Comm. Korean Math. Soc. 15 (2000), No. 2, pp. 349–355

ON HOMOTOPY EQUIVALENCE OF NONNILPOTENT SPACES AND ITS APPLICATIONS

SANG-EON HAN

ABSTRACT. In this paper we generalize the Whitehead theorem which says that a homology equivalence implies a homotopy equivalence for nilpotent spaces. We make some theorems on a homotopy equivalence of non-nilpotent spaces, e.g., the solvable space or space satisfying the condition (T^{**}) or space X with $\pi_1(X)$ Engel, or locally nilpotent space with some properties. Furthermore we find some conditions that the Wall invariant will be trivial.

1. Introduction

For nilpotent spaces ([1], p.58) X, Y we know that a homology equivalence between X and Y makes a homotopy equivalence ([2, 3]). But there is not much information on the relations between homology equivalence and homotopy equivalence for non-nilpotent spaces.

In this paper, we make the homotopy equivalence of non-nilpotent spaces clear with relation to the homology equivalence. The spaces, e.g., the solvable space or locally nilpotent space under some conditions are studied (see Theorem 3.1). We work in the category of topological spaces having the homotopy type of connected pointed CW-complexes with base point and denote it by T.

2. Some properties of the non-nilpotent space and condition (T^{**})

In this section, we study a locally nilpotent space and its properties

Received March 5, 1999. Revised October 29, 1999.

2000 Mathematics Subject Classification: Primary 55N10; Secondary 55N91.

Key words and phrases: condition (T^{**}), (locally) nilpotent space, solvable space, homotopy equivalence.

The author wishes to acknowledge the financial support of the Korea Research Foundation made in the program of 1998.

with respect to conditions (T^*) and (T^{**}) respectively.

We recall that a locally nilpotent group is the group whose all finitely generated subgroups are nilpotent groups ([10]).

And let T_N be the category of nilpotent spaces and continuous maps.

Now we recall the concept of a locally nilpotent space as follows:

A space $X(\in T)$ is said to be a locally nilpotent space ([5, 6]) if

(1) $\pi_1(X)$ is a locally nilpotent group, and
(2) the action $\pi_1(X) \times \pi_n(X) \to \pi_n(X)$ is nilpotent for all $n \geq 2$ ([1]).

And the category of locally nilpotent spaces and continuous maps is denoted by T_{LN}.

We know that the category T_N is a full subcategory of T_{LN}. We say that a space $X(\in T)$ satisfies the condition (T^*) ([5]) if for all $g, t \in \pi_1(X)$ either $g[g, \pi_1(X)] = t[t, \pi_1(X)]$ or $g[g, \pi_1(X)] \bigcap t[t, \pi_1(X)] = \phi$.

Now we define an effective concept with respect to the locally nilpotent space.

We say that X satisfies the condition (T^{**}) ([5, 6]) if for all $g(\neq 1) \in \pi_1(X)$, then $g \notin [g, \pi_1(X)]$.

Since the group $[g, \pi_1(X)]$ is a normal subgroup of $\pi_1(X)$, the condition (T^{**}) has a homotopy invariant property.

Let's say that a group action G on H is called solvable if there exists a finite chain: $H = H_1 \supset H_2 \supset H_3 \supset \cdots \supset H_j \supset \cdots \supset H_n = \{e\}$ such that for each j

(1) H_j is closed under the action of G, and
(2) H_{j+1} is normal in H_j and H_j/H_{j+1} is abelian.

DEFINITION 2.1. We define that a space $X(\in T)$ is solvable if

(1) $\pi_1(X)$ is solvable, and
(2) there is the solvable action $\pi_1(X) \times \pi_n(X) \to \pi_n(X)$ for all $n \geq 2$.

Even though the group $\pi_n(X)$ for $n \geq 2$ is commutative, the condition (2) of definition 2.1 is meaningful with respect to the closed property of the solvable action.

And the category of solvable spaces and continuous maps is denoted by T_S. We get the following easily.

THEOREM 2.2. *The category T_S has the finite product property, i.e., for set $\{X_\alpha | \alpha \in M : finite\}$, $X_\alpha \in T_S$ for any α if and only if $\prod_{\alpha \in M} X_\alpha \in T_S$.*

And the category T_N is a full subcategory of T_S.

In a fibration $F_f \to E \xrightarrow{f} B$, if the reduced homology group $\tilde{H}_*(F_f) = 0$, $* \geq 0$ we recall that f is an acyclic map, where F_f is a homotopy fiber of f.

In a fibration $F \to E \xrightarrow{p} B$, for any path $\alpha : I \to B$ and singular q-complex $g : \Delta^q \to p^{-1}(\alpha(0))$ determines a map $G : \Delta^q \times I \to E$ over $\alpha \circ pr_2 : \Delta^q \times I \to I \to B$ and extending $G_0 = g : \Delta^q \times \{ 0 \} \to E$ by the homotopy lifting property, where pr_2 means a second projection. If α is a loop, then $G_1 : \Delta^q \times \{ 1 \} \to E$ is a q-simplex in $p^{-1}(\alpha(1)) = p^{-1}(\alpha(0))$. Now elements of $\pi_1(B)$ operate on $H_q(F)$. Thus we have the following [8].

