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THE PRODUCT FORMULA FOR NIELSEN ROOT NUMBER

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ABSTRACT. In [6], Cheng-Ye You gave a condition equivalent to the Nielsen number product formula for fiber maps. And Jerzy Jezierski also gave a similar condition for coincidences of fiber maps. The main purpose of this paper is to find the condition for which holds the product formula for Nielsen root numbers $N(f;a) = N(\bar{f};\bar{a})N(f_b;a)$.

1. Introduction

The topological theories of Nielsen number for fixed points, coincidences and roots are all concerned with properties of spaces. Root theory was found in an interesting special case, that in which $g : X \to Y$ is the constant map $g(x) = a$ for some $a \in Y$.

The computation of the Nielsen coincidence number is one of the central issues in Nielsen coincidence theory. For fiber-preserving maps between fibrations, a product formula for the Nielsen number is therefore a desirable result to obtain. In [6], You gave conditions equivalent to the product formula $N(f) = N(f_b) \cdot N(\bar{f})$ for the Nielsen numbers of fibre maps. And in [2] Jezierski found similar conditions for coincidences of fibre maps.

The main object of this paper is to give necessary and sufficient conditions for such a formula to hold in the root case. As a root theory resembles coincidence theory in the sense that precise results have only been obtained for maps that are defined on manifolds, we extend our attention to the more general spaces. We will investigate the situation where the spaces are not necessarily manifolds.

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A very clear presentation of many results in root theory can be found in Kiang's book ([5]). In [5], the Nielsen root number is defined without using index; Let $f : X \to Y$ be a mapping under $H : f \simeq H(\cdot,1) : X \to Y$ a homotopy. If the root class $A \in \Gamma'_a(f)$ corresponds to a root class $\in \Gamma'(H(\cdot,1),a)$ under any such H, then A is called an essential root class. The Nielsen root number is the number of the essential root classes of f for $a \in Y$. In §2, we define an index of Nielsen root class for the map $f : M \to N$ of compact spaces which satisfies the conditions $(*)$. The definition of $N(f;a)$ using index is given by the cardinality of the essential root classes of f for $a \in Y$ with non-zero index. Then this number $N(f;a)$ also has the usual basic properties.

In §3, most results are revisions of [2]. So we assume that the reader is familiar with the theorems for maps of Jerzy Jezierski ([2]). We consider the root Reidemeister classes and establish the relation between the Nielsen root classes and the root Reidemeister classes.

In the final section we consider the condition for which holds the product formula. The maps of Theorem 4.5 also satisfy the condition $(**)$ for which the spaces are extended to more general spaces. And to get the product formula we find the condition; for any class $A \in \Gamma'_a(f)$ such that $A \subset p^{-1}(\bar{A}_i)$, $A \cap p^{-1}(b_i)$ is a single class of f_{b_i} $(b_i \in \bar{A}_i)$.

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2. Definition of index for Nielsen root classes

Let $f : M \to N$ be a map where M, N are compact spaces which satisfy the following conditions;

$(*)$
(1) there is a subspace $M_e \subset M$ which is an n-dimensional orientable manifold such that $M - M_e$ is an ANR and $\dim(M - M_e) \leq n-2$.
(2) there is a neighborhood $a \in V \subset N$ such that V is homeomorphic to $\mathbb{R}^n$.

Let $a \in N$ be fixed. We denote the set of roots of $f : M \to N$ by $\Gamma_a(f) = \{x \in M; f(x) = a\}$.

Two points x, $x' \in \Gamma_a(f)$ are said to be *Nielsen equivalent* if there is a path p in M from x to x' such that $[f \circ p] = [a]$ (Here $[f \circ p]$ denotes the fixed-end-point homotopy class containing $f \circ p$ and a is used both to denote the point $a \in N$ as well as the constant path at $a \in N$). This equivalence induces an equivalent relation; an equivalence class of roots is called a root class and the set of root classes is denoted by $\Gamma'_a(f)$.

To define an index for a root class $A \in \Gamma'_a(f)$ let us consider the following two cases.

(1) If $\Gamma_a(f) \subset M_e$, then take a neighborhood U of A satisfying $U \cap \Gamma_a(f) = A$ and $U \approx \mathbb{R}^n$.

