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FINSLER METRICS COMPATIBLE WITH A SPECIAL RIEMANNIAN STRUCTURE

HONG-SUH PARK, HA-YONG PARK* AND BYUNG-DOO KIM

ABSTRACT. We introduce the notion of the Finsler metrics compatible with a special Riemannian structure f of type (1,1) satisfying $f^6 + f^2 = 0$ and investigate the properties of Finsler space with them.

1. Introduction

A Finsler space F^n admitting a Finsler metric $L(x, y)$ and an almost complex structure J satisfying the Rizza condition ([3], [10]) is called an almost Hermitian Finsler manifold or simply a Rizza manifold. The Rizza manifold has been studied by G. B. Rizza [10], Y. Ichijyō [4] and M. Fukui [1]. The f-structure in a Riemannian manifold was introduced and studied by K. Yano [11]. Recently, in [4], Y. Ichijyō introduced the Finsler metrics compatible with f-structure and they were studied by some authors ([4], [7], [8]). On the other hand, $\varphi(4, 2)$-structure in a Riemannian manifold was introduced and studied by K. Yano, C. S. Houh and B. Y. Chen [12], and the Finsler metrics compatible with a $\varphi(4, 2)$-structure were studied by the first two authors ([9]).

The present paper is the consecutive study of [9]. We investigate the Finsler metrics compatible with a special structure $f^i{}_j$ ($\neq 0$) in the Riemannian manifold of type (1,1) satisfying $f^i{}_r f^r{}_k f^k{}_h f^h{}_l f^l{}_t f^t{}_j + f^i{}_r f^r{}_j = 0$.

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2. Preliminaries

Let ${f^i}_j$ ($\neq 0$) be a tensor field of type (1,1) and class C^∞ satisfying

$$(2.1)\qquad {f^i}_r {f^r}_k {f^k}_h {f^h}_l {f^l}_t {f^t}_j + {f^i}_r {f^r}_j = 0,\ rank\ ({f^i}_r) = 2r \leq n.$$

If we put

$$(2.2)\qquad {\ell^i}_j = -{f^i}_r {f^r}_k {f^k}_h {f^h}_j,\ {m^i}_j = {f^i}_r {f^r}_k {f^k}_h {f^h}_j + \delta^i_j,$$

where δ^i_j is the Kronecker delta, we have

$$(2.3)\qquad \begin{aligned} {\ell^i}_j + {m^i}_j &= \delta^i_j,\ {\ell^i}_r {\ell^r}_j = {\ell^i}_j,\ {\ell^i}_r {m^r}_j = {m^i}_r {\ell^r}_j = 0,\\ {m^i}_r {m^r}_j &= {m^i}_j,\ {f^k}_j {f^r}_k {\ell^i}_r = {\ell^k}_j {f^r}_k {f^i}_r = {f^r}_j {f^i}_r,\\ {f^i}_r {f^r}_k {m^k}_j &= {m^i}_r {f^r}_k {f^k}_j = 0,\ {f^i}_r {f^r}_k {f^k}_h {f^h}_l {\ell^l}_j = -{\ell^i}_j. \end{aligned}$$

Hence, ${\ell^i}_j$ and ${m^i}_j$ are complementary projection operators on the tangent space $T_p(M)$ at each point p of M^n. Let $\mathcal{L}$ and $\mathcal{M}$ be the distributions corresponding to ${\ell^i}_j$ and ${m^i}_j$ respectively. $\mathcal{L}$ is a $2r$-dimensional distribution and $\mathcal{M}$ is an $(n-2r)$-dimensional distribution. The tangent space $T_p(M)$ is expressed by $\mathcal{L} \oplus \mathcal{M}$. For any $y \in T_p(M)$, $y = u + v$ for $u \in \mathcal{L}$ and $v \in \mathcal{M}$, that is, the local components of u and v are expressed as $u^i = {l^i}_j y^j$ and $v^i = {m^i}_j y^j$ for $y^i \in T_p(M)$. The tensor ${f^i}_r {f^r}_j$ acts on $\mathcal{L}$ as an almost complex structure operator and on $\mathcal{M}$ as a null operator. If rank of ${f^i}_j$ is n, then ${l^i}_j = {\delta^i}_j$ and ${m^i}_j = 0$, so ${f^i}_j$ satisfies ${f^i}_k {f^k}_h {f^h}_l {f^l}_j = -{\delta^i}_j$, that is, ${f^i}_k {f^k}_j$ is an almost complex structure. It is well known ([11]) that, in a manifold with the structure ${f^i}_j$ satisfying (2.1), there exists a positive definite Riemannian metric a_{ij} with respect to which the distributions $\mathcal{L}$ and $\mathcal{M}$ are orthogonal and

