi-universal family is performed by successive extensions of families of finite order deformations and the problem of the actual construction is reduced to an analytic problem, where a family of $n$-th order deformations is a family whose parameter space has the structure local ring $\mathcal{O}_{T_n} = \mathcal{O}_T/\mathfrak{m}_T^{n+1}$ with denoting $\mathfrak{m}_T$ the maximal ideal of $\mathcal{O}_T$. The extension is completely controlled by the following two classes represented by the cohomology classes in $H^1(X, \Theta_X)$ and in $H^2(X, \Theta_X)$ respectively; the *infinitesimal deformation class* which represents a family of first order deformations modulo first order biholomorphisms and the *obstruction class* which represents an obstruction for the extension of a family of finite order deformations. The method of successive extensions of finite order deformations is closely related to the Kodaira-Spencer's technique which will be stated in subsection 2.2. The relation between the concept of a family of actual deformations and that of a family of finite order deformations is similar to that between a holomorphic function and a finite segment of its Taylor series.

### 2.2. Holomorphic description and differentiable description

In the case of complex manifolds, both of the viewpoints are directly connected. From the holomorphic view point, a complex manifold is determined by patchings of holomorphic local charts, while from the differentiable view point, it is determined by a complex structure on the underlying differentiable manifold. Both of the descriptions of complex manifolds are equivalent to each other by the theorem of A. Newlander-L. Nirenberg ([27]).

From the holomorphic view point, a family of deformations is given by a holomorphic parameterization of the patchings of holomorphic local charts, which directly provides a family of deformations of a complex manifold as was defined in subsection 2.1. In this manner, the semi-universal family of a compact complex manifold was constructed by O. Forster and K. Knorr ([13]).

From the differentiable view point, there are two ways to treat the moduli of compact complex manifolds. First way is due to K. Kodaira and D. C. Spencer and the other due to M. Kuranishi. In both of the ways, the construction of the semi-universal family relies on the fact that a perturbation of the complex structure on a complex manifold $X$ is given by a $\phi \in A_X^{0,1}(T^{1,0}X)$ satisfying the integrability condition

$$\bar{\partial}\phi - \frac{1}{2}[\phi, \phi] = 0. \tag{2.2.1}$$

In the way due to K. Kodaira and D. C. Spencer (cf. [18] or [17]), a family of complex structures is given as a powerseries $\phi(t)$ in $A_X^{0,1}(T^{1,0}X)[[t_1, \ldots, t_d]]$ satisfying

(1) $\phi(0) = 0$,

(2) $\bar{\partial}\phi(t) - \frac{1}{2}[\phi(t), \phi(t)] = 0$,

(3) $\phi(t)$ is convergent with respect to the Sobolev norm $|| \ ||_k$.

This description of a family of deformations of the complex structure is not formulated in the general deformation theoretic context (i.e., not formulated by a deformation functor). But $H^1(X, T^{1,0}X)$ is considered as the space of infinitesimal deformation classes and the obstruction class to the extension of a family $\phi^{(\mu-1)}(t)$ satisfying $\bar{\partial}\phi^{(\mu-1)}(t) - \frac{1}{2}[\phi^{(\mu-1)}(t), \phi^{(\mu-1)}(t)] \equiv 0 \mod \mathfrak{m}^{\mu}$ to a family $\phi^{(\mu)}(t)$ satisfying $\bar{\partial}\phi^{(\mu)}(t) - \frac{1}{2}[\phi^{(\mu)}(t), \phi^{(\mu)}(t)] \equiv 0 \mod \mathfrak{m}^{\mu+1}$ appears in $H^2(X, T^{1,0}X)$, where we denote by $\mathfrak{m}^{\mu}$ the ideal of all polynomials of $t = (t_1, \ldots, t_d)$ of degree greater than or equals to $\mu$. As a counterpart of the semi-universal family of deformations of $X$, K. Kodaira and D. C. Spencer constructed a powerseries $\phi(t)$ in $A_X^{0,1}(T^{1,0}X)[[t_1, \ldots, t_d]]$ satisfying (1)–(3) as above and

(4) $\phi(t) = \sum_{\sigma=1}^{d} \phi_\sigma t_\sigma + \ldots$ where $\bar{\partial}\phi_\sigma = 0 \, (\sigma = 1, \ldots, d)$, $[\phi_1], \ldots, [\phi_d]$ is a base of $H^1(T^{1,0}X)$ and $+\ldots$ denotes the terms in $\mathfrak{m}^2$.

In their argument, (3) is proved by the method of majorants based on the elliptic estimate for the Green's operator $||Gu||_k \leq C||u||_{k-2}$.

Then, by the theorem of A. Newlander-L. Nirenberg ([27]) and the Kodaira-Spencer's completeness theorem (cf. [19] or [17] Theorem 6.1), $\phi(t)$ as above provides the semi-universal family of deformations of $X$.

REMARK 2.2.2. Though the Kodaira-Spencer's construction as above was done under the assumption $H^2(X, T^{1,0}X) = 0$ (cf. [17]), it can be generalized to the case of $H^2(X, T^{1,0}X) \neq 0$.

There is another way due to Kuranishi (cf. [21]). He directly considered the orbit space of complex structures on $X$ as the moduli space of complex manifolds diffeomorphic to $X$;

$$\left\{\phi \in A_X^{0,1}(T^{1,0}X) \,|\, \bar{\partial}\phi - \frac{1}{2}[\phi, \phi] = 0\right\} / \mathrm{Diff}(X).$$

But, in general, it is impossible to put a complex analytic structure on it even locally (like as the universal family need not exist in the holomorphic view point) since the dimension of the automorphism group of the complex structure (corresponding to) $\phi$ which acts as the isotopy group at $\phi$ may change with $\phi$. To avoid this difficulty, he considered a modified orbit space

(2.2.3)

$$\left\{\phi \in A_X^{0,1}(T^{1,0}X) \,|\, ||\phi||_k : \text{small}, \ \bar{\partial}\phi - \frac{1}{2}[\phi, \phi] = 0\right\} / {}^{\perp}C^{\infty}(X, T^{1,0}X),$$

where ${}^{\perp}C^{\infty}(X, T^{1,0}X)$ denotes the orthogonal complement of $H^0(X, T^{1,0}X)$ and we use the same notation for the set of diffeomorphisms induced from ${}^{\perp}C^{\infty}(X, T^{1,0}X)$ by the exponential map, and put a complex structure on it by taking a slice (called the Kuranishi slice) to the action of ${}^{\perp}C^{\infty}(X, T^{1,0}X)$;

(2.2.4)

$$\left\{\phi \in A_X^{0,1}(T^{1,0}X) \,|\, ||\phi||_k : \text{small}, \ \bar{\partial}\phi - \frac{1}{2}[\phi, \phi] = 0, \ \bar{\partial}^*\phi = 0\right\}.$$

Then, it is proved that the Kuranishi slice is isomorphic to the parameter space of the semi-universal family of deformations of $X$ (at least, as reduced complex spaces, cf. (2.2.3) or [21]). In this case, $H^1(X, T^{1,0}X)$ is realized as the tangent space at the origin of the space (2.2.4). In these arguments, the main analytical tool is an application of the Banach inverse mapping theorem based on the estimate $||Gu||_k \leq C||u||_{k-2}$.