Since M_e is oriented, there is a generator $z_{M,A}$ of $H_n(M_e, M_e - A)$. Consider the homomorphism

$$\begin{array}{ccc} H_n(M_e, M_e - A) \ni z_{M,A} & & \\ \wr\| & & \\ H_n(U, U - A) & \xrightarrow{f_*} & H_n(N, N - a) \ni z_{N,a} \end{array}$$

If $f_*(z_{M,A}) = k \cdot z_{N,a}$, then we define the *index for* $A \in \Gamma'_a(f)$ $\ ind(f; A) = k$.

(2) More generally if $\Gamma_a(f) \not\subset M_e$, then we replace f with f' which is homotopic to f and satisfies $\Gamma_a(f') \subset M_e$. And we define

$$ind\ (f; A) = ind\ (f'; A')$$

where A' is the Nielsen root class of f' corresponding to $A \in \Gamma'_a(f)$.

The next lemma shows that such map f' satisfying $\Gamma_a(f') \subset M_e$ exists.

LEMMA 2.1. *Any map $f : M \to N$ is homotopic to a map f' such that $\Gamma_a(f') \subset M_e$.*

PROOF. Consider $f^{-1}(\bar{V}) \cap (M - M_e) = \{x \in M \mid x \notin M_e,\ f(x) \in \bar{V}\}$ where V is a neighborhood of $a \in N$ satisfying $\bar{V} \approx D^n$. Then since $\dim(M - M_e) \leq n - 2$ by the assumption

$$\dim(f^{-1}(\bar{V}) \cap (M - M_e)) \leq n - 2.$$

Then there is a homotopy

$$\tilde{f}_t : (f^{-1}(\bar{V}) \cap (M - M_e), f^{-1}(bd\ V) \cap (M - M_e)) \longrightarrow (\bar{V}, bd\ V) = (D^n, S^{n-1})$$

rel $f^{-1}(bd\ V) \cap (M - M_e)$ to a map into *bd* V.
Then we define

$$f_t(x) = \begin{cases} \tilde{f}_t(x) & \text{if } x \in f^{-1}(\bar{V}) \cap (M - M_e) \\ f(x) & \text{otherwise.} \end{cases}$$

Take $f' = f_1$, and now prove that $\Gamma_a(f') \subset M_e$. Let $x \in \Gamma_a(f')$ and suppose that $x \notin M_e$. If $x \in f^{-1}(\bar{V})$ then $x \in (M - M_e) \cap f^{-1}(\bar{V})$. Hence $f'(x) = \tilde{f}_1(x) \in bd\ \bar{V}$. So $f'(x) \neq a$. If $x \notin f^{-1}(\bar{V})$, then $f'(x) = f_1(x) = f_0(x) = f(x) \notin \bar{V}$. So $f'(x) \neq a$. This contradiction proves our claim. □

Now consider the following lemma which establishes our definition well defined.

Lemma 2.2. *If $f_0, f_1 : M \to N$ are homotopic and satisfy $\Gamma_a(f_0) \cup \Gamma_a(f_1) \subset M_e$, then the indices of corresponding Nielsen root classes are equal.*

Proof. Let $F : M \times I \to N$ be a homotopy between f_0 and f_1, and $A_0 \in \Gamma'_a(f_0)$ correspond to $A_1 \in \Gamma'_a(f_1)$. Then there is a homotopy F' rel $M \times \{0,1\}$ such that $F'(\cdot, 0) = f_0, F'(\cdot, 1) = f_1$ and $\Gamma_a(F') \subset M_e \times I$. Let $A \in \Gamma'_a(f)$ and let $f_0, f_1 \simeq f$ satisfy $\Gamma_a(f_0), \Gamma_a(f_1) \subset M_e$ and let $A_0 \in \Gamma'_a(f_0), A_1 \in \Gamma'_a(f_1)$ are corresponded to A. Then $f_0 \simeq f_1$ and A_0 corresponds to A_1. Now $ind(f_0; A_0) = ind(f_1; A_1)$ since all roots of the homotopy are lying inside $M_e \times I$. □

Let $f : M \to N$ be a map satisfying the condition $(*)$. Then for any $A \in \Gamma'_a(f)$ we define $ind(f; A)$ as the index of a root class A of f . A root class $A \in \Gamma'_a(f)$ is said to be an *essential root class* if $ind(f; A) \neq 0$. The number of essential root classes of f is called the *Nielsen root number* and denoted by $N(f; a)$. This definition of $N(f; a)$ yields a positive integer. Now we show that $N(f; a)$ has the usual basic properties.