$$(2.4)\qquad a_{ij} = a_{pq} {f^p}_r {f^r}_i {f^q}_k {f^k}_j + a_{ip} {m^p}_j,\ a_{ip} {f^p}_r {f^r}_j = -a_{jp} {f^p}_r {f^r}_i.$$

In a Finsler space F^n, the metric tensor $g_{ij}(x,y)$ and C-tensor $C_{ijk}(x,y)$ are introduced by

$$g_{ij}(x,y) = (1/2)\dot{\partial}_i \dot{\partial}_j L^2(x,y),\ C_{ijk}(x,y) = (1/4)\dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L^2(x,y),$$

where $\dot{\partial}_i = \partial/\partial y^i$.

3. (f^2, L)-manifold

Let $T_p(M)$ be a tangent space at any point p of a Finsler space F^n with the metric $L(x,y)$. If we define the norm of $y \in T_p(M)$ as

$$\|y\| = L(x,y), \tag{3.1}$$

then $T_p(M)$ becomes a normed linear space. Since $f^i{}_r f^r{}_j$ are almost complex structures on the subspace $\mathcal{L}$ in $T_p(M)$, we define the scalar product of a complex number $\bar{c} = |\bar{c}|(\cos\theta + i\sin\theta)$ and any $\ell^i{}_r y^r$ on $\mathcal{L}$ as follows:

$$\bar{c}\ell^i{}_r y^r = |\bar{c}|(\delta^i_j \cos\theta + f^i{}_r f^r{}_j \sin\theta)\ell^j{}_k y^k.$$

Then we have the following from the properties of Finsler metric $L(x,y)$:

(1) $\|\ell y\| = L(x, \ell y) \geq 0$,

(2) $\|\ell y\| = L(x, \ell y) = 0$ if and only if $y = 0$,

(3) $\|\ell y_1 + \ell y_2\| = L(x, \ell y_1 + \ell y_2) \leq L(x, \ell y_1) + L(x, \ell y_2)$
$= \|\ell y_1\| + \|\ell y_2\|$,

(4) since a finite normed space is complete, $\{\ell y \mid \|\ell y\| = L(x, \ell y)\}$ is complete.

Therefore, if $\|\bar{c}\ell y\| = |\bar{c}|\|\ell y\|$ for any complex number $\bar{c}$ and $\ell y \in \mathcal{L}$, then $\mathcal{L}$ is a complex Banach space. If we put the components of f_θ as $f_\theta{}^i{}_j = \delta^i_j \cos\theta + f^i{}_r f^r{}_j \sin\theta$, then

$$\|\bar{c}\ell y\| = L(x, \bar{c}\ell y) = |\bar{c}|L(x, f_\theta \ell y)$$

and hence $\|\bar{c}\ell y\| = |\bar{c}|\|\ell y\|$ is equivalent to

$$L(x, f_\theta \ell y) = L(x, \ell y). \tag{3.2}$$

In a Finsler space $F^n(M,L)$ with the norm defined by (3.1) and the Riemannian structure $f^i{}_j$ satisfying (2.1), if the Finsler metric $L(x,y)$ satisfies (3.2), L is said to be *compatible with* $f^i{}_j$ and F^n is called an *(f^2, L)-manifold.*

Thus we have:

PROPOSITION 3.1. *In an (f^2, L)-manifold, the distribution $\mathcal{L}$ in the tangent space $T_p(M)$ of a Finsler space $F^n(M, L)$ is a complex Banach space.*

