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CR INVARIANTS OF WEIGHT 6

KENGO HIRACHI

ABSTRACT. All scalar CR invariants of weight ≤ 6 are explicitly given for 3-dimensional strictly pseudoconvex CR structures, as an application of Fefferman's ambient metric construction and its generalization by the author.

1. Introduction

In [5], Fefferman initiated a program of writing down all local invariants of strictly pseudoconvex real hypersurfaces in $\mathbb{C}^n$, in explicit, computable form (we here call such invariants CR invariants). His aim was to express the asymptotic expansion of the Bergman kernel of a strictly pseudoconvex domain in terms of these invariants. This program has been continued by the works ([6], [7], [3], [1], [9]). In particular, Bailey, Eastwood and Graham ([1]) gave a complete description of CR invariants of weight $\leq n$ (for weight $\leq n-19$, it had been obtained in [5]). This result was applied to express the Bergman kernel up to the logarithmic singularity.

Recently, the author ([8]) has generalize their result so that we can also express the logarithmic singularity. The main idea of [8] is to consider the local invariants of the pair (M, r), where M is a strictly pseudoconvex hypersurface and r is a defining function of M. We here assume that r is normalized by a complex Monge–Ampère equation; this normalization determines r uniquely modulo $O^{n+2}(M)$, see Sections 3 and 4. CR invariants are then characterized as invariants of the pair (M, r) that are independent of r. It is shown that all invariants of the pair (M, r) are expressed as Weyl invariants, that is, linear combinations of

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complete contractions of curvature tensors of a metric associated with r. For weight $\leq n+1$, or ≤ 5 in case $n=2$, all Weyl invariants are shown to be independent of r, and we obtain an expression of CR invariants in terms of Weyl invariants. This is a natural generalization of the result of [1] mentioned above. For higher weight, however, we do not know a practical way of constructing CR invariants.

In this note, we compute CR invariants of weight ≤ 6 in case $n=2$ by two different methods. The result for weight 6 is new and is obtained by using a computer algebra program.

The first method, explained in Section 2, is based on a direct computation of Moser's normal form. By definition, CR invariants are polynomials in Moser's normal form coefficients $A=(A^{\ell}_{p\bar{q}})$ that are invariant under the action of the structure group H. For each weight, it is shown that all CR invariants are linear combinations of a finite list of monomials of A. Thus we can determine all CR invariants of given weight by computing the action of H on each monomials in the list. Such computation for weight ≤ 5 has been done in Graham [6] and [9]. A straightforward generalization of this procedure also gives the result for weight 6 (Theorem 2.1), while it is much longer and a use of computer is inevitable.

The second method is an application of the theory for the invariants of the pair (M,r) in [8], explained in Sections 3 and 4. We reduce the problem of finding all CR invariants to that of determining Weyl invariants that are independent of r. For each weight, the vector space of all Weyl invariants is of finite dimension. Thus dependence of Weyl invariants on r is completely known by writing down generators of Weyl invariants in terms of the Taylor coefficients of r. For weight ≤ 5, such computation has been done in [9]. For weight 6, we can proceed similarly, while the computation is much longer and we have used a computer. The result is described in Section 5.

The expression of CR invariants obtained by the second method is much simpler than that for the first one and the construction itself is geometric. It is hopeful that the basis of CR invariants given here could be obtained by an invariant-theoretic argument, without a computer-aided computation. For $n>3$ and weight $\leq n-1$, [1] has explicitly given basis of CR invariants in terms of Weyl invariants (see Theorem 3.2 below). Our result for $n=2$ and weight ≤ 6 is analogous to their result. We include an observation about this analogy at the end of Section 5.

2. Definition of CR invariants

We start by recalling the definition of (local scalar) CR invariants of strictly pseudoconvex real hypersurfaces in $\mathbb{C}^n$. We here follow [5] and utilize Moser's theory of normal form [2] to describe the curvatures of the surfaces.

Let M be a real-analytic strictly pseudoconvex hypersurface in $\mathbb{C}^n$. It is shown in [2] that, for each point $p \in M$, there exists a local coordinate system z such that $p = 0$ and M is locally given by the equation

$$\rho(z) = 2u - |z'|^2 - \sum_{|\alpha|,|\beta|\geq 2, \ell\geq 0} A^{\ell}_{\alpha\overline{\beta}} z'_\alpha \overline{z}'_\beta v^\ell = 0, \tag{2.1}$$

where $z' = (z_1, \ldots, z_{n-1})$, $z_n = u + i\,v$, and $z'_\alpha = z_{\alpha_1} \ldots z_{\alpha_p}$. Here the coefficients $A^{\ell}_{\alpha\overline{\beta}}$ satisfy the following three conditions:

(i) $\overline{A^{\ell}_{\alpha\overline{\beta}}} = A^{\ell}_{\beta\overline{\alpha}}$;

(ii) for each p, q, ℓ,

$$\mathbf{A}^{\ell}_{p\overline{q}} = (A^{\ell}_{\alpha\overline{\beta}})_{|\alpha|=p,|\beta|=q}$$

is a bisymmetric tensor of type (p, q) on $\mathbb{C}^{n-1}$;

(iii) for $p, q \leq 3$, $\ell \geq 0$,

$$\operatorname{tr}^{p+q-3} \mathbf{A}^{\ell}_{p\overline{q}} = 0, \tag{2.2}$$

where the trace is taken for the standard metric on $\mathbb{C}^{n-1}$.

We denote the (germ of) surface (2.1) by $N(A)$, where $A = (A^{\ell}_{p\overline{q}})$ is the list of coefficients of ρ, and call it a *normal form* of (M, p) with respect to the *normal coordinates* z. Note that a normal form and coordinates for (M, p) are not uniquely determined, that is, two different surfaces in normal form may be biholomorphically equivalent. Keeping this fact in mind, we make the following definition of CR invariants.

Definition. A *CR invariant of weight* $w \in \mathbb{N}$ is a polynomial $P(A)$ in the normal form coefficients $A = (A^{\ell}_{p\overline{q}})$ that satisfies the following transformation law: If Φ is a local biholomorphic map which preserves 0 and maps a surface in normal form $N(A)$ to another surface in normal form $N(\widetilde{A}) = \Phi(N(A))$, then

$$P(\widetilde{A}) = |\det \Phi'(0)|^{-2w/(n+1)} P(A), \tag{2.3}$$

where Φ' is the holomorphic Jacobi matrix of Φ.

REMARK 2.1. A CR invariant $P(A)$ gives an assignment to each real-analytic strictly pseudoconvex hypersurface M of a function $P_M : M \to \mathbb{C}$: for $p \in M$ take a local biholomorphic map Φ such that $\Phi(p) = 0$ and $\Phi(M)$ is in normal form $N(A)$, and set

$$P_M(p) := |\det \Phi'(p)|^{2w/(n+1)} P(A).$$

This definition is independent of the choice of Φ because of (2.3). The functionals P_M naturally appear in several problems of Several Complex Variables, e.g. in the expansion of the Bergman kernel. Note also that the functional P_M can be also defined for C^∞ surfaces M, and the assumption of real-analyticity of M is not essential in the definition of CR invariants. See [9].

In order to verify that a given polynomial $P(A)$ is a CR invariant, we need to know all the equivalence classes of surfaces in normal form together with the maps Φ. In case $M = N(0)$, the boundary of the Siegel domain $D_0 = \{2u > |z'|^2\}$, the set of all normal coordinates for $(\partial D_0, 0)$ is just the isotropy group H of D_0 at 0. For general $M = N(A)$, the set of all normal coordinates for $(N(A), 0)$ is naturally parameterized by H. If we denote the change of coordinates corresponding to $h \in H$ by $\Phi_{(h,A)}$, then $\{\Phi_{(h,A)}(N(A)) : h \in H\}$ gives the set of all surfaces in normal form that are equivalent to $N(A)$.

Let $\mathcal{N}$ be the space of all surfaces in normal form

$$\mathcal{N} = \Big\{ A = (A^\ell_{\alpha\overline{\beta}})_{|\alpha|,|\beta|\geq 2, \ell \geq 0} : A \text{ satisfies (i), (ii), and (iii) } \Big\}.$$

For $(h, A) \in H \times \mathcal{N}$, take $\widetilde{A} \in \mathcal{N}$ such that $N(\widetilde{A}) = \Phi_{(h,A)}(N(A))$ and set $h.A := \widetilde{A}$. Then the map

$$H \times \mathcal{N} \ni (h, A) \mapsto h.A \in \mathcal{N}$$

defines an action of H on $\mathcal{N}$. Clearly, the H-orbits in $\mathcal{N}$ are the equivalence classes of surfaces in normal form.

In terms of this H-action, we may rewrite (2.3) as

$$P(h.A) = \sigma_w(h) P(A) \tag{2.4}$$

for any $(h, A) \in H \times \mathcal{N}$, where $\sigma_w(h) = |\det \Phi'_{(h,A)}(0)|^{-2w/(n+1)}$ (the right-hand side is shown to be independent of A). Thus we can say that CR invariants are H-invariant polynomials on $\mathcal{N}$.

Using this expression, Graham [6] (see also [9]) determined all CR invariants of weight ≤ 5 in case $n = 2$. We recall his result together with its generalization to weight 6, which is based on a computer-aided calculation.