## Part II. The Kuranishi program for constructing the semi-universal family of normal isolated singularities

### §3 Deformations of a normal isolated singularity

The moduli of a (even compact) singular variety contains a new factor which does not appear in the manifolds-case. It is because the local structure of a singular variety has non-trivial moduli, while any local neighborhood of a complex manifold has only trivial moduli. In this paper, we consider only the moduli of germs of complex spaces with only isolated singular points.

Let $V$ be an irreducible and reduced complex analytic space with $\mathrm{Sing}(V) = \{0\}$. Since we consider a germ of $V$ at $0 \in V$, we may assume that $V$ is an analytic subspace of a neighborhood of 0 in $\mathbf{C}^N$ defined by a finite number of holomorphic equations $h_1(w) = \cdots = h_m(w) = 0$. There are two natural ways to perturb $V$. The one way is to perturb the defining equations and the other is to perturb the complex structure on its regular part (or on a neighborhood of the boundary). Though any of them produces a new singularity, these ways perturbing $V$ are so naive that, without any restriction, we may be leaded to the following pathological phenomena.

(I) Deformation of the defining equations.

EXAMPLE 3.1. Let $f : \mathbf{C}^2 \to \mathbf{C}^4$ be a holomorphic map given by $(w_0, w_1, w_2, w_3) = (z^3, z^2w, zw^2, w^3)$. Then $V_0 := f(\mathbf{C}^2)$ is a singular surface defined by holomorphic equations $w_0w_3 - w_1w_2 = w_0w_2 - w_1^2 = w_1w_3 - w_2^2 = 0$. Let $V_t$ be a subvariety defined by $w_0w_3 - w_1w_2 - t = w_0w_2 - w_1^2 = w_1w_3 - w_2^2 = 0$. Then, since $V_t = \{w \in \mathbf{C}^4 \,|\, w_1 = w_2 = w_0w_3 - t = 0\}$ for $t \neq 0$, we have $\dim_{\mathbf{C}} V_t = 1$ $(t \neq 0)$ whereas $\dim_{\mathbf{C}} V_0 = 2$.

In order to avoid this pathological behavior, the correct definition of a family of deformations of $V$ requires an additional condition;

(3.2) any linear relation among $h_1(w), \ldots, h_m(w)$, $\sum_{\gamma=1}^{m} q_\gamma(w)h_\gamma(w) = 0$, is lifted to a linear relation among $h_1(w,t), \ldots, h_m(w,t)$.

This additional requirement is generalized as follows; a family of deformations of a singular variety is defined as a flat surjective holomorphic map

$$\pi : \mathcal{V} \to D \text{ such that } \pi^{-1}(0) = V.$$

In the case of deformations of the germ of a complex space, the flatness condition is equivalent to the requirement (3.2) and it assures the constancy of dimensions and some other cohomological invariants.

Like as the case of deformations of compact complex manifolds, a family is called *semi-universal* if any other family of deformations of $V$ is obtained from that family by a parameters-change which is uniquely determined up to the first order term.

In the theory of deformations of $V$, the space of infinitesimal deformation classes (resp. of obstruction classes) is $\mathrm{Ext}^1(\Omega^1_V, \mathcal{O}_V)$ (resp. $\mathrm{Ext}^2(\Omega^1_V, \mathcal{O}_V)$) (cf. [35] and [29]), and the formal construction of the semi-universal family of deformations of the germ of an isolated singularity was done by [32]. The actual semi-universal family was constructed by G. Tjurina ([35] under the assumption that $V$ is normal and $\mathrm{Ext}^2(\Omega^1_V, \mathcal{O}_V) = 0$) and H. Grauert ([15] in general).

(II) Deformation of the complex structure on the regular part.

A point $p \in V'$ of a singular variety $V'$ is *normal* if its structure local ring $\mathcal{O}_{V',p}$ is integrally closed in its quotient field. In our case, it is equivalent to the property; the restriction map $H^0(V, \mathcal{O}) \to H^0(V \setminus K, \mathcal{O})$ is an isomorphism for any Stein neighborhood $V$ of $p$ and any holomorphically convex compact subset $K$. Therefore the normality of the singular variety is required in order to control the singularities by mean of its regular part.

Let us consider deformations of the complex structure on the regular part of $V$. It produces a family of deformations of a tubular neighborhood of the boundary; $\pi_U : \mathcal{U} \to D$. As at the end of the proof of Theorem 1.2.2, by [31] or [36], each complex manifold $\mathcal{U}_t := \pi_U^{-1}(t)$ $(t \in D)$ is completed to a normal Stein space $V_t'$ which is uniquely determined by $\mathcal{U}_t$. However, in the following example due to O. Riemenschneider we can read that the family $\pi_U : \mathcal{U} \to D$ itself need not to be completed to a family of deformations of $V$.

EXAMPLE 3.3. ([30]). For any compact Riemann surface $C$ of genus $\geq 2$, there exists a negative line bundle $L_0$ on $C$, an affine normal subvariety $V_0 \subset \mathbf{C}^N$ being the contraction of the zero-section of $L_0$, and a family $\pi : \mathcal{X} \to \Delta := \{t \in \mathbf{C} \,|\, |t| < \delta\}$ of deformations of

$L_0$ with $f = \overline{f}\zeta \in H^0(L_0, \mathcal{O})$ ($\overline{f} \in H^0(C, L_0^{-1})$) being impossible to be extended to a holomorphic function $\tilde{f} \in H^0(\mathcal{X}, \mathcal{O})$, where we denotes by $\zeta$ a (local) fibre coordinate of $L_0$.