From the definition of the metric tensor we have

$$\begin{aligned} \dot\partial_i \dot\partial_j L^2(x, \ell y) &= 2g_{pq}(x, \ell y)\ell^p{}_i \ell^q{}_j, \\ \dot\partial_i \dot\partial_j L^2(x, f_\theta \ell y) &= 2g_{pq}(x, f_\theta \ell y) f_\theta{}^p{}_k f_\theta{}^q{}_h \ell^k{}_i \ell^h{}_j, \end{aligned} \tag{3.3}$$

where the components of $f_\theta \ell y$ are $(\delta^i_r \cos\theta + f^i{}_l f^l{}_r \sin\theta)\ell^r{}_k y^k$. From (3.2) and (3.3) we have

$$g_{pq}(x, f_\theta \ell y) f^p_{\theta r} f^q_{\theta s} \ell^r{}_i \ell^s{}_j = g_{pq}(x, \ell y)\ell^p{}_i \ell^q{}_j. \tag{3.4}$$

Since $L(x, y)$ is homogeneous of degree one in y, (3.4) leads to (3.2). That is, (3.2) is equivalent to (3.4).

Now, differentiating (3.4) with respect to θ and using

$$(f_\theta \ell y)_{\theta=0} = \ell y, \ (f^i_{\theta j})_{\theta=0} = \delta^i_j, \ (df^i_{\theta j}/d\theta)_{\theta=0} = f^i{}_k f^k{}_j,$$

then we have

$$\begin{aligned} &2C_{pqt}(x, \ell y) f^t{}_n f^n{}_m \ell^m{}_d y^d \ell^p{}_i \ell^q{}_j + g_{pq}(x, \ell y) f^p{}_c f^c{}_r \ell^r{}_i \ell^q{}_j \\ &+ g_{pq}(x, \ell y) f^q{}_n f^n{}_s \ell^s{}_j \ell^p{}_i = 0. \end{aligned} \tag{3.5}$$

On the other hand, we have

$$\begin{aligned} &\frac{d}{d\theta}\left\{ g_{pq}(x, f_\theta \ell y) f^p_{\theta r} f^q_{\theta s} \ell^r{}_i \ell^s{}_j \right\} \\ &= f^r_{\theta i} f^s_{\theta j} \{ 2C_{pqt}(x, f_\theta \ell y) f^t{}_n f^n{}_a \ell^a{}_e f^l_{\theta m} y^m \ell^p{}_r \ell^q{}_s \\ &+ g_{pm}(x, f_\theta \ell y) f^m{}_a f^a{}_q \ell^p{}_r \ell^q{}_s + g_{mq}(x, f_\theta \ell y) f^m{}_l f^l{}_p \ell^p{}_r \ell^q{}_s \} = 0 \end{aligned}$$

by virtue of $f^p_{\theta r}\ell^r{}_i = \ell^p{}_r f^r_{\theta i}$ and (3.5). Therefore $g_{pq}(x, f_\theta \ell y) f^p_{\theta r} f^q_{\theta s} \ell^r{}_i \ell^s{}_j$ is independent of θ, so (3.4) holds. Thus (3.4) is equivalent to (3.5).

Next, transvecting (3.5) by $y^i y^j$, we get

$$g_{pq}(x, \ell y) f^p{}_k f^k{}_r \ell^r{}_i \ell^q{}_j y^i y^j = 0. \tag{3.6}$$

Differentiating (3.6) with respect to y^h, we have

$$2C_{pqr}(x,\ell y)\ell^r{}_h f^p{}_k f^k{}_l \ell^l{}_i \ell^q{}_j y^i y^j \\ + g_{pq}(x,\ell y) f^p{}_k f^k{}_l \ell^l{}_h \ell^q{}_j y^j + g_{pq}(x,\ell y) f^p{}_k f^k{}_l \ell^l{}_i y^i \ell^q{}_h = 0,$$

from which

$$\{g_{pq}(x,\ell y) f^p{}_k f^k{}_l \ell^l{}_h \ell^q{}_j + g_{pq}(x,\ell y) f^p{}_k f^k{}_l \ell^l{}_j \ell^q{}_h\} y^j = 0. \tag{3.7}$$

Conversely we assume that (3.7) holds. Differentiating (3.7) with respect to y^i, we easily get (3.5). Thus (3.5) is equivalent to (3.7).