THEOREM 2.3. **(Homotopy invariance)** *Let $f_0, f_1 : M \to N$ be homotopic maps satisfying the condition $(*)$. Then $N(f_0; a) = N(f_1; a)$.*

PROOF. Let f_0', f_1' be homotopic under the same homotopy such that $\Gamma_a(f_0')$, $\Gamma_a(f_1') \subset M_e$ and let A_0, A_1 correspond to A_0', A_1' respectively. Then by the definition and Lemma 2.2, we have

$$ind(f_0; A_0) = ind(f_0'; A_0') = ind(f_1'; A_1') = ind(f_1; A_1).$$

This implies $N(f_0; a) = N(f_1; a)$. □

THEOREM 2.4. **(Lower bound property)** *Let $f : M \to N$ be a map satisfying the condition $(*)$. Then every map homotopic to f has at least $N(f; a)$ roots.*

PROOF. Let f have $N(f; a)$ essential root classes. Then it has at least $N(f; a)$ roots. If f' is homotopic to f which has $N(f'; a)$ roots, then by homotopy invariance (Theorem 2.3) we have $N(f'; a) = N(f; a)$. □

Now we give an example for map $f : M \to N$ satisfying the condition $(*)$.

EXAMPLE 1.5. Let K, L be n-dimensional pseudomanifolds and let $M_e = |K| - |K|^{(n-2)}$and $a \in int\ \sigma \subset |L|$ where σ is n-dim simplex. Then $f : |K| \to |L|$ satisfies our assumptions and also we can show this map f satisfies above two lemmas.

REMARK. In [5], the Nielsen root number $N(f; a)$ is defined without using index. It is very general but hard to calculate. And if we denote it as $\bar{N}(f; a)$, that is, the number of root classes which do not disappear after any homotopy of f, then we have $N(f; a) \leq \bar{N}(f; a)$.

3. The root Reidemeister classes

Let X, Y be path-connected space and let $f : X \to Y$ be continuous map. Fix a point $x \in X$. Define an action of $\pi_1(X, x)$ on $\pi_1(Y, f(x))$ by

$$(\langle d \rangle \cdot \langle \alpha \rangle) = \langle f(d) \cdot \alpha \rangle.$$

Denote the quotient set $\nabla(f;x,r) = \pi_1(Y, f(x))/im f_\sharp(\pi_1(X,x))$ as the set of Reidemeister classes of f where r is a path from $f(x)$ to $a \in Y$.

Suppose that $H \subset \pi_1(Y, f(x))$ is a normal subgroup. Then we may define the action of $\pi_1(X,x)$ on $\pi_1(Y, f(x))$ as above, and we get $\nabla_H(f;x,r)$.

Now we define an injection map $\rho_{(x,r)} : \Gamma'_a(f) \to \nabla(f;x,r)$. Let $x_0 \in \Gamma_a(f)$ and we fix a path u from x to x_0. Then $fu - r$ is a loop based at $f(x)$. We put $\rho_{(x,r)}(x_0) = [\langle fu - r\rangle]$. Then the next lemma shows that $\rho_{(x,r)}$ is well-defined injection map. A pair (x,r) with $x \in X$ and r a path in Y from $f(x)$ to a is called a reference pair for f.

LEMMA 3.1. *Let (x,r) be a reference pair for f.*

a) *$\rho_{(x,r)}$ is independent of the choice of u.*

b) *If $x_0, x_1 \in \Gamma_a(f)$ are in the same Nielsen class, then*

$$\rho_{(x,r)}(x_0) = \rho_{(x,r)}(x_1).$$

c) *$\rho_{(x,r)}$ is injective.*

PROOF. a) If u' is another path from x to x_0, then

$$\langle fu' - r\rangle = \langle fu' - fu + fu - r\rangle = \langle f(u'-u) + fu - r\rangle.$$