By this example, O. Riemenschneider concluded that the family $\pi : \mathcal{X} \to \Delta$ cannot be blown down to any family of deformations of $V$. Now, we remark that the function $f$ in Example 3.3 cannot be extended to a holomorphic function $\tilde{f}' \in H^0(\mathcal{U}, \mathcal{O})$ for any open part $\mathcal{U}$ of $\mathcal{X}$ such that $\mathcal{U} \cap \pi^{-1}(0) = U := V \setminus K$ with a Stein neighborhood $V$ and a holomorphically convex compact subset $K$ as above. In fact, since the obstruction to the extension of $f$ is an algebraic condition about $\overline{f} \in H^0(C, L_0^{-1})$, it is also an obstruction to the extension to a function in $H^0(\mathcal{U}, \mathcal{O})$ as well. We shall see that we can also read in this example that there exists a family of deformations of $U$ as above such that it is not completed to a family of deformations of $V_0$. For, if a family $\pi_U : \mathcal{U} \to \Delta$ with $\pi_U^{-1}(0) = U$ is completed to a family $\pi : \mathcal{V} \to \Delta$ of deformations of $V_0$ then the restriction map $H^0(\mathcal{U}, \mathcal{O}) \to H^0(U, \mathcal{O})$ is surjective by the following commutative diagram

$$\begin{array}{ccc} H^0(\mathcal{U}, \mathcal{O}) & \longrightarrow & H^0(U, \mathcal{O}) \\ \uparrow & & \uparrow \simeq \\ H^0(\mathcal{V}, \mathcal{O}) & \longrightarrow & H^0(V_0, \mathcal{O}) \end{array}$$

since $\mathcal{V}$ is Stein.

A reasonable condition to assure the surjectivity of $H^0(\mathcal{U}, \mathcal{O}) \to H^0(U, \mathcal{O})$ is the stable embeddability; a family of complex manifolds is *stably embeddable* if it is embeddable in $\mathbf{C}^N$ by a family of holomorphic embeddings. In Theorem 5.1.4 below, we will see that the stable embeddability is a correct requirement in order for a family $\pi_U : \mathcal{U} \to \Delta$ to be completed to a family of deformations of $V_0$.

## §4 The Kuranishi program

In the rest of this paper, we study the moduli of varieties with only normal isolated singularities from deformation of CR structures on a link of the singularity.

## 4.1. The Germ of a normal isolated singularity and the CR structure on its link

Let $V'$ be a normal analytic subvariety of $\mathbf{C}^N$ as in the introduction and $V := V' \cap B(c)$. Then, $M := V' \cap S_c^{2N-1}$ is called a link of the singularity $0 \in V'$. $M$ is an oriented differentiable manifold (the orientation on $V'$ induces the one on $M$) and moreover the complex structure of (the regular part of) $V'$ induces a strongly pseudoconvex CR structure on $M$. Therefore we have a compact strongly pseudoconvex CR manifold $M$ which is embedded in $\mathbf{C}^N$.

On the contrary, by Theorem 1.2.1, any compact strongly pseudoconvex embeddable CR manifold is a real hypersurface of a normal Stein space. We remark that the normal Stein space is uniquely determined by the CR manifold $M$ because, by the Lewy extension theorem, we have

$$H^\infty(\overline{V}, \mathcal{O}) \simeq H^0_{CR}(M)$$

where $H^\infty(\overline{V}, \mathcal{O})$ denotes the space of all holomorphic functions on $V$ which are extendable to $C^\infty$-functions on $V'$ across $M$ and $H^0_{CR}(M) := \{f \in C^\infty(M) \,|\, \overline{\partial}_b f = 0\}$.

This is the basic correspondence between compact strongly pseudoconvex embeddable CR manifolds and normal isolated singularities, that we will rely on in our approach to (the semi-universal family of) deformations of germs of normal isolated singularities.

## 4.2. The Kuranishi program

In [22], M. Kuranishi proposed the following program for constructing the semi-universal family of deformations of the germ of $V$ of $\dim_{\mathbf{C}} V \geq 3$, relying on the basic correspondence between germs of normal isolated singularities and compact strongly pseudoconvex CR manifolds, stated in subsection 4.1: (1) First, construct a theory of moduli (or of deformations) of CR structures on $M$ so that it controls the moduli (or deformations) of the regular part of $V$. (2) Second, find a way singular points are added to complete it.

In order to accomplish the Kuranishi program, we have to clear the following three geometric or analytic difficulties:

(i) Under the basic correspondence, infinitely many non-isomorphic CR manifold correspond to the same singularity. Hence, we have to construct a deformation theory of CR structures based on some equivalence relation coarser than the CR equivalence.

(ii) In the case of compact complex manifolds, the construction of the semi-universal family due to K. Kodaira, D. C. Spencer or M. Kuranishi (cf. subsection 2.2) heavily relied on the fact that the $\bar{\partial}$-equation is elliptic; in particular, on the elliptic estimate $||Gu||_k \leq C||u||_{k-2}$ with denoting by $|| \ ||_k$ the $k$-th order Sobolev norm. We have to clear the difficulty that the $\bar{\partial}_b$-equation is not elliptic.

(iii) We have to find a CR-condition corresponding to the flatness condition on families of singularities.

### 4.3. A partial answer ([22], [3], [24], [8])

By [22], [3], [24] and [8], the following partial answer was obtained.

THEOREM 4.3.1. *Suppose that* $\dim_{\mathbf{C}} V \geq 4$ *and* $\mathrm{depth}\mathcal{O}_V \geq 3$ *hold. Then there exists a family of deformations of the CR structure on* $M$ *such that*

(1) *its parameter space is isomorphic to the semi-universal family of deformations of the germ of* $V$,

(2) *moreover, that family of CR structures is realized as a family of real hypersurfaces of the semi-universal family of deformations of the germ of* $V$.

In the rest of this section, we will give a brief review on the proof of this theorem and single out the ideas which are needed for the complete accomplishment of the Kuranishi program.

(4.3.2) [22] cleared the difficulty (i) by removing the effect (on perturbations of the CR structure) of wiggles of $M$ in a tubular neighborhood. In order to clear the difficulty (ii), it is proposed to apply the Kuranishi's technique stated in subsection 2.2 with replacing the Banach inverse mapping theorem by the Nash-Moser iteration method. In [22], a $C^\infty$-slice $\{\phi\}$ having the following property is constructed; for any perturbation $U_\omega$ of the regular part of $V$ there exists an embedding

$$f_\omega : M \to U_\omega$$

such that

(1) $f_\omega$ is CR with respect to some CR structure $\phi$ in that slice,

(2) the assignment $\omega \to \phi$ is of class $C^\infty$.

This property is called the *Kuranishi versality* and it implies that the slice $\{\phi\}$ contains all deformations of the regular part of $V$ through the basic correspondence stated in subsection 4.1.