Consequently we have:

THEOREM 3.1. *The condition* (3.2) *is equivalent to one of the following assertions:*

(1) $2C_{pqt}(x,\ell y) f^t{}_n f^n{}_m \ell^m{}_d y^d \ell^p{}_i \ell^q{}_j + g_{pq}(x,\ell y) f^p{}_c f^c{}_r \ell^r{}_i \ell^q{}_j$
$+ g_{pq}(x,\ell y) f^q{}_n f^n{}_s \ell^s{}_j \ell^p{}_i = 0.$

(2) $g_{pq}(x,\ell y) f^p{}_k f^k{}_r \ell^r{}_i \ell^q{}_j y^i y^j = 0.$

(3) $\{g_{pq}(x,\ell y) f^p{}_k f^k{}_l \ell^l{}_h \ell^q{}_j + g_{pq}(x,\ell y) f^p{}_k f^k{}_l \ell^l{}_j \ell^q{}_h\} y^j = 0.$

Now, in an (f^2, L)-manifold, we assume that

$$g_{ij}(x,\ell y) = g_{pq}(x,\ell y) f^p{}_r f^r{}_i f^q{}_l f^l{}_j, \tag{3.8}$$

where $f^i{}_j(x)$ is a structure satisfying (2.1).

Differentiating (3.8) with respect to y^k, we get

$$C_{ijr}(x,\ell y)\ell^r{}_k = C_{pqr}(x,\ell y) f^p{}_l f^l{}_i f^q{}_h f^h{}_j \ell^r{}_k. \tag{3.9}$$

Transvecting (3.9) by $\ell^j{}_s$ and using (2.3), we have

$$C_{ijr}(x,\ell y)\ell^r{}_k \ell^j{}_s = C_{pqr}(x,\ell y) f^p{}_l f^l{}_i f^q{}_h f^h{}_s \ell^r{}_k. \tag{3.10}$$

Since $C_{ijr}(x,\ell y)$ is symmetric in all indices, from (3.10) we get

$$C_{pqr}(x,\ell y) f^p{}_l f^l{}_i f^q{}_h f^h{}_s l^r{}_k = C_{pqr}(x,\ell y) f^p{}_l f^l{}_i f^q{}_h f^h{}_k \ell^r{}_s. \tag{3.11}$$

From (3.9) and (3.11), we obtain

$$C_{ijr}(x,\ell y)\ell^r{}_k = C_{pqr}(x,\ell y)f^p{}_l f^l{}_i f^q{}_h f^h{}_k \ell^r{}_j. \tag{3.12}$$

If we make use of (2.3), (3.12) leads to

$$\begin{aligned} &C_{ijk}(x,\ell y) - C_{ijr}(x,\ell y)m^r{}_k \\ &=C_{pqj}(x,\ell y)f^p{}_l f^l{}_i f^q{}_h f^h{}_k - C_{pqr}(x.\ell y)f^p{}_l f^l{}_i f^q{}_h f^h{}_k m^r{}_j. \end{aligned} \tag{3.13}$$

Transvecting (3.13) by $f^i{}_m f^m{}_t f^j{}_n f^n{}_s \ell^k{}_d$ and using (2.3), we get

$$C_{ijk}(x,\ell y)f^i{}_m f^m{}_t f^j{}_n f^n{}_s \ell^k{}_d = -C_{pqj}(x,\ell y)\ell^p{}_t f^q{}_r f^r{}_d f^j{}_n f^n{}_s.$$

Hence $C_{ijk}(x,\ell y)f^i{}_m f^m{}_t f^j{}_n f^n{}_s \ell^k{}_d = 0$ from (3.11). Therefore from (3.9) $C_{ijr}(x,\ell y)\ell^r{}_k = \dot\partial_k g_{ij}(x,\ell y) = 0$, that is, $g_{ij}(x,\ell y)$ is a Riemannian metric.