Since $u' - u$ is a loop based at x

$$\langle fu' - r\rangle = \langle fu - r\rangle.$$

b) Let v be a path joining x_0 to x_1 such that $fv \simeq a$. Then $\rho_{(x,r)}(x_0) = [\langle fu - r\rangle]$. On the other hand,

$$\rho_{(x,r)}(x_1) = [\langle fu' - r\rangle] \qquad (*1)$$

where u' is a path from x to x_1. But we may take $u' = u + v$. Then

$$\begin{aligned}(*1) &= [\langle f(u+v) - r\rangle] = [\langle fu + fv - r\rangle] \\ &= [\langle fu - r\rangle] = \rho_{(x,r)}(x_0).\end{aligned}$$

c) Suppose that $\rho_{(x,r)}(x_0) = \rho_{(x,r)}(x_1)$. Now show that x_0, x_1 are Nielsen related. Our assumption (i.e., $[\langle fu - r\rangle] = [\langle fu' - r\rangle] \in \nabla(f;x,r)$) means that there is a loop w based at x such that

$$\langle -fw + fu' - r\rangle = \langle fu - r\rangle \in \pi_1(Y, f(x)),$$

$$\langle -fu - fw + fu' \rangle = \langle f(-u - w + u') \rangle = 1 \in \pi_1(Y, f(x_0)).$$

Thus $-u - w + u'$ establishes Nielsen relation between x_0 to x_1. □

By above Lemma 3.1. b), we also obtain a map $\rho_{(x,r)}$ defined on $\Gamma'_a(f)$, i.e., $\rho_{(x,r)} : \Gamma'_a(f) \to \nabla(f; x, r)$, which assigns to the class of $x_0 \in \Gamma_a(f)$ the class $[\langle fu - r \rangle]$ where u is path from x to x_0.

Let $x_0, x_1 \in X$ and let r_0, r_1 be paths in Y from $f(x_0)$ to a and $f(x_1)$ to a respectively. Then the next lemma establishes a relationship between the set $\nabla(f; x_0, r_0)$ and $\nabla(f; x_1, r_1)$.

LEMMA 3.2 ((1.3) in [2]). *Let w be a path from x_0 to x_1 and let $\nu : \nabla(f; x_0, r_0) \to \nabla(f; x_1, r_1)$ be a transformation defined by $\nu[\langle \alpha \rangle] = [\langle -fw + \alpha + r_0 - r_1 \rangle]$.*

a) *ν is independent of the choice of the path w.*

b) *If $x_0 = x_1$ and $r_0 = r_1$, then ν is the identity.*

c) *If (x_2, r_2) is another reference pair, then the diagram*

$$\begin{array}{ccc} \nabla(f; x_0, r_0) & \xrightarrow{\nu} & \nabla(f; x_1, r_1) \\ & \nu \searrow & \downarrow \nu \\ & & \nabla(f; x_2, r_2) \end{array}$$

commutes.

d) *The diagram*

$$\begin{array}{ccc} \Gamma'_a(f) & \xrightarrow{\rho_{(x_0,r_0)}} & \nabla(f; x_0, r_0) \\ & \rho_{(x_1,r_1)} \searrow & \downarrow \nu \\ & & \nabla(f; x_1, r_1) \end{array}$$

commutes.

The conclusion b) and c) show that ν is bijective and that we can identify all $\nabla(f; x, r)$ by ν to get an abstract set denoted by $\nabla(f)$. Then d) shows that we can identify all $\rho_{(x,r)}$ to get an injection $\rho : \Gamma'_a(f) \to \nabla(f)$. Thus each Nielsen root class may be identified with an Reidemeister class. On the other hand, we say that a class $A \in \nabla(f)$ equals $A' \in \Gamma'_a(f)$

as a set $\rho(A') = A$, and that A is empty if it does not lie in the image of ρ.

Now we define transformations between the sets of Reidemeister (or Nielsen) classes of maps appearing in a homotopy commutative diagram. Consider a commutative diagram of path-connected topological spaces.