We first introduce a notation which is specific to the 2-dimensional case. In case $n = 2$, the tensor $\mathbf{A}^{\ell}_{p\bar{q}}$ has only one component. If we denote the component by $A^{\ell}_{p\bar{q}}$, then the trace conditions (2.2) are reduced to

$$A^{\ell}_{2\bar{2}} = A^{\ell}_{2\bar{3}} = A^{\ell}_{3\bar{2}} = A^{\ell}_{3\bar{3}} = 0 \quad \text{for } \ell \geq 0. \tag{2.5}$$

Thus we may write CR invariants as polynomials in the variables

$$A^{\ell}_{2\bar{4}},\ A^{\ell}_{4\bar{2}},\ \text{and } A^{\ell}_{p\bar{q}} \text{ with } p+q \geq 7,\ \ell \geq 0.$$

THEOREM 2.1. *Let $n = 2$ and I^{CR}_w be the vector space of all CR invariants of weight w. Then a basis of I^{CR}_w for $w \leq 6$ is given by*

weight	*base of* I^{CR}_w
0	1
1 *or* 2	0
3	$A^{0}_{4\bar{4}}$
4	$\lvert A^{0}_{2\bar{4}}\rvert^2 = A^{0}_{2\bar{4}}A^{0}_{4\bar{2}}$
5	$P_1(A), P_2(A)$
6	$P_3(A), P_4(A), (A^{0}_{4\bar{4}})^2$

Table I

Here P_1, P_2, P_3, P_4 are given by

$$\begin{aligned}
P_1(A) =& \lvert A^{0}_{5\bar{2}}\rvert^2 + 18\,\lvert A^{0}_{4\bar{3}}\rvert^2 + \mathrm{Re}\left(18\,A^{0}_{3\bar{5}}A^{0}_{4\bar{2}} - 5i\ A^{1}_{2\bar{4}}A^{0}_{4\bar{2}}\right),\\
P_2(A) =& \lvert A^{0}_{5\bar{2}}\rvert^2 + \frac{171}{20}\,\lvert A^{0}_{4\bar{3}}\rvert^2 + \mathrm{Re}\left(\frac{15}{2}\,A^{0}_{3\bar{5}}A^{0}_{4\bar{2}} - \frac{37}{20}\,i\,A^{1}_{2\bar{4}}A^{0}_{4\bar{2}}\right),\\
P_3(A) =& \lvert A^{0}_{2\bar{6}}\rvert^2 - 10\,\lvert A^{0}_{3\bar{5}}\rvert^2 + \frac{3}{2}\,\lvert A^{1}_{2\bar{4}}\rvert^2 + \frac{61}{5}\,\lvert A^{0}_{4\bar{4}}\rvert^2\\
&+ \mathrm{Re}\left(48\,A^{0}_{4\bar{3}}\,A^{0}_{4\bar{5}} + 26\,A^{0}_{4\bar{2}}\,A^{0}_{4\bar{6}} - 28\,A^{0}_{3\bar{6}}\,A^{0}_{5\bar{2}} - \frac{5}{2}\,A^{0}_{4\bar{2}}\,A^{2}_{2\bar{4}}\right)\\
&+ \mathrm{Im}\left(2\,A^{3}_{5\bar{0}}\,A^{1}_{2\bar{4}} - 10\,A^{0}_{5\bar{2}}\,A^{1}_{2\bar{5}} + 12\,A^{0}_{4\bar{3}}\,A^{1}_{3\bar{4}} + 9\,A^{0}_{4\bar{2}}\,A^{1}_{3\bar{5}}\right),
\end{aligned}$$

$$\begin{aligned} P_4(A) =& |A^0_{2\bar{6}}|^2 + 11\,|A^0_{3\bar{5}}|^2 - \frac{1}{4}\,|A^1_{2\bar{4}}|^2 - \frac{2}{5}\,|A^0_{4\bar{4}}|^2 \\ &+ \mathrm{Re}\left(- 36\,A^0_{4\bar{3}}\,A^0_{4\bar{5}} - 16\,A^0_{4\bar{2}}\,A^0_{4\bar{6}} + 14\,A^0_{3\bar{6}}\,A^0_{5\bar{2}} + A^0_{4\bar{2}}\,A^2_{2\bar{4}}\right) \\ &+ \mathrm{Im}\left(2\,A^0_{5\bar{3}}\,A^1_{2\bar{4}} + 4\,A^0_{5\bar{2}}\,A^1_{2\bar{5}} - 9\,A^0_{4\bar{3}}\,A^1_{3\bar{4}} - 5\,A^0_{4\bar{2}}\,A^1_{3\bar{5}}\right). \end{aligned}$$

REMARK 2.2. There are many choice of a basis of I^{CR}_w, and $P_j(A)$ above have no special meaning as CR invariants. These $P_j(A)$ will appear again in the computation in Section 5. This is the only reason we take them as a basis.