DEFINITION 2.3. A fibration $F \to E \to B$ is said to be quasi-nilpotent if the action of $\pi_1(B)$ on $H_n(F)$ is nilpotent, $n \geq 0$. Furthermore the fibration $F \to E \to B$ is strong quasi-nilpotent if it is quasi-nilpotent and if, in addition, $\pi_1(B)$ is nilpotent.

We recall that a group G satisfies the maximal condition if it has no infinite strictly increasing chain of subgroups ([10]).

We recall the Engel group G ([10]), i.e., the group which has a relation of the form $[\cdots[[x,y],y],\cdots,y] := [x,y,y,\cdots,y] := [x,_n y] = 1$, where $[x,y] := x^{-1}y^{-1}xy$, the commutator of x and y. The number of entries of y's in the formula above depends on both x, $y (\in G)$. We do not need to bound it uniformly.

3. Main Theorems

Let's check the homotopy equivalence of the various cases:

THEOREM 3.1. *If $f : X \to Y$ is an acyclic map and X satisfies one of the followings:*

1. *$X (\in T_S)$,*
2. *X is the space satisfying condition (T^*) or (T^{**}) with $\pi_1(X)$ finite,*

(3) $X(\in T_{LN})$ *such that* $\pi_1(X)$ *is torsion-free with all proper subgroups of* $\pi_1(X)$ *nilpotent,*

(4) X *such that* $\pi_1(X)$ *is an Engel and whether* $\pi_1(X)$ *is finite or* $\pi_1(X)$ *is infinite with the maximal condition,*

then f *is a homotopy equivalence.*

PROOF. By the classical homotopy exact sequence of a fibration: $F_f \to X \xrightarrow{f} Y$, we have an epimorphism $\pi_1(f)$. And we get $\tilde{H}_1(F_f)$ trivial from an acyclic map property of f, i.e., $\pi_1(F_f)$ is a perfect group. Furthermore the homomorphic image of a perfect group is also perfect. Hence $\pi_1(Y) \cong \frac{\pi_1(X)}{P\pi_1(X)}$ where $P\pi_1(X)$ means a perfect normal subgroup of $\pi_1(X)$. Now let's check each cases

For case (1): since $X \in T_S$, by definition of the solvability of the space X we have

$$(\pi_1(X))^{(n)} = [(\pi_1(X))^{(n-1)}, (\pi_1(X))^{(n-1)}]$$

trivial for some n where $[,]$ means the commutator subgroup. Since we get

$$P(\pi_1(X)) = (P(\pi_1(X)))^{(n)} \leq (\pi_1(X))^{(n)}$$

trivial, $P(\pi_1(X))$ is finally zero.

For case (2): from the fact that X satisfies the condition (T^{**}) with $\pi_1(X)$ finite we get the space X as a nilpotent space ([5, Lemma 3.1]).

Next, if X satisfies the condition (T^*) then we know that X also satisfies the condition (T^{**}). If not, $g \in [g, \pi_1(X)]$ for some $g(\neq 1) \in \pi_1(X)$. Then $g^{-1} \in [g, \pi_1(X)]$ and $1 \in g[g, \pi_1(X)]$. Thus $g[g, \pi_1(X)] \bigcap 1[1, \pi_1(X)] \neq \phi$. Since X satisfies the condition (T^*), $g[g, \pi_1(X)] = 1$. Since $g(\neq 1) \in g[g, \pi_1(X)]$, we have a contradiction. Thus $\pi_1(X)$ is nilpotent even for the case X satisfying the condition (T^*).

For case (3): for $X(\in T_{LN})$ if $\pi_1(X)$ is torsion-free with all proper subgroups of $\pi_1(X)$ nilpotent then $X \in T_N$ ([11]).

For case (4): if $\pi_1(X)$ is finite with $\pi_1(X)$ an Engel group we get $\pi_1(X)$ nilpotent and even when $\pi_1(X)$ is infinite. Furthermore $\pi_1(X)$ has the maximal condition and we get a nilpotent group $\pi_1(X)$. Thus $P\pi_1(X)$ is trivial.

With any of the four cases above, we have $\pi_1(f)$ as an isomorphism. Thus f is a homotopy equivalence from the fact that f is an acyclic map and by the classical Whitehead theorem. □

COROLLARY 3.2. *If $f : X \to Y$ is a quasi-nilpotent homology equivalence and the space X satisfies one of the cases* (1) ~ (4) *from* Theorem 3.1, *then f is a homotopy equivalence.*

PROOF. A quasi-nilpotent homology equivalence is equivalent to an acylic map. □

REMARK. For the case (1) or (2) from Theorem 3.1, even though $X \in T_S$, X need not satisfy the condition (T^*). For example, BS_3 is a solvable space but BS_3 does not satisfy the condition (T^*) where B means Milnor's classifying space and S_3 means symmetric group.