(4.3.3) The difficulty caused by the adoption of the Nash-Moser iteration method was cleared by T. Akahori under the assumption that $\dim_{\mathbf{C}} V \geq 4$. Therefore, the difficulty (ii) was cleared under that dimensional condition. The main idea of [3] is that if we restrict the argument in the subspace $A_b^{0,1}({}^\circ T') \subset A_b^{0,1}(T')$ and if we employ a norm which measures the derivative-loss only in the direction of ${}^\circ T' \oplus {}^\circ T''$ then the second difficulty does not appear. This is based on the fact that if $\phi$ is in $A_b^{0,1}({}^\circ T')$ then the integrability condition $\overline{\partial}_b\phi - \frac{1}{2}[\phi,\phi] - R_3(\phi) = 0$ contains derivations only in ${}^\circ T' \oplus {}^\circ T''$. By applying the Kodaira-Spencer's technique stated in subsection 2.2 to a new subcomplex $(\Gamma(M, E_q), \bar{\partial}_b)$ (cf. [3]) of the Kuranishi's complex $(A_b^{0,q}(T'), \bar{\partial}_b)$ and employing a norm weaker than the Sobolev norm, a family of CR structures with holomorphic parameters was constructed such that it has the same property as the Kuranishi versality in the context of holomorphic parameterization. (There can be an alternative way; we can apply the Kuranishi's technique in stead of the Kodaira-Spencer's one to the Akahori's new subcomplex and obtain a complex analytic slice ([23]); and we can also employ the Folland-Stein norm instead of the Akahori's norm.)

(4.3.4) [3] together with [24] accomplished the first step of the Kuranishi program for $\dim_{\mathbf{C}} V \geq 4$. It is proved in [24] that the Akahori's family of CR structures on $M$ is realized as a family of CR structures induced on a family of real hypersurfaces of a family of deformations of a tubular neighborhood of $M$.

(4.3.5) [33] essentially cleared the difficulty (iii) and accomplished the second step under the assumption depth $\mathcal{O}_V \geq 3$. In [33], it is shown that if depth $\mathcal{O}_V \geq 3$ then the representation hull of functor of deformations of $U$ and that of $V$ are isomorphic to each other. This implies that if depth $\mathcal{O}_V \geq 3$ then the parameter spaces of the formal semi-universal families of deformations of $U$ and of $V$ are isomorphic to each other as formal complex spaces. Then, by the Artin approximation theorem ([4]), we have (1) of Theorem 4.3.1. (2) of Theorem 4.3.1 follows from a further comparison of the Akahori's family of CR structures and the semi-universal family of deformations of $V$ by the argument of [24] taking account of (1).

(4.3.6) There is another way to prove (1) of Theorem 4.3.1. R. O. Buchweitz and J. J. Millson ([8]) proved (1) as an application of the following general comparison method due to W. M. Goldman and J. J. Millson: Let $(L, d)$ be a differential graded Lie algebra and choose a complement $C^1(L)$ to the 1-coboundary $dL^0 \subset L^1$. We construct a functor $Y_L : \mathcal{A} \to \mathbf{Sets}$ as follows where $\mathcal{A}$ denotes the category of Artin local $\mathbf{C}$-algebras: Let $A \in \mathrm{Obj}(\mathcal{A})$ and $\mathfrak{m} \subset A$ the maximal ideal. Then we define $Y_L(A) := \{\eta \in C^1(L) \otimes \mathfrak{m} \mid d\eta + \frac{1}{2}[\eta, \eta] = 0\}$.

THEOREM 4.3.7 ([14] THEOREM 4.1). *Suppose $f : L_1 \to L_2$ is a homomorphism of differential graded Lie algebras such that $f$ induces an isomorphism on first cohomology and an injection on second cohomology. Then the representation hulls $R_{L_1}$ and $R_{L_2}$ of $Y_{L_1}$ and $Y_{L_2}$ respectively are isomorphic to each other.*

### 4.4. Remaining problem

In the assumptions of Theorem 4.3.1, the dimensional condition was required in order for the Akahori's analysis to work and the depth-condition was required in order to rule out the difference between deformations of a tubular neighborhood of $M$ and that of $V$. And, at the beginning, the Kuranishi program was proposed for normal isolated singularities of $\dim_{\mathbf{C}} V \geq 3$ since all three dimensional compact strongly pseudoconvex CR manifolds need not to be boundaries of singular varieties. Then, the following questions naturally arise:

Is there any CR approach to treat deformations of normal isolated surface singularities?

Is there any way to weaken the dimensional condition?

What is the CR analogue of the flatness condition?

The answer of this paper is that all of these questions are affirmatively answered if we consider only stably embeddable deformations of the CR structure on $M$.

## Part III. Accomplishment of the Kuranishi program

### §5 Stably embeddable deformations of CR structures

Let $V'$ and $M$ be as in §4. In this section, we consider how to remove the depth-condition of Theorem 4.3.1 and what is the CR analogue of the semi-universal family of deformations of $V$.

### 5.1. Deformations of a tubular neighborhood of $M$ and deformations of $V$

Since the condition depth $\mathcal{O}_V \geq 3$ in Theorem 4.3.1 was assumed in order to rule out the difference between deformations of a tubular neighborhood of $M$ and deformations of $V$, we first compare families of deformations of a tubular neighborhood of $M$ and that of deformations of $V$ (cf (4.3.5)), at the first order deformation level. We recall that the spaces of first order deformation classes of $U := V \setminus 0$ and of $V$ are $H^1(U, T^{1,0}U)$ (cf. [18]) and $\mathrm{Ext}^1(\Omega^1_V, \mathcal{O}_V)$ (cf. [35] or [29]) respectively.

PROPOSITION 5.1.1.

$$\mathrm{Ext}^1(\Omega^1_V, \mathcal{O}_V) \simeq \mathrm{Ker}\{H^1(U, T^{1,0}U) \xrightarrow{\tilde{F}_0} H^1(U, T^{1,0}\mathbf{C}^N{}_{|U})\}$$

*where $\tilde{F}_0$ denotes the homomorphism induced from the bundle homomorphism $df_0 : T^{1,0}U \to T^{1,0}\mathbf{C}^N{}_{|U}$ with denoting $f_0 : U \to \mathbf{C}^N$ the natural embedding.*

For the proof, see [25] Proposition 1.7.

We understand the implication of this isomorphism as follows. From the exact sequence associated to the embedding $f_0 : V \to \mathbf{C}^N$;

$$0 \to T^{1,0}U \xrightarrow{\tilde{F}_0} T^{1,0}\mathbf{C}^N{}_{|U} \to N_{U/\mathbf{C}^N} \to 0, \tag{5.1.2}$$

we have

$$\begin{aligned} &\mathrm{Ker}\{H^1(U, T^{1,0}U) \xrightarrow{\tilde{F}_0} H^1(U, T^{1,0}\mathbf{C}^N{}_{|U})\} \\ &\simeq \mathrm{Im}\{H^0(U, N_{U/\mathbf{C}^N}) \to H^1(U, T^{1,0}U)\}. \end{aligned} \tag{5.1.3}$$

We note that $H^0(U, N_{U/\mathbf{C}^N})$ represents the space of first order displacements of $U$ in $\mathbf{C}^N$ (cf. [18]). Then, Proposition 5.1.1 together with (5.1.3) implies that a family of first order deformations of $U$ corresponds to a family of first order deformations of $V$ if and only if it is embedded in $\mathbf{C}^N$ as a family of submanifolds. A family of deformations of $U$ which is realized as a family of submanifolds of $\mathbf{C}^N$ is called a family of stably embeddable deformations of $U$.