This theorem for weight ≤ 5 has been obtained in [6] and [9]. For $w = 6$, this result is new, while the procedure of computation is exactly same as that of [6]. We here only explain the procedure of the computation.

We first recall that H is generated by the following two subgroups:

$$\begin{aligned} H_0 &= \{\phi_\lambda : \lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}\}, \\ H_1 &= \{\psi_{(\xi,r)} : (\xi, r) \in \mathbb{C} \times \mathbb{R}\}, \end{aligned}$$

where

$$\begin{aligned} \phi_\lambda(z_1, z_2) &= (\lambda z_1, |\lambda|^2 z_2), \\ \psi_{(\xi,r)}(z_1, z_2) &= \frac{(z_1 - \overline{\xi}\, z_2, z_2)}{1 + \xi\, z_1 - \eta\, z_2} \qquad \left(\eta = -\frac{|\xi|^2}{2} + i\, r\right). \end{aligned}$$

The action of H_0 on $\mathcal{N}$ is explicitly given by

$$\widetilde{A}^\ell_{p\bar{q}} = \lambda^{1-p-\ell}\overline{\lambda}^{1-q-\ell} A^\ell_{p\bar{q}}, \quad \text{where } \widetilde{A} = \phi_\lambda.A, \tag{2.6}$$

which is equivalent to $\phi_\lambda(N(A)) = N(\phi_\lambda.A)$. In view of this formula, we define *biweight* of a monomial $P(A)$ of $A = (A^\ell_{p\bar{q}})$ to be the pair of integers (w', w'') such that

$$P(\widetilde{A}) = \lambda^{-w'}\overline{\lambda}^{-w''} P(A) \quad \text{for } A \text{ and } \widetilde{A} \text{ in (2.6)}.$$

In particular, $A^\ell_{p\bar{q}}$ has biweight $(p + \ell - 1, q + \ell - 1)$. We also define *weight* to be the average of biweight $(w' + w'')/2$. For $\Phi = \phi_\lambda$, the transformation law (2.3) is written as

$$P(\widetilde{A}) = |\lambda|^{-2w} P(A) \quad \text{for every } \lambda \in \mathbb{C}^*.$$

This is equivalent to the condition that each monomial of $P(A)$ has biweight (w, w).

We next consider the action of H_1. It is shown that $\sigma_w(h) = 1$ for $h \in H_1$. Thus (2.4) for $h \in H_1$ is reduced to

$$P(h.A) = P(A) \quad \text{for } h \in H_1.$$

Therefore, we can give all CR invariants by determining all H_1-invariant polynomials $P(A)$ of homogeneous biweight.

For $w \leq 2$, there are no monomials of biweight (w, w). Hence there are no CR invariants of weight ≤ 2.

For $w = 3$, the only monomial of biweight $(3,3)$ is $A^0_{4\bar{4}}$ (up to a constant multiple). By computing the action of H_1 on this component, we see that $A^0_{4\bar{4}}$ is a CR invariant.

For $w = 4$, there are 3 monomials of biweight $(4,4)$:

$$A^0_{2\bar{4}}A^0_{4\bar{2}},\ A^0_{5\bar{5}},\ A^1_{4\bar{4}}.$$

Again, by computing the action of H_1 on the components appearing here, we see that $const.A^0_{2\bar{4}}A^0_{4\bar{2}}$ is the only H_1-invariant linear combinations of these monomials.

For $w = 5$, there are 9 monomials of biweight $(5,5)$:

$$A^0_{3\bar{5}}A^0_{4\bar{2}},\ A^1_{2\bar{4}}A^0_{4\bar{2}},\ A^0_{5\bar{3}}A^0_{2\bar{4}},\ A^1_{4\bar{2}}A^0_{2\bar{4}},$$
$$|A^0_{2\bar{5}}|^2,\ |A^0_{3\bar{4}}|^2,\ A^0_{6\bar{6}},\ A^1_{5\bar{5}},\ A^2_{4\bar{4}}.$$

In this case, the condition that a linear combination of these monomials to be H_1-invariant is given by a system of 10 linear equations of 9 variables. (The number of equations is the number of monomials of biweight $(5,4)$ or $(4,5)$; see Section 4 of [9].) The space of solutions is two dimensional and $\{P_1(A), P_2(A)\}$ gives a basis.