Thus we have:

THEOREM 3.2. *If an (f^2, L)-manifold satisfies (3.8), then $g_{ij}(x,\ell y)$ is a Riemannian metric, that is, the distribution $\mathcal{L}$ is a Riemannian space.*

4. Vanishing h-covariant derivatives of the structure $f^i{}_j$

In the even dimensional Riemannian manifold M^n, the *Nijenhuis tensor* of an almost complex structure $J^i{}_j(x)$ is represented by

$$N^i{}_{jk} = (\partial_r J^i{}_j)J^r{}_k - (\partial_r J^i{}_k)J^r{}_j + J^i{}_r(\partial_j J^r{}_k - \partial_k J^r{}_j), \tag{4.1}$$

where $\partial_k = \partial/\partial x^k$.

Now, in an (f^2, L)-manifold, let $F\Gamma = (\Gamma^i{}_{jk}, G^i{}_j, C^i{}_{jk})$ and ∇_k be the Cartan connection ([6]) and the h-covariant derivative with respect to $F\Gamma$ respectively. Therefore the h-covariant derivative of the structure tensor $f^i{}_j$ satisfying (2.1) with respect to $F\Gamma$ gives

$$\nabla_k f^i{}_j = \partial_k f^i{}_j + \Gamma^i{}_{rk} f^r{}_j - f^i{}_r \Gamma^r{}_{jk}. \tag{4.2}$$

From (2.3), $f^i{}_m f^m{}_j$ acts on $\mathcal{L}$ as an almost complex structure operator, which implies that $(f^i{}_k f^k{}_h f^h{}_r f^r{}_j)u^j = -\delta^i{}_j u^j$, where $u^i = \ell^i{}_j y^j$ for any $y^i \in T_p(M)$. Hence the Nijenhuis tensor $N^i{}_{jk}(x, \ell y)$ on $\mathcal{L}$ is easily given by

$$
\begin{aligned}
N^i{}_{jk}(x,\ell y) = {} & \{(\nabla_k f^i{}_m) f^m{}_j - f^i{}_m \nabla_k f^m{}_j\} f^r{}_a f^a{}_k \\
& - \{(\nabla_r f^i{}_m) f^m{}_k - f^i{}_m \nabla_r f^m{}_k\} f^r{}_a f^a{}_j \\
& + f^i{}_a f^a{}_r \{(\nabla_j f^r{}_m) f^m{}_k + f^i{}_m \nabla_j f^m{}_k\} \\
& - (\nabla_k f^r{}_m) f^m{}_j + f^r{}_m \nabla_k f^m{}_j\}
\end{aligned}
\tag{4.3}
$$

by virtue of (4.1), (4.2) and $\Gamma^i{}_{jk} = \Gamma^i{}_{kj}$. Hence we obtain:

THEOREM 4.1. *In an (f^2, L)-manifold, the distribution $\mathcal{L}$ is complex manifold if the h-covariant derivative of a structure $f^i{}_j$ satisfying* (2.1) *with respect to Cartan connection vanishes.*

Let us represent $\bar{\nabla}_k$ the h-covariant derivative with respect to the Berwald connection ([6]) $B\Gamma = (G^i{}_{jk}, G^i{}_j, 0)$. If $G^i{}_{jk}$ are functions of position alone, that is, $\dot{\partial}_h G^i{}_{jk} = 0$ holds, then the Finsler space F^n is said to be a *Berwald space.*

Let us suppose the h-covariant derivative of a structure $f^i{}_j$ satisfying (2.1) with respect to the Cartan connection $F\Gamma$ vanishes, that is,

$$\nabla_k f^i{}_j = \partial_k f^i{}_j + \Gamma^i{}_{rk} f^r{}_j - f^i{}_r \Gamma^r{}_{jk} = 0.$$

From $\Gamma^i{}_{km} y^m = G^i{}_k$, we have

$$y^m \partial_m f^i{}_j + G^i{}_m f^m{}_j - f^i{}_m G^m{}_j = 0. \tag{4.4}$$

Differentiating (4.4) partially with respect to y^k, we have

$$\bar{\nabla}_k f^i{}_j = \partial_k f^i{}_j + G^i{}_{rk} f^r{}_j - f^i{}_r G^r{}_{jk} = 0. \tag{4.5}$$

Next, let $H_h{}^i{}_{jk}$ be the h-curvature of $B\Gamma$. That is

$$H_h{}^i{}_{jk} = \delta_k G^i{}_{hj} + \delta_j G^i{}_{hk} + G^i{}_{rk} G^r{}_{hj} - G^i{}_{rj} G^r{}_{hk}, \tag{4.6}$$

where $\delta_k = \partial_k - G^r{}_k \dot{\partial}_r$.