This correspondence, between deformations of $V$ and stably embeddable deformations of the complex structure on $U$, holds not only at the first order deformation level but also in the higher order deformations. In fact, we can prove the following theorem by a similar way as Theorems 4.1, 4.2 and 5.1 of [6].

THEOREM 5.1.4. (1) *Let* $\phi(t) \in A_U^{0,1}(T^{1,0}U)[[t_1, \ldots, t_d]]$ *and* $f(t) \in A_U^0(T^{1,0}\mathbf{C}^N{}_{|U})[[t_1, \ldots, t_d]]$ *satisfy*
(a) $\phi(0) = 0$, $f(0) = 0$,
(b) $\overline{\partial}\phi(t) - \frac{1}{2}[\phi(t), \phi(t)] = 0$,
(c) $\overline{\partial}(f_0 + f(t)) - \phi(t) \cdot (f_0 + f(t)) = 0$.
*Then there exists* $h(t) \in \oplus^m H^0(B(c), \mathcal{O})[[t_1, \ldots, t_d]]$ *satisfying*
(d) $h_\gamma(0) = h_\gamma$ $(\gamma = 1, \ldots, m)$,
(e) $h_\gamma(t) \circ (f_0 + f(t)) = 0$ $(\gamma = 1, \ldots, m)$,
(f) *any relation* $\sum_{\gamma=1}^m p_\gamma h_\gamma = 0$ *is lifted to* $p(t) \in \oplus^m H^0(B(c), \mathcal{O})[[t_1, \ldots, t_d]]$ *with* $\sum_{\gamma=1}^m p_\gamma(t) h_\gamma(t) = 0$.
(2) *If* $h(t) \in \oplus^m H^0(B(c), \mathcal{O})[[t_1, \ldots, t_d]]$ *satisfies (d)–(f) as above, then there exist* $\phi(t) \in A_U^{0,1}(T^{1,0}U)[[t_1, \ldots, t_d]]$ *and* $f(t) \in A_U^0(T^{1,0}\mathbf{C}^N{}_{|U})[[t_1, \ldots, t_d]]$ *satisfying (a)–(c).*

Therefore, not all but only stably embeddable deformations of $U$ correspond to deformations of $V$, at least in the formal deformation level.

### 5.2. Stably embeddable deformations of CR structures

The CR analogue of the stably embeddable deformations of the complex structure on $U$ is deformations of the CR structure on $M$ which can be simultaneously embedded into $\mathbf{C}^N$.

First, we give some fundamental definitions in the theory of stably embeddable deformations of the CR structure on $M$. For simplicity, we assume that the parameter space $D$ of the family is non-singular.

DEFINITION 5.2.1. A family of deformations of the CR structure on $M$ is a $\phi(t)$ in $A_b^{0,1}(T')[[t_1, \ldots, t_d]]$ satisfying
(1) $\phi(0) = 0$,
(2) $\overline{\partial}_b\phi(t) - \frac{1}{2}[\phi(t), \phi(t)] - R_3(\phi(t)) = 0$,
(3) $\phi(t)$ is convergent with respect to the Folland-Stein norm.

DEFINITION 5.2.2. A family of deformations of the CR structure on $M$, $\phi(t)$ is stably embeddable in $\mathbf{C}^N$ if there exists $f(t) \in A_b^{0,1}(T^{1,0}\mathbf{C}^N{}_{|M})$ $[[t_1, \ldots, t_d]]$ satisfying

(1) $f(0) = 0$,

(2) $\overline{\partial}_b^{\phi(t)}(f_0 + f(t)) = 0$,

(3) $f(t)$ is convergent with respect to the Folland-Stein norm.

THEOREM 5.2.3. *The space of first order deformation classes of stably embeddable deformations of CR structures*

$$\simeq \mathrm{Ker}\{H^1(T^{'}) \xrightarrow{F_0} H^1(T^{1,0}\mathbf{C}^N{}_{|M})\}$$

*where $F_0$ is the homomorphism induced from the bundle homomorphism $\rho^{1,0} \circ df_0 : T^{'} \to T^{1,0}\mathbf{C}^N{}_{|M}$.*

*Proof.* First, we note that

$$\begin{cases} \overline{\partial}_b(\phi t) - \frac{1}{2}[\phi t, \phi t] - R_3(\phi t) \equiv 0 \bmod \mathfrak{m}^2 \\ \overline{\partial}_b^{\phi t}(f_0 + ft) \equiv 0 \bmod \mathfrak{m}^2 \end{cases}$$

holds if and only if $\overline{\partial}_b\phi = 0$ and $\overline{\partial}_b f - F_0\phi = 0$ hold. Next, let $(\phi_1, f_1)$ and $(\phi_2, f_2)$ satisfy $\overline{\partial}_b\phi_1 = \overline{\partial}_b\phi_2 = 0$ and $\overline{\partial}_b f_1 - F_0\phi_1 = \overline{\partial}_b f_2 - F_0\phi_2 = 0$. Since the families of first order deformations of CR structures $\phi_1 t$ and $\phi_2 t$ are obtained from wiggles in a family of deformations of a tubular neighborhood of $M$ if and only if $\phi_1 - \phi_2 \in \overline{\partial}_b A_b^0(T^{'})$ (this is calculated in [22]), the space of stably embeddable first order deformation classes is

$$\{(\phi, f) \in A_b^{0,1}(T^{'}) \oplus A_b^0(T^{1,0}\mathbf{C}^N{}_{|M}) \,|\, \overline{\partial}_b\phi = 0, \overline{\partial}_b f - F_0\phi = 0\}/\sim$$

where $(\phi_1, f_1) \sim (\phi_2, f_2)$ if $\phi_1 - \phi_2 \in \overline{\partial}_b A_b^0(T^{'})$ holds. Clearly it is isomorphic to $\mathrm{Ker}\{H^1(T^{'}) \xrightarrow{F_0} H^1(T^{1,0}\mathbf{C}^N{}_{|M})\}$. □

Taking account of Proposition 5.1.1, the following proposition indicates that stably embeddable deformations of the CR structure on $M$ correspond to deformations of $V$ at the first order deformation level.