For $w = 6$, there are 24 monomials of biweight $(6,6)$. In this case, the condition to be H_1-invariant is reduced to a system of 31 linear equations of 24 variables. The computation for making the equations is just a straightforward generalization of the method of [6]. This procedure is purely algebraic and it is not hard to implement it on a computer algebra program. We have used *Mathematica* to obtain P_3 and P_4.

3. Fefferman's Weyl invariants

In this section we describe a geometric procedure of constructing CR invariants, which is called ambient metric construction, by following [5] and [1]. This procedure is shown to produce all CR invariants of weight $\leq n$.

Let Ω be a strictly pseudoconvex domain in $\mathbb{C}^n$. It is shown by Fefferman ([4]) that there exists a C^∞ defining function r of Ω, positive in Ω, satisfying

$$J[r] := (-1)^n \det \begin{pmatrix} r & r_j \\ r_{\overline{k}} & r_{j\overline{k}} \end{pmatrix} = 1 + O^{n+1}(\partial\Omega),$$

where

$$r_{j\overline{k}} = \frac{\partial^2 r}{\partial z_j \partial \overline{z}_k} \quad \text{for} \quad 1 \leq i, k \leq n,$$

and that such defining function is unique up to $O^{n+2}(\partial\Omega)$. Here $O^\ell(\partial\Omega)$ denotes a term which vanishes to ℓ th order along the boundary $\partial\Omega$. We call such a defining function a *Fefferman's defining function* and denote by r^{F}. One of the most important property of Fefferman's defining function is its transformation law: if $\Phi\colon \Omega \to \widetilde{\Omega}$ is a biholomorphic map, then

$$\widetilde{r}^{\mathrm{F}} \circ \Phi = r^{\mathrm{F}} \,|\det \Phi'|^{-1/(n+1)} + O^{n+2}(\partial\Omega), \tag{3.1}$$

where r^{F} and $\widetilde{r}^{\mathrm{F}}$ are Fefferman's defining functions of Ω and $\widetilde{\Omega}$, respectively. Using this transformation law, Fefferman gave a procedure of constructing CR invariants, which is called *ambient metric construction.* Introducing a new variable $z_0 \in \mathbb{C}^*$, we define a Lorentz-Kähler metric $g = g[r^{\mathrm{F}}]$ in a neighborhood of $\mathbb{C}^* \times M$ in $\mathbb{C}^{n+1}$ by

$$g_{j\overline{k}} = \frac{\partial^2 r^{\mathrm{F}}_{\#}}{\partial z_j \partial \overline{z}_k}, \quad \text{where} \quad r^{\mathrm{F}}_{\#}(z_0, z) := |z_0|^2 r^{\mathrm{F}}(z).$$

We call $g[r^{\mathrm{F}}]$ the *ambient metric* for $\partial\Omega$. Let $R[r^{\mathrm{F}}]$ be the curvature tensor of $g[r^{\mathrm{F}}]$ and let $R^{(p,q)}[r^{\mathrm{F}}] = \overline{\nabla}^{q-2}\nabla^{p-2}R$ be its covariant derivatives, where ∇ (resp. $\overline{\nabla}$) is the covariant derivative of type $(1,0)$ (resp. $(0,1)$). Then make complete contractions of tensor product of several curvature tensors:

$$W = \operatorname{contr}\left(R^{(p_1,q_1)} \otimes \cdots \otimes R^{(p_s,q_s)}\right). \tag{3.2}$$

We define the *weight* of W to be

$$w = \sum_{j=1}^{s} (p_j + q_j)/2 - s$$

and define *Weyl invariants of weight* $w \in \mathbb{N}$ to be linear combinations of complete contractions of the form (3.2) of weight w.

Now we consider the case $\partial\Omega$ is in normal form $N(A)$ in a neighborhood of 0. Then, for a Weyl invariant of weight $\leq n$, the value $W|_{(z_0,z)=(1,0)}$ is given by a polynomial of A, which is denoted by $P_W(A)$. This polynomial is shown to be a CR invariant of weight w by using the transformation law (3.1). The following theorem states that this procedure gives all CR invariants of weight $\leq n$.

THEOREM 3.1. ([5], [1]) *For $w \leq n$, every CR invariant of weight w is expressed as $P_W(A)$ for a Weyl invariant W of weight w.*

The restriction on the weight comes from the fact that r^{F} is defined only up to $O^{n+2}(\partial\Omega)$, while the estimate is not optimal. We can replace n by $n+1$ (or 5 in case $n=2$) in Theorem 4.2 below.