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ M' & \xrightarrow{f'} & N' \end{array}$$

Let (x,r) be a reference pair for f and let $H \subset \pi_1(N), H' \subset \pi_1(N')$ be normal subgroups such that $k_\sharp(H) \subset H'$. Then the map

$$\kappa : \nabla_H(f;x,r) \to \nabla_{H'}(f';h(x),k(r))$$

given by $\kappa[\langle a\rangle_H] = [\langle ka\rangle_{H'}]$ defines a transformation $\kappa : \nabla_H(f) \to \nabla_{H'}(f)$ which does not depend on the choice of (x,r).

THEOREM 3.3 ((2.1) in [2]). *Consider the diagram of path-connected topological spaces*

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ M' & \xrightarrow{f'} & N' \end{array}$$

Let $H \subset \pi_1(N), H' \subset \pi_1(N')$ be normal subgroups such that $k_\sharp(H) \subset H'$.

a) *If our diagram is commutative then it determines a map $\kappa : \nabla_H(f) \to \nabla_{H'}(f)$ given by $\kappa : \nabla_H(f;x,r) \to \nabla_{H'}(f';h(x),k(r))$, $\kappa[\langle a\rangle_H] = [\langle ka\rangle_{H'}]$ where (x,r) is a reference pair for f.*

b) *If the diagram is only homotopy commutative (with some homotopies $F : kf \simeq f'h$) but $k_\sharp H = H'$ and k,h are homeomorphisms then it determines a bijective transformation $\eta_F : \nabla_H(f) \to \nabla_{H'}(f)$ which may be represented by*

$$\eta_F : \nabla_H(f;x,r) \to \nabla_{H'}(f';h(x),-F(x,\cdot)+kr)$$

$$\eta_F[\langle\alpha\rangle_H] = [\langle -F(x,\cdot) + k\alpha + F(x,\cdot)\rangle_H].$$

The transformation η_F depends on the choice of the homotopy F. But if there exists a map $F : M \times I \times I \to N'$ such that $F(x,0,t) = (k_t \circ f)(t)$, $F(x,1,t) = (f' \circ h_t)(t)$, then it does not distinguish between homotopic homotopies.

Let $p : E \to B$, $p' : E' \to B'$ be locally trivial bundles such that the total spaces, base spaces and fibres are path connected. We assume that B and B' are paracompact. Then the above bundles are Hurewicz fibrations; denote by λ, λ' their lifting functions. Let $E_b = p^{-1}(b)$ be the fibre over $b \in B$.

Let $b_0, b_1 \in \Gamma_{\bar{a}}(\bar{f})$ and let $\bar{u}$ be a path in B from b_0 to b_1 and $\bar{f}\bar{u} \simeq$ *const.* we denote by $\tau_{\bar{u}} : E_{\bar{u}(0)} \to E_{\bar{u}(1)}$. This map is given by the formula $\tau_{\bar{u}}(x) = \lambda(\bar{u}, x)(1)$.

Since the considered bundles are locally trivial we may assume that the maps $\tau_{\bar{u}} : E_{\bar{u}(0)} \to E_{\bar{u}(1)}$ are homeomorphisms.

Suppose we are given two fibre maps, i.e., a commutative diagram:

$$\begin{array}{ccc} E & \xrightarrow{\;f\;} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{\;f'\;} & B' \ni \bar{a} \end{array}$$

Now we show how a path joining points from on Nielsen class of $\Gamma_{\bar{a}}(\bar{f})$ induces a map between the Nielsen classes on the corresponding fibres. For $b \in B$, we denote by $f_b : E_b \to E'_{\bar{f}(b)}$ the restriction of f on $p^{-1}(b)$ i.e., $f_b = f|_{p^{-1}(b)} : p^{-1}(b) \to (p')^{-1}(\bar{f}(b))$.

For a fixed $b' \in B'$ we denote by K the normal subgroup of $\pi_1(E'_{b'})$ given by the formula

$$K(x') = ker(\pi_1(E'_{b'}, x') \to \pi_1(E', x').$$

Let $b \in \Gamma_{\bar{a}}(\bar{f})$. Then $f|_{p^{-1}(b)} = f_b : p^{-1}(b) \to (p')^{-1}(\bar{f}(b)) = (p')^{-1}(\bar{a})$. So we get the set $\nabla_K(f_b)$.