PROPOSITION 5.2.4 ([25], PROPOSITION 4.5).

$$\mathrm{Ker}\{H^1(U, T^{1,0}U) \xrightarrow{\tilde{F}_0} H^1(U, T^{1,0}\mathbf{C}^N{}_{|U})\}$$
$$\simeq \mathrm{Ker}\{H^1(T') \xrightarrow{F_0} H^1(T^{1,0}\mathbf{C}^N{}_{|M})\}.$$

Like as Theorem 5.1.4, we can prove that stably embeddable deformations of the CR structure on $M$ correspond to deformations of $V$ not only at the first order deformation level but also in the formal deformation level (it is proved in Theorems 4.1 and 5.1 of [6] for the surface singularity case).

The following definition is the stably embeddable version of the Kuranishi versality stated in (4.3.2).

DEFINITION 5.2.5. A family $\phi(t)$ of stably embeddable deformations of the CR structure on $M$ is Kuranishi versal if it has the following property; for any family of stably embeddable deformations of a tubular neighborhood of $M$, $\pi : \mathcal{U} \to S$, there exists a holomorphic map $\tau : S \to D$ and a family of embeddings $F : M \times S \to \mathcal{U}$ satisfying

(1) $\tau(0) = 0$, $F_{|M\times 0} = \mathrm{id}_M$,

(2) $\pi \circ F = p_2$ where $p_2 : M \times S \to S$ denotes the projection onto the second factor,

(3) for any $p \in M \subset \pi^{-1}(0)$ and any holomorphic function $f$ on a neighborhood of $p$, $\overline{\partial}_b^{\phi(\tau(s))} f \circ F(z, s) = 0$ holds.

A Kuranishi versal family is *Kuranishi semi-universal* if $d\tau_0 : T_0 S \to T_0 D$ in the above is uniquely determined.

The following theorem indicates that the Kuranishi semi-universal family of stably embeddable deformations of the CR structure on $M$ is indeed the CR analogue of the semi-universal family of deformations of $V$.

THEOREM 5.2.6 ([25] THEOREM 11.1). *If the Kuranishi semi-universal family of stably embeddable deformations of the CR structure on $M$ exists, then*

(1) *its parameter space is isomorphic to the parameter space of the semi-universal family of deformations of $V$,*

(2) *moreover, that family of deformations of the CR structure on $M$ is realized as a family of real hypersurfaces of the semi-universal family of deformations of $V$.*

## §6 Construction of the Kuranishi semi-universal family

We construct the Kuranishi semi-universal family by applying the Kodaira-Spencer's technique stated in the subsection 2.2, which consists of the following two components:

(1) Deformation complex; a differential complex whose first cohomology is isomorphic to the space of infinitesimal deformation classes and the second cohomology is isomorphic to the space of obstructions classes.

(2) Harmonic analysis on the deformation complex; we can replace it by some weaker homotopy formula.

### 6.1. Deformation complex

In order to find the deformation complex, we describe the spaces of infinitesimal deformation classes and of obstruction classes.

PROPOSITION 6.1.1.

(1) *The space of infinitesimal deformation classes*

$$\simeq \mathrm{Ker}\{H^1(T') \xrightarrow{F_0} H^1(T^{1,0}\mathbf{C}^N{}_{|M})\}.$$

(2) *The space of obstruction classes*

$$\simeq \mathrm{Ker}\{H^1(N_{U/\mathbf{C}^N|M}) \xrightarrow{H} H^1(\oplus^m 1_M)\}$$

*where $H$ denotes the homomorphism induced from the restriction of the bundle homomorphism $H : T^{1,0}\mathbf{C}^N \to \oplus^m 1_{\mathbf{C}^N}$ given by $H(v) := (v(h_1), \ldots, v(h_{m_1}))$.*

For the proof, see Remark 3.11 of [25].

Then the total simple complex $(K^\bullet, d)$ of the following double complex $K^{\bullet,\bullet}$ is the deformation complex of stably embeddable deformations of the CR structure on $M$;

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & H_b^0(T^{1,0}\mathbf{C}^N{}_{|M}) & \xrightarrow{H} & H_b^0(\oplus^m 1_M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \\
0 & \longrightarrow & K^{0,0} = A_b^0(T') & \xrightarrow{F_0} & A_b^0(T^{1,0}\mathbf{C}^N{}_{|M}) & \xrightarrow{H} & A_b^0(\oplus^m 1_M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \bar\partial_b} & & \downarrow{\scriptstyle \bar\partial_b} & & \downarrow{\scriptstyle \bar\partial_b} & & \\
0 & \longrightarrow & A_b^{0,1}(T') & \xrightarrow{F_0} & A_b^{0,1}(T^{1,0}\mathbf{C}^N{}_{|M}) & \xrightarrow{H} & A_b^{0,1}(\oplus^m 1_M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \bar\partial_b} & & \downarrow{\scriptstyle \bar\partial_b} & & \downarrow{\scriptstyle \bar\partial_b} & & \\
0 & \longrightarrow & A_b^{0,2}(T') & \xrightarrow{F_0} & A_b^{0,2}(T^{1,0}\mathbf{C}^N{}_{|M}) & \xrightarrow{H} & A_b^{0,2}(\oplus^m 1_M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \bar\partial_b} & & \downarrow{\scriptstyle \bar\partial_b} & & \downarrow{\scriptstyle \bar\partial_b} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

where $H^0(T^{1,0}\mathbf{C}^N{}_{|M}) := \{\eta \in A_b^0(T^{1,0}\mathbf{C}^N{}_{|M}) \,|\, \bar\partial_b\eta = 0\}$ and $i$ denotes the natural inclusion map. $(K^\bullet,\, d)$ is given by

$$K^q := K^{0,q} + K^{1,q-1} + K^{2,q-2}$$

$$d(a_q, b_{q-1}, c_{q-2}) := (\bar\partial_b a_q,\, \bar\partial_b b_{q-1} + (-1)^q F_0 a_q,\, \bar\partial_b c_{q-2} + (-1)^{q-1} H b_{q-1})$$

where we denote $K^{p,q} := 0$ for $q \le -2$ and $\bar\partial_b b_{-1} := i b_{-1}$ and $\bar\partial_b c_{-1} := i b_{-1}$.

Then, the following proposition together with Proposition 6.1.1 implies that $(K^\bullet,\, d)$ is the deformation complex of the stably embeddable deformations of the CR structure on $M$.

PROPOSITION 6.1.2.

(1) $H^1(K^\bullet) \simeq \mathrm{Ker}\{H^1(T') \overset{F_0}{\to} H^1(T^{1,0}\mathbf{C}^N{}_{|M})\}$.