The proof of Theorem 3.1 in [1] also gives a basis of CR invariants. To state their result, we recall some definitions from [1]. We say that a complete contraction (3.2) is *traceless* if it does not involve any internal traces, that is, no two indices on the same tensor are contracted together. We say that two complete contractions are *equivalent* if they can be made to coincide by permuting the tensors and by permuting the indices of each of the tensors. Choose a representative from each such equivalence class of traceless complete contractions and call the chosen contractions *allowable.*

THEOREM 3.2. ([1]) *Let $w \leq n-1$. If $\{W_1, \dots, W_d\}$ is a list of allowable complete contractions, then $P_{W_1}(A), \dots, P_{W_d}(A)$ form a basis of I_w^{CR}.*

Note that the restriction on weight is sharp. For $w = n$, there is a relation among allowable Weyl invariants, which is obtained by skew symmetrizing n indices in (3.2).

REMARK 3.1. In the proof of Theorem 3.2, the following argument is used. First make linear combinations of complete contractions of the form

$$\operatorname{contr}\left(\mathbf{A}^0_{p_1\overline{q}_1} \otimes \cdots \otimes \mathbf{A}^0_{p_s\overline{q}_s}\right) \tag{3.3}$$

that involves no internal traces. Then formally replace each $\mathbf{A}^0_{p\overline{q}}$ by $R^{(p,q)}$ and replace the trace for the standard metric $\delta_{i\overline{j}}$ on $\mathbb{C}^{n-1}$ by the ambient metric $g_{i\overline{j}}$. This replacement gives an injection from the space of linear combinations of complete contractions of the form (3.3) to the vector space of Weyl invariants. Thus the study of linear relations among allowable Weyl invariants is reduced to study of that for complete contractions (3.3). For such complete contractions, we can apply the invariant theory for $U(n-1)$ and obtain the desired linear independence.

4. Generalized Weyl invariants

For weight $w > n$, Fefferman's ambient metric construction breaks down. In particular, if $n = 2$, Theorems 3.1 and 3.2 give no information; there are no CR invariants of weight ≤ 2 (see Theorem 2.1). In this section, following [8], we generalize Fefferman's method by introducing parameters that describe the ambiguity of r^{F}. We apply this result, in the next section, to obtain Table I in Theorem 2.1.

We lift the equation $J[r] = 1$ to the $\mathbb{C}^*$-bundle, on which the ambient metric is defined,

$$J_{\#}[u] := (-1)^n \det \left(u_{j\overline{k}}\right)_{0\leq j,k\leq n} = |z_0|^2$$

and consider its asymptotic solutions along $\mathbb{C}^* \times N(A) \subset \mathbb{C}^* \times \mathbb{C}^n$ of the form

$$u = r_{\#} + r_{\#} \sum_{j=1}^{\infty} \eta_j \cdot \left(r^{n+2} \log r_{\#}\right)^j,$$

where r is a smooth defining function of $N(A) \subset \mathbb{C}^n$, $r_{\#} := |z_0|^2 r$, and η_j are smooth functions in a neighborhood of $0 \in \mathbb{C}^n$. Such an asymptotic solution exists for any $N(A)$ and is uniquely determined (modulo flat function along $\partial\Omega$) by the additional initial condition:

$$\left.\frac{\partial^{n+2} r}{\partial \rho^{n+2}}\right|_{\rho=0} = f(z', \overline{z}', v) \in C^\infty(N(A)).$$

Here ρ-partial differentiation is defined with respect to the real coordinates $(z', \overline{z}', \rho, v)$. If we write the Taylor expansion of $f(z', \overline{z}', v)$ as

$$f(z', \overline{z}', v) = \sum_{|\alpha|,|\beta|,\ell\geq 0} C^{\ell}_{\alpha\overline{\beta}} z'^{\alpha} \overline{z}'^{\beta} v^{\ell},$$

then the Taylor expansion of r at 0 is determined by A and $C = (C^{\ell}_{\alpha\overline{\beta}})$. Thus we may write the germ of r at 0 as $r[A, C]$.

Now we follow Fefferman's ambient metric construction and define Weyl invariants W for the metric $g[A, C] = (\partial^2 r[A, C]/\partial z_j \partial \overline{z}_k)$, which is a germ of Lorentz-Kähler metric at $(1,0) \in \mathbb{C}^* \times \mathbb{C}^n$. In this case, the value of a Weyl invariant W at $(z_0, z) = (1,0)$ is given by a polynomial in (A, C) for any weight. We denote this polynomial by $P_W(A, C)$. In case $P_W(A, C)$ is independent of C, we say that W is *C-independent*; then $P_W(A)$ gives a CR invariant. The following theorem claims that all CR invariants are given by C-independent Weyl invariants.

THEOREM 4.1. ([8]) *For each CR invariant $P(A)$, there exists a C-independent Weyl invariant W such that $P(A) = P_W(A)$.*

For $n \geq 3$ (resp. $n = 2$), all Weyl invariant W of weight $\leq n+1$ (resp. ≤ 5) are C-independent. We thus obtain a generalization of Theorem 3.1.