Let $b_0, b_1 \in \Gamma_{\bar{a}}(\bar{f})$ be Nielsen equivalent and $\bar{u}$ be a path joining them such that $\bar{f}\bar{u} \simeq$ *const.*. We are going to define a bijective transformation between the sets $\nabla_K(f_{b_0})$ and $\nabla_K(f_{b_1})$ induced by $\bar{u}$.

Consider the diagram:

$$\begin{array}{ccc} E_{b_0} & \xrightarrow{f_{b_0}} & E'_{\bar{f}(b_0)} \\ {\scriptstyle \tau_{\bar{u}}}\downarrow & & \downarrow{\scriptstyle \tau'_{\bar{f}\bar{u}} \simeq id} \\ E_{b_1} & \xrightarrow{f_{b_1}} & E'_{\bar{f}(b_1)} \end{array}$$

Then above diagram is homotopy commutative by means of

$$F(x,t) = f_{\bar{u}(t)}\tau_{\bar{u}_0^t}(x).$$

Now we may apply definition η_F to above diagram and we obtain a map

$$T_{\bar{u}} : \nabla_K(f_{b_0}) \to \nabla_K(f_{b_1}).$$

According to the definition of η_F and the similar calculation from [2] the formula representing this map takes the form

$$T_{\bar{u}} : \nabla_K(f_{b_0}; x, r) \to \nabla_K(f_{b_1}; x', r')$$

given by $T_{\bar{u}}[\langle\alpha\rangle_K] = [\langle\alpha'\rangle_K]$, where $x' = u(1)$ and r', α' are paths in $E'_{\bar{f}(b_1)}$homotopic to $-fu + r$ and $-fu + \alpha + fu$ respectively.

Let $b \in \Gamma_{\bar{a}}(\bar{f})$. Then $(i_b)_\nabla : \nabla_K(f_b) \to \nabla(f)$ will stand for the map induced by the commutative diagram

$$\begin{array}{ccc} E_b & \xrightarrow{f_b} & E'_{\bar{f}(b)} \\ \downarrow & & \downarrow \\ E & \xrightarrow{f} & E' \end{array}$$

with constant homotopy.

LEMMA 3.4 ((4.7) in [2]). *Let $b_0, b_1 \in \Gamma_{\bar{a}}(\bar{f})$ and let $\bar{u}$ be a path joining them such that $\bar{f}\bar{u} \simeq$ const.. Then the diagram is commute.*

$$\begin{array}{ccc} \nabla_K(f_{b_0}) & \xrightarrow{T_{\bar{u}}} & \nabla_K(f_{b_1}) \\ (i_{b_0})_\nabla \searrow & & \downarrow (i_{b_1})_\nabla \\ & & \nabla(f). \end{array}$$

PROOF. Choose a reference pair (x_0, r_0) for f_{b_0}. Then

$$(i_{b_0})_\nabla[\langle\alpha\rangle_K] = [\langle\alpha\rangle] \in \nabla(f; x_0, r_0)$$

$$(i_{b_0})_\nabla T_{\bar{u}}[\langle\alpha\rangle_K] = [\langle\alpha'\rangle] \in \nabla(f; x_1, r_1),$$

(where $x_1 = u(1), r_1 = -fu + r_0, \alpha' = -fu + \alpha + fu$ and $u = \lambda(\bar{u}, x_0)$).
Since $\nu : \nabla(f; x_0, r_0) \to \nabla(f; x_1, r_1)$

$$\begin{aligned} \nu[\langle\alpha\rangle] &= [\langle -fu + \alpha + r_0 - r_1\rangle] \\ &= [\langle -fu + \alpha + r_0 - r_0 + fu\rangle] \\ &= [\langle -fu + \alpha + fu\rangle] = [\langle\alpha'\rangle]. \end{aligned}$$

Thus above two elements represent the same element in $\nabla(f)$. □

LEMMA 3.5 ((4.11) in [2]). *Let $b_0, b_1 \in \Gamma_{\bar{a}}(\bar{f})$ and let $A_i \in \nabla_K(f_{b_i})$ $(i = 0, 1)$. Then $(i_{b_0})_\nabla(A_0) = (i_{b_1})_\nabla(A_1)$ iff there exists a path $\bar{w}$ from b_0 to b_1 such that $\bar{f}\bar{w} \simeq$ const. in B' and $T_{\bar{w}}(A_0) = A_1$.*