(2) $H^2(K^\bullet) \simeq \mathrm{Ker}\{H^1(N_{U/\mathbf{C}^N|M}) \overset{H}{\to} H^1(\oplus^m 1_M)\}$.

Proof is direct from the definition of $(K^\bullet,\, d)$.

### 6.2. Homotopy formula

THEOREM 6.2.1 ([25], THEOREMS 5.1, 5.2, 6.1, 12.2). *For* $q = 1, 2$, *there exist operators* $Z_q : K^q \to K^q \cap Ker\ d$ *and* $Q_q : K^q \cap Ker\ d \to K^{q-1}$ *satisfying*

(1) $Z_{q|\mathrm{Ker}\ d} = id_{|\mathrm{Ker}\ d}$,

(2) $d \circ Q_q \circ d = d$.

Then, $\rho_q := (1 - Q_q \circ d) \circ Z_q$ is the projection operator onto $\mathrm{Im}\rho_q \simeq H^q_d(K^\bullet)$ and we have a homotopy formula (a substitute of the Hodge-Kodaira decomposition);

$$u = \rho_q u + dQ_q Z_q u + (1 - Z_q)u,\ u \in K^q. \tag{6.2.2}$$

For these homotopy operators, we have the following estimates.

THEOREM 6.2.3 ([25], PROPOSITIONS 7.1, 7.2, 12.2).

(1) *Let* $(a_1, b_0, c_{-1}) \in K^1$. *If* $Z_1(a_1, b_0, c_{-1}) = (a_1', b_0', c_{-1}')$ *and* $Q_1(a_1', b_0', c_{-1}') = (a_0'', b_{-1}'')$, *then*

$$||a_0''||_{(k+1)} \leq C||a_1'||_{(k)} \leq C'||a_1||_{(k)}.$$

(2) *Let* $(a_2, b_1, c_0) \in K^2$. *If* $Z_2(a_2, b_1, c_0) = (a_2', b_1', c_0')$ *and* $Q_2(a_2', b_1', c_0') = (a_1'', b_0'', c_{-1}'')$, *then*

$$||a_1''||_{(k)} + ||b_0''||_{(k+1)} \leq C||b_1'||_{(k)} \leq C'||b_1||_{(k)}.$$

### 6.3. Construction of the Kuranishi semi-universal family

The construction of the Kuranishi semi-universal family is carried out by applying the Kodaira-Spencer's technique to the following setting; we consider

$$(\phi, f, k) \in K^1 = K^{0,1} + K^{1,0} + K^{2,-1}$$

and its integrability condition

$$\left(\overline{\partial}_b\phi - \frac{1}{2}[\phi, \phi] - R_3(\phi),\ \overline{\partial}_b^{\phi}(f_0 + f),\ (h + \widetilde{k}) \circ (f_0 + f)\right) = (0, 0, 0)$$

in $K^2 = K^{0,2} + K^{1,1} + K^{2,0}$, where $\widetilde{k}(t) \in H^0(\overline{B(c)}, \mathcal{O}_{\mathbf{C}^N})$ denotes a holomorphic extension of $k$ and we consider $(h + \widetilde{k}(t)) \circ (f_0 + f(t))$ as the Taylor series with powers of $f(t)$ and centered at $f_0$.

REMARK 6.3.1. In a strict sense, $(h+\widetilde{k})\circ(f_0+f)$ is not an element of $K^{2,0}$. But this causes no trouble in the following formal construction (I).

We give a brief explanation of this integrability condition:

(i) $\overline{\partial}_b\phi - \frac{1}{2}[\phi,\phi] - R_3(\phi) = 0$ is the condition for $\phi \in A_b^{0,1}(T')$ to represent a (integrable) CR structure.

(ii) For $f = \sum_{\beta=1}^N f^\beta \frac{\partial}{\partial w^\beta} \in A_b^{0,}(T^{1,0}\mathbf{C}^N{}_{|M})$, $(f_0^1+f^1,\dots,f_0^N+f^N) \in \oplus^N C^\infty(M)$ defines an embedding $f_0+f : M \to \mathbf{C}^N$ and the condition $\overline{\partial}_b^\phi(f_0+f) = 0$ is the condition so that the embedding $(f_0+f)$ is CR with respect to the CR structure ${}^\phi T''$.

(iii) Since holomorphic equations $h+\widetilde{k} := (h_1+\widetilde{k}_1,\dots,h_m+\widetilde{k}_m) = (0,\dots,0)$ defines a subvariety of $\overline{B(c)}$, the condition $(h+\widetilde{k})\circ(f_0+f) = 0$ implies that the image $(f_0+f)(M)$ is the boundary of that subvariety.

In the rest of this section, we shall briefly review the construction of the Kuranishi semi-universal family. For simplicity of the argument, we assume that $H^2(K^\bullet) = 0$.

(I) Formal construction.

We use the notation that $(\phi^{(\mu)}(t), f^{(\mu)}(t), k^{(\mu)}(t))$ denotes the polynomial part of $(\phi(t), f(t), k(t))$ of degree $\mu$ and $(\phi_\mu(t), f_\mu(t), k_\mu(t))$ the homogeneous polynomial part of the same degree.

$(\phi^{(1)}(t), f^{(1)}(t), k^{(1)}(t))$ is defined by

$$(\phi_1(t), f_1(t), k_1(t)) = \sum_{\sigma=1}^{d}(\phi_\sigma, f_\sigma, k_\sigma)t_\sigma$$

where $d(\phi_\sigma, f_\sigma, k_\sigma) = (0,0,0)$ $(\sigma = 1,\dots,d)$ and $\{(\phi_\sigma, f_\sigma, k_\sigma)\}_{1\le\sigma\le d}$ is a base of $H^1(K^\bullet)$. (This definition of $(\phi_1(t), f_1(t), k_1(t))$ assures that $d\tau_0$ in Definition 5.2.5 is uniquely determined if $\tau$ exists.)

$(\phi_\mu(t), f_\mu(t), k_\mu(t))$ is defined inductively by the following way.

Let

$$\begin{aligned}(a_\mu(t), b_\mu(t), c_\mu(t)) = {} & \text{the } \mu\text{-th homogeneous term of} \\ & -(\overline{\partial}_b\phi^{(\mu-1)}(t) - \frac{1}{2}[\phi^{(\mu-1)}(t), \phi^{(\mu-1)}(t)] - R_3(\phi^{(\mu-1)}(t)), \\ & \overline{\partial}_b^{\phi^{(\mu-1)}(t)}(f_0 + f^{(\mu-1)}(t)),\ (h + \widetilde{k}^{(\mu-1)}(t))\circ(f_0 + f^{(\mu-1)}(t)))\end{aligned}$$

(note that the right-hand side is well defined by the Taylor polynomials of $h$ and $\widetilde{k}^{(\mu-1)}(t)$ of degree up to $\mu$).