THEOREM 4.2. ([8]) *Let $n \geq 3$ (resp. $n = 2$). Then, for $w \leq n+1$ (resp. ≤ 5), every CR invariant is expressed as $P_W(A)$ for a Weyl invariant W of weight w.*

The restriction on weight is optimal: there is a *C-dependent* Weyl invariant of weight $n+2$ (or weight 6 in case $n = 2$). See also the computations in the next section.

Note that $r = r[A, C]$ satisfies $J[r] = 1 + O^{n+1}(N(A))$; hence r is a Fefferman's defining function. We thus see, for weight $\leq n$, that the CR invariants $P_W(A)$ in the previous section coincide with those given in Theorem 4.2.

REMARK 4.1. The terminology used here is different from those of [8]. In [8] we first consider (3.2) as a formal expression and call it *Weyl polynomial*; then define *Weyl functional* to be the assignment of functions obtained by evaluating a Weyl polynomial for ambient metrics. Two different Weyl polynomials may give the same Weyl functional. Thus the distinction is essential in the proof of the theorems.

5. Explicit computation in case $n = 2$

In this section we always assume $n = 2$. Let

$$I_w^{\mathrm{Weyl}} = \{P_W(A, C) : \ W \text{ is a Weyl invariant of weight } w\}$$

and identify a Weyl invariant W with the polynomial $P_W(A, C)$. Then Theorem 4.2 is written as

$$I_w^{\mathrm{CR}} = I_w^{\mathrm{Weyl}} \quad \text{for } w \leq 5.$$

As an application of this theorem, we give an alternative proof Theorem 2.1 for weight ≤ 5.

Let $E^{(p,q)}$ be a tensor of type (p,q) defined by

$$E^{(p,q)} := \mathrm{tr}^4 \left(R^{(p+4,p+4)}\right),$$

where the trace is taken for the metric g. In particular, $E^{(0,0)}$ is a scalar. It is then easy to show that I_w^{Weyl} is generated by traceless complete contractions of the tensor products of

$$R^{(p,q)} \quad p,q \geq 2 \quad \text{and} \quad E^{(p,q)} \quad p,q \geq 0.$$

(Follow the arguments in Section 5 of [8].) Using this fact, we list up generators of I_w^{Weyl}. The table for weight ≤ 6 is

weight	generators of I_w^{Weyl}
1	0
2	$W_{2,2}$
3	$W_{2,3}$, $E^{(0,0)}$
4	$W_{2,4}$, $W_{3,3}$
5	$W_{2,5}$, $W_{3,4}$
6	$W_{2,6}$, $W_{3,5}$, $W_{4,4}$, $\widetilde{W}_{2,2}$, $(E^{(0,0)})^2$

Table II

Here $W_{p,q}$ and $\widetilde{W}_{p,q}$ are traceless complete contractions of the from

$$W_{p,q} := \mathrm{contr}\left(R^{(p,q)} \otimes R^{(q,p)}\right) \quad \text{and} \quad \widetilde{W}_{p,q} := \mathrm{contr}\left(R^{(p,q)} \otimes E^{(q,p)}\right).$$

There are several ways to make such complete contractions. The results of this section are independent of the choice.

To select a basis from the generators in Table II, we need to determine linear relations among these Weyl invariants. A computation in [9] gives

$$W_{2,2} = W_{2,3} = 0,$$
$$E^{(0,0)} = -(4!)^2 A^0_{4\bar{4}},$$
$$W_{2,4} = \frac{3}{7} W_{3,3} = 7 \cdot 2^8 \, |A^0_{2\bar{4}}|^2$$

and

$$\|R^{(2,5)}\|^2 = -4\,(5!)^2\,P_1(A), \qquad \|R^{(3,4)}\|^2 = -\frac{16\,(5!)^2}{3}\,P_2(A),$$

where P_1 and P_2 are defined in Theorem 2.1. We thus obtain:

weight	basis of $I_w^{\mathrm{Weyl}} = I_w^{\mathrm{CR}}$
1	0
2	0
3	$E^{(0,0)}$
4	$W_{2,4}$ (or $W_{3,3}$)
5	$W_{2,5}$, $W_{3,4}$