Let $b \in \Gamma_{\bar{a}}(\bar{f})$ and let $\bar{v}$ be a loop based at b such that $\bar{f}\bar{u} \simeq$ *const.* Then we have the map $T_{\bar{u}}; \nabla(f_b) \to \nabla(f_b)$. So two Nielsen classes A_0, A_1 $\in \nabla(f_b)$ give the same class in $\nabla(f)$ iff there is a loop $\bar{u}$, as above, such that $T_{\bar{u}}(A_0) = A_1$. Thus to get a product formula, we need to assume that $C_0(f) = ker f_\sharp = \{\langle\alpha\rangle \in \pi_1(E) : f_\sharp(\langle\alpha\rangle) = 1 \in \pi_1(E')\}$ acts trivially on $\nabla(f_b)$ (it means for any $\bar{u} \in C_0(f), T_{\bar{u}} =$ *identity*).

4. The product formula for Nielsen root number

Consider the commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{\bar{f}} & B' \end{array}$$

which satisfies the following conditions;
(i) (E, p, B) satisfies that $E_e (\subset E)$ is n-dimensional manifold

such that $dim(E - E_e) \leq dimE - 2$.

(ii) (E', p', B') satisfies that for $a \in E', p'(a) \in B'$, $a \in U \subset E'$,

(**) and $p'(a) \in V \subset B'$ such that U, V are homeomorphic to Euclidean space.

(iii) $dimE = dimE', dimB = dimB'$ (hence the dimensions of all fibres are equal). And the orientations of the base spaces, total spaces and fibers are compatible.

We consider the root index for $f, \bar{f}$ and f_b with respect to these orientations.

THEOREM 4.1. *Let* $A_0 \in \Gamma'_a(f)$ *and* $p(A_0) \subset \bar{A}_0 \in \Gamma'_{\bar{a}}(\bar{f}), b \in \bar{A}_0$. *Then* $ind(f; A_0) = ind(\bar{f}; \bar{A}_0) \cdot ind(f_b; A_0 \cap p^{-1}(b))$.

PROOF. Suppose we are given a continuous family of commutative diagram.

$$\begin{array}{ccc} E & \xrightarrow{f_t} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{\bar{f}_t} & B' \end{array}$$

Find a fibre homotopy such that $f_0 = f$ and the roots of $f_1, \bar{f}_1$ lie in Euclidean part of E, F, B. Then

$$\begin{aligned} ind(f; A_0) &= ind(f_0; A_0) = ind(f_1; A_1) \\ &= ind(\bar{f}_1; \bar{A}_1) \cdot ind(f_{1_{b_1}}; A_1 \cap p^{-1}(b_1)) \\ &= ind(\bar{f}_0; \bar{A}_0) \cdot ind(f_{0_{b_0}}; A_0 \cap p^{-1}(b)) \\ &= ind(\bar{f}; \bar{A}_0) \cdot ind(f_{b_0}; A_0 \cap p^{-1}(b)). \end{aligned}$$ □

Now we find the fibre homotopy. First we note that for $\bar{f}$ there is a homotopy $\bar{f}_t$ such that $\bar{f}_0 = f$ and $\Gamma_{\bar{a}}(\bar{f}_1) \subset B_e$. We may assume that $\Gamma_{\bar{a}}(\bar{f}_1)$ is finite. Then by Homotopy Covering Property, there is homotopy $f_t : E \to E$ such that $f_0 = f$ covering $\bar{f}_t$. Let $b \in \Gamma_{\bar{a}}(\bar{f}_1)$. Consider $f_b : p^{-1}(b) \to p^{-1}(\bar{f}(b))$. Then we find a homotopy f'_t such that $f'_0 = f_b$ and $\Gamma_e(f'_1) \subset (p^{-1}(b))_e$. So there is a homotopy f_t $(1 \leq t \leq 2)$ such that the restriction to the fibre $p^{-1}(b)$ is the previous homotopy f'_t. Then the map f_2 satisfies the claim.