Then $(\phi_\mu(t), f_\mu(t), k_\mu(t))$ is defined by

$$(\phi_\mu(t), f_\mu(t), k_\mu(t)) := Q_2 Z_2(a_\mu(t), b_\mu(t), c_\mu(t))$$

and we choose a holomorphic extension $\widetilde{k_\mu}(t) \in H^\infty(\overline{B(c)}, \mathcal{O})[[t_1, \dots, t_d]]$ of $k_\mu(t)$ due to [1] or Theorem A.1 of [6].

PROPOSITION 6.3.2. *$(\phi(t), f(t), k(t))$ defined as above satisfies*

$$\Big(\overline{\partial}_b\phi(t) - \frac{1}{2}[\phi(t), \phi(t)] - R_3(\phi(t)),$$
$$\overline{\partial}_b^{\phi(t)}(f_0 + f(t)),\ (h + \widetilde{k}(t)) \circ (f_0 + f(t))\Big) = (0, 0, 0).$$

*Proof.* We prove, by induction on $\mu$, that it satisfies

$$\text{(6.3.3:}\mu\text{)}\quad (\overline{\partial}_b\phi(t) - \frac{1}{2}[\phi(t), \phi(t)] - R_3(\phi(t)),$$
$$\overline{\partial}_b^{\phi(t)}(f_0 + f(t)),\ (h + \widetilde{k}(t)) \circ (f_0 + f(t))) \equiv (0, 0, 0) \bmod \mathfrak{m}^{\mu+1}.$$

For $\mu = 1$, it is clearly satisfied by the definition of $(\phi^{(1)}(t), f^{(1)}(t), k^{(1)}(t))$.

We suppose that (6.3.3:$\mu$-1) holds.

LEMMA 6.3.4 ([25], PROPOSITION 8.4).

$$d(a_\mu(t), b_\mu(t), c_\mu(t)) = (0, 0, 0).$$

From Lemma 6.3.4 and the assumption $H^2(K^\bullet) = 0$, we infer that

$$d(\phi_\mu(t), f_\mu(t), k_\mu(t)) = (a_\mu(t), b_\mu(t), c_\mu(t)).$$

Therefore, we have

$$\begin{aligned}
&(\overline{\partial}_b\phi(t) - \frac{1}{2}[\phi(t), \phi(t)] - R_3(\phi(t)),\\
&\qquad \overline{\partial}_b^{\phi(t)}(f_0 + f(t)),\ (h + \widetilde{k}(t)) \circ (f_0 + f(t)))\\
&\equiv (\overline{\partial}_b\phi^{(\mu)}(t) - \frac{1}{2}[\phi^{(\mu)}(t), \phi^{(\mu)}(t)] - R_3(\phi^{(\mu)}(t)),\\
&\qquad \overline{\partial}_b^{\phi^{(\mu)}(t)}(f_0 + f^{(\mu)}(t)),\ (h + \widetilde{k^{(\mu)}}(t)) \circ (f_0 + f^{(\mu)}(t)))\\
&\equiv (0, 0, 0) \bmod \mathfrak{m}^{\mu+1}.
\end{aligned}$$

□

(II) Convergence of the formal solution. The convergence of $\phi(t)$ and $f(t)$ under the assumption $\phi(t) \in A_b^{0,1}({}^\circ T')[[t_1, \ldots, t_d]]$ follows by applying the Kodaira-Spencer's technique for convergence (cf. [17]). Let $A(t) := \frac{b}{16c}\sum_{\mu\geq 1}\frac{c^\mu}{\mu^2}(t_1 + \cdots + t_d)^\mu$ be a convergent powerseries satisfying

$$A(t)^2 << \frac{b}{c}A(t)$$

where we employ the following notation; for power series $A(t) := \sum_{I=(i_1,\ldots,i_d)} a_I t^I$ and $B(t) := \sum_{I=(i_1,\ldots,i_d)} b_I t^I$ in $\mathbf{R}_{\geq 0}[[t_1, \ldots, t_d]]$, we denote $A(t) << B(t)$ if $a_I \leq b_I$ holds for all $I$.

PROPOSITION 6.3.5. *Suppose that $\phi(t) \in A_b^{0,1}({}^\circ T')[[t_1, \ldots, t_d]]$ holds. Then, for sufficiently large $b$ and $c$, we have*

$$||\phi||_{(k)}(t) + ||f||_{(k+1)}(t) << A(t)$$

*where we denote $||\phi||_{(k)}(t) := \sum_I ||\phi_I||_{(k)} t^I$ if $\phi(t) := \sum_I \phi_I t^I$.*

*Proof.* We shall prove by induction on $\mu$ that

$$||\phi^{(\mu)}||_{(k)}(t) + ||f^{(\mu)}||_{(k+1)}(t) << A(t) \tag{6.3.6:$\mu$}$$

holds for all $\mu$.

It is clear that

$$||\phi^{(1)}||_{(k)}(t) + ||f^{(1)}||_{(k+1)}(t) << A(t) \tag{6.3.6:1}$$

holds if $b$ is chosen sufficiently large.

Suppose that (6.3.6:$\mu$-1) holds. By the assumption $\phi(t) \in A_b^{0,1}({}^\circ T')$ $[[t_1, \ldots, t_d]]$, we have

$$\begin{aligned}||b_\mu||_{(k)}(t) &<< ||\phi^{(\mu-1)}||_{(k)}(t)||f^{(\mu-1)}||_{(k+1)}(t)\\ &<< C_1 A(t)^2 << \frac{C_1 b}{c}A(t).\end{aligned}$$

Hence, by the estimate of homotopy operators (cf. Theorem 6.2.2 (2)) we have

$$||\phi_\mu||_{(k)}(t) + ||f_\mu||_{(k+1)}(t) << \frac{C_2 b}{c}A(t).$$

Here we note that the constants $C_1$ and $C_2$ as above are independent of $\mu$. Therefore, if we choose $c$ so that $\frac{C_2 b}{c} < 1$ holds then (6.3.6:$\mu$) follows. $\square$

(III) The technical assumption $\phi(t) \in A_b^{0,1}(^\circ T')[[t_1, \ldots, t_d]]$ in Proposition 6.3.5 is satisfied by modifying the construction of the homotopy operators $Z_2$ and $Q_2$ in Theorem 6.2.1. This is the main part to clear analytical difficulty stated in (ii) of subsection 4.2 and see [25] §§5–6 and §12 for the detailed proof.

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