4. Applications to the wall invariants

We found some properties of the Wall invariant for the space satisfying the condition (T^{**}) ([7]).

For a space X, we consider the group ring $\mathbb{Z}\pi_1(X)$. Let $K_0(\mathbb{Z}\pi_1(X))$ denote the Grothendieck group of the group ring $\mathbb{Z}\pi_1(X)$.

A space is said to be a type FP if the singular chain complex $C_i\tilde{X}$ of the universal covering $\tilde{X}$ of X is chain homotopy equivalent, (as $\mathbb{Z}\pi_1(X)$-complex) to a finite projective complex, i.e., a complex $\bar{C}_i$ with $\bar{C}_i = 0$ for i big enough, and with each $\bar{C}_i$ a finitely generated projective $\mathbb{Z}\pi_1(X)$ module.

If X is of type FP, the Wall obstruction $\omega(X)$ is defined by

$$\omega(X) = \Sigma(-1)^i[\bar{C}_i] \in \tilde{K}_0(\mathbb{Z}\pi_1(X))$$

where $\bar{C}_i$ is a finite projective complex equivalent to $C_i\tilde{X}$, and $[\bar{C}_i]$ denotes the class of $\bar{C}_i$ in the projective class group $\tilde{K}_0(\mathbb{Z}\pi_1(X))$. It is evident that $w(X)$ is independent of the choice of $\bar{C}_i$.

Furthermore, a space X of type FP is dominated by a finite complex if and only if $\pi_1(X)$ is finitely presented.

We know the fact that: if $\pi_1(X)$ is nilpotent then X is type FP if and only if X is finitely dominated ([9]).

LEMMA 4.1 ([8, Theorem 2.1]). *Let* $F \xrightarrow{j} E \to B$ *be a fibration with* F *a finitely dominated complex and* B *a finite complex. Then* E *is a finitely dominated complex and* $w(E) = j_* w(F)\chi(B)$, *where* $j_* : K_0(\mathbb{Z}\pi_1(F)) \to K_0(\mathbb{Z}\pi_1(E))$ *is a group homomorphism and* χ *means the Euler characteristic.*

THEOREM 4.2. *In a fibration* $F \xrightarrow{j} E \to B$ *with* F *a finitely dominated space if the finite space* $B(\in T_{LN})$ *satisfies one of the following:*

(1) $\pi_1(B)(\neq 0)$ *is finite,*
(2) $\pi_1(B)$ *is torsion-free with all proper subgroups of* $\pi_1(X)$ *nilpotent,*

then $\omega(E) = 0$.

PROOF. For case (1), we know that B satisfies the condition (T^{**})([5, 6]) and furthermore from the finiteness of $\pi_1(B)$ we get B as a nilpotent space.

For case (2), B is also nilpotent from (3) in Theorem 3.1.

Thus in any cases above we have $\chi(B) = 0$ ([4]). Finally we have $\omega(E) = 0$ from Lemma 4.1. □

References

[1] A. K. Bousfield and D. M. Kan, *Homotopy Limits, Completions and Localization,* L. N. S. , vol. 304, Springer-Verlag, Berlin, 1972.

[2] E. Dror, *A generalization of the Whitehead theorem,* L. N. M., vol. 249, Springer-Verlag, Berlin and New York, 1971, pp. 13–22.

[3] S. M. Gersten, *The Whitehead theorem for nilpotent spaces,* Proc. of A. M. S. **47** (1975), no. 1, 259–260.

[4] Sang-Eon Han, *On the Euler characteristic of the space satisfying condition* (T^{**}), Tamkang Jour. of Math. **29** (1998), no. 2, 115–120.

[5] ———, *On the R-good property of the space satisfying condition* (T^{**}), Kyushu J. Math. **52** (1998), no. 2, 279–286.

[6] ———, *On the space satisfying conditions* (T^*) *or* (T^{**}) *and its applications II,* Topology and its applications (to appear).

[7] ———, *Wall invariant for the space satisfying the condition* (T^{**}), Tsukuba J. of Math. **22** (1998), no. 1, 157–164.

[8] P. J. Hilton and J. Roiterberg, *On the Zeeman comparison theorem for the homology of quasi-nilpotent fibrations,* Quart. J. Math. Oxford **27** (1976), 433–444.

[9] G. Mislin, *Wall obstruction of the nilpotent spaces*, Topolgy **14** (1975), 311–317.
[10] D. J. S. Robinson, *A Courses in the Theory of Groups*, Springer-Verlag, New York, 1980.
[11] H. Smith, *Groups with few nonnilpotent subgroups*, Glasgow Math. J. **39** (1997), 141–151.

Department of Mathematics
College of Natural Sciences
Honam University
Kwangju 506-714, Korea
E-mail: sehan@honam.honam.ac.kr