Table III

We can also apply the same procedure to express Weyl invariants of weight 6 in terms of (A, C). A computation using *Mathematica* gives

$$\begin{aligned}
&W_{2,6} = F_1(A) + 51840\,Q(A,C) + 14929920\,A^0_{4\bar{4}}\,C^0_{0\bar{0}},\\
&W_{3,5} = F_2(A) + 25920\,Q(A,C) + 7257600\,A^0_{4\bar{4}}\,C^0_{0\bar{0}},\\
&W_{4,4} = F_3(A) + 20736\,Q(A,C) + 5723136\,A^0_{4\bar{4}}\,C^0_{0\bar{0}},\\
&\widetilde{W}_{2,2} = F_4(A) - 576\,A^0_{4\bar{4}}\,C^0_{0\bar{0}},\\
&(E^{(0,0)})^2 = (4!)^4\,(A^0_{4\bar{4}})^2,
\end{aligned}$$

where

$$Q(A,C) = (C^0_{0\bar{0}})^2 - 64\,\mathrm{Re}\,A^0_{4\bar{3}}\,C^0_{0\bar{1}} - \frac{16}{3}\,\mathrm{Re}\,A^0_{4\bar{2}}\,C^0_{0\bar{2}}$$

and F_j are polynomials of A. From these formulas, we see that C-independent Weyl invariants are generated by

$$\begin{gathered}
W_{2,6} - \frac{5}{2}\,W_{4,4} + 1080\,\widetilde{W}_{2,2},\\
W_{3,5} - \frac{5}{4}\,W_{4,4} + 180\,\widetilde{W}_{2,2},\\
(E^{(0,0)})^2.
\end{gathered} \tag{5.1}$$

By computing $F_j(A)$ explicitly, we see that the first two Weyl invariants are respectively given by

$$F_1 - \frac{5}{2}F_3 + 1080\,F_4 = 1382400\,P_3(A),$$
$$F_2 - \frac{5}{4}F_3 + 180\,F_4 = -345600\,P_4(A),$$

where P_3 and P_4 are given in Theorem 2.1. Thus by Theorem 4.1 we have

Proposition 5.1. *The Weyl invariants* (5.1) *form a basis of* I_6^{CR}.

We close this note by giving an analogy of Theorem 3.2 in case $n = 2$. It is clear that the allowable Weyl invariants in the sense of Section 3 do not give basis (as $W_{2,2} = 0$). To define allowable Weyl invariants for $n = 2$, in view of Remark 3.1, we first define *allowable* monomial in A of weight w to be the monomials of $A^0_{p\bar q}$ of biweight (w, w). All allowable (monic) monomials of $A^0_{p\bar q}$ for weight ≤ 6 are

weight	allowable monomials
1 or 2	0
3	$A^0_{4\bar 4}$
4	$\lvert A^0_{2\bar 4}\rvert^2$
5	$\lvert A^0_{2\bar 5}\rvert^2$, $\lvert A^0_{3\bar 4}\rvert^2$
6	$\lvert A^0_{2\bar 6}\rvert^2$, $\lvert A^0_{3\bar 5}\rvert^2$, $(A^0_{4\bar 4})^2$

Table IV

For weight ≤ 5, we can relate Tables III and IV by replacing $A^0_{p\bar q}$ by $R^{(p,q)}$ and taking complete contractions. This is an analogy of the formal replacement explained in Remark 3.1. If we call the Weyl invariants obtained by this replacement *allowable Weyl invariants*, then we obtain the analogy of Theorem 3.2 for $n = 2$ and weight ≤ 5.

For weight 6, the same replacement gives three Weyl invariants:

$$W_{2,6}, \quad W_{3,5}, \quad (E^{(0,0)})^2.$$

In this case, we can obtain three linearly independent CR invariants by adding $W_{4,4}$ and $\widetilde{W}_{2,2}$ (which are missing in the list above) to the first two Weyl invariants as in (5.1). So, we can say that there is a basis of

I_6^{CR} each of them corresponds to an allowable monomial. This result is obtained by a direct computation and we do not know the reason why such a correspondence holds.

REMARK 5.2. For weight 7, there are nontrivial cubic Weyl invariants and for such invariants the discussion about allowable invariants breaks down. In fact, there are 7 cubic allowable monomials of A of weight 7

$$A^0_{4\overline{4}}|A^0_{2\overline{4}}|^2, \quad A^0_{2\overline{4}}A^0_{2\overline{4}}A^0_{6\overline{2}}, \quad A^0_{3\overline{4}}A^0_{3\overline{4}}A^0_{4\overline{2}}, \quad A^0_{\overline{2}4}A^0_{\overline{2}4}A^0_{\overline{6}2}$$

and their conjugates, but only CR invariant of degree three is const. $A^0_{4\overline{4}}|A^0_{2\overline{4}}|^2$, which is the product of invariants of weight 3 and weight 4. Therefore other cubic allowable monomials have no relation to CR invariants.

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Graduate School of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro
Tokyo 153-8914
JAPAN
E-mail:hirachi@ms.u-tokyo.ac.jp