Let $\bar{A}_1, \cdots, \bar{A}_s$ denote all essential root classes of $\bar{f} : B \to B'$. Then the essential classes of f lie over $\bar{A}_1 \cup \cdots \cup \bar{A}_s$. For any $i = 1, \cdots, s$, let

$$C_i = \sharp\{A \in \Gamma'_a(f) \mid ind(f; A) \neq 0,\ p(A) \subset \bar{A}_i\}.$$

Then $N(f; a) = C_1 + \cdots + C_s$ and $N(\bar{f}; \bar{a}) = s$.

Consider an essential Nielsen class $A \in \Gamma'_a(f)$. Let $p(A) \subset \bar{A}_1$ and let $b_1 \in \bar{A}_1$. Consider the set $A \cap p^{-1}(b_1)$. How many Nielsen classes of the restriction $f|_{p^{-1}(b_1)} = f_{b_1}$ are contained in $A \cap p^{-1}(b)$? According to Lemma 3.5, we have the next lemma.

LEMMA 4.2. *The two points x_0, x_1 are Nielsen related as roots of f iff there exists a loop $\bar{u}$ based at $b_0 \in B$ such that $\bar{f}\bar{u}$ is contractible and $T_{\bar{u}} : \nabla_K(f) \longrightarrow \nabla_K(f)$ carries A_0 into A_1 ($\bar{u} \in ker f_\sharp$).*

COROLLARY 4.3. *ker $f_\sharp$ is acting on the set $\Gamma(f_{b_0})$ by $T_{\bar{u}}$. And two classes $A_0, A_1 \in \Gamma'_a(f_{b_0})$ belong to the same Nielsen class of $A \in \Gamma'_a(f)$ iff they lie in the same orbit of the above action*

$$T_{\bar{u}} : \nabla_K(f_{b_0}) \longrightarrow \nabla_K(f_{b_0}),\ T_{\bar{u}}[\langle\alpha\rangle_K] = [(T_{\bar{u}}\langle\alpha\rangle); \bar{u} \in ker\ \ f_\sharp].$$

Now we discuss the product formula of the Nielsen root numbers of a fibre map.

THEOREM 4.4. a) *If $N(\bar{f}; \bar{a}) = 0$, then $N(f; a) = 0$.*

b) *If $N(f_b; a) = 0$ for all b lying in an essential class, then $N(f; a) = 0$.*

c) *If $N(f_b; a) = 1$ for all b lying in an essential class, then $N(f; a) = N(\bar{f}; \bar{a})$.*

PROOF. If the assumption of a) or b) is satisfied, then every class of f is inessential. The assumption of c) implies that $C_i = 1$ for all $i = 1, \cdots, s$. □

THEOREM 4.5. *Suppose we are given a commutative diagram which satisfies condition (**) and $N(f; a) \neq 0$.*

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ p\downarrow & & \downarrow p' \\ B & \xrightarrow{\bar{f}} & B' \end{array}$$

Then the formula $N(f;a) = N(f_{b_1};a) + \cdots + N(f_{b_s};a)$ *holds iff the following two conditions hold;*

(a) $N_K(f_b;a) = N(f_b;a)$.

(b) *For any class* $A \in \Gamma'_a(f)$ *such that* $A \subset p^{-1}(\bar{A}_i)$, $A \cap p^{-1}(b_i)$ *is a single class of* $f_{b_i} = f|_{p^{-1}(b_i)}$, $b_i \in \bar{A}_i$.

PROOF. Under the assumption (b), for $A_i \in \Gamma'_{\bar{a}}(\bar{f})$ the action on $\nabla_K(f_{b_i})$ is trivial (i.e., $T_{\bar{u}} =$ identity). Then we have

$$C_i = N_K(f_{b_i};a), \quad i = 1, \cdots, s.$$

So

$$\begin{aligned} N(f;a) = C_1 + \cdots + C_s &= N_K(f_{b_1};a) + \cdots + N_K(f_{b_s};a) \\ &= N(f_{b_1};a) + \cdots + N(f_{b_s};a). \end{aligned}$$ □

If moreover the Nielsen numbers of all restriction to fibre are equal to $N_K(f_{b_1};a)$, then above product formula gives

$$N(f;a) = s \cdot N_K(f_{b_1};a) = N(\bar{f};\bar{a}) \cdot N_K(f_{b_1};a),$$

that is, $N(f;a) = N(\bar{f};\bar{a}) \cdot N(f_b;a)$.

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