Applying the Ricci identity of $B\Gamma$ to $f^i{}_h f^h{}_j$ ([6]), we have

$$f^r{}_l f^l{}_i H_r{}^h{}_{jk} - f^h{}_m f^m{}_r H_i{}^r{}_{jk} = 0, \tag{4.7}$$

by virtue of (4.5) and $T^r{}_{jk} = G^i{}_{jk} - G^i{}_{kj} = 0$.

On the other hand, if an n-dimensional Finsler space F^n $(n \geq 3)$ satisfies $H_i{}^h{}_{jk} = K(g_{ij}\delta^h_k - g_{ik}\delta^h_j)$, then F^n is called *a Finsler space of constant curvature* ([6]). In this case, (4.7) can be written as

$$K\{f^r{}_l f^l{}_i(g_{rj}\delta^h_k - g_{rk}\delta^h_j) - f^h{}_m f^m{}_r(g_{ij}\delta^r_k - g_{ik}\delta^r_j)\} = 0. \tag{4.8}$$

Contracting (4.8) with respect to h and j and using (2.4), we have

$$K\{(1-n)g_{rk}f^r{}_l f^l{}_i - g_{ri}f^r l f^l{}_k\} = 0.$$

We assume that $g_{ri}f^r{}_l f^l{}_j$ is symmetric in indices i and j. Then we have $Kg_{rk}f^r{}_l f^l{}_i = 0$. If $g_{rk}f^r{}_l f^l{}_i = 0$, then $f^r{}_l f^l{}_i = 0$. This is a contradiction. Therefore $K = 0$. Thus we have:

THEOREM 4.2. *Let $F^n (n \geq 3)$ be an (f^2, L)-manifold with constant curvature. If the h-covariant derivative of a structure $f^i{}_j$ satisfying (2.1) with respect to Cartan connection vanishes and $g_{ri}f^r{}_l f^l{}_j$ is symmetric in indices i and j, then h-curvature tensor of Berwald connection vanishes.*

We put $G_i{}^h{}_{jk} = \dot{\partial}_i G^h{}_{jk}$ and $G_{ij} = G_i{}^r{}_{jr}$. It is noted that $G_i{}^h{}_{jk}$ and G_{ij} are symmetric in indices i, j, k and i, j respectively.

By Euler's theorem on homogeneous function in y, we have

$$(1) \quad G^h{}_{i0} = G^h{}_{0i} = G^h{}_i, \qquad (2) \quad G_{i0} = G_{0i} = 0, \tag{4.9}$$

where the index 0 denotes the contraction with the element of support y. From (2) of (4.9), we have

$$(\dot{\partial}_r G_{jm})y^m = -G_{jr}. \tag{4.10}$$

On the other hand, the Douglas tensor $D_i{}^h{}_{jk}$ ([5]) is given by

$$D_i{}^h{}_{jk} = G_i{}^h{}_{jk} - \frac{1}{n+1}(y^h \dot{\partial}_k G_{ij} + \delta^h_i G_{jk} + \delta^h_j G_{ki} + \delta^h_k G_{ij}).$$

In a Finsler space F^n, if $D_i{}^h{}_{jk} = 0$ and $\nabla_k f^i{}_j = 0$, from (4.5) we have

$$(\partial_k f^i{}_l) f^l{}_j + f^i{}_l \partial_k f^l{}_j + G^i{}_{mk} f^m{}_l f^l{}_j - f^i{}_l f^l{}_m G^m{}_{jk} = 0, \tag{4.11}$$

$$G_i{}^h{}_{jk} = \frac{1}{n+1}(y^h \dot{\partial}_k G_{ij} + \delta_i^h G_{jk} + \delta_j^h G_{ki} + \delta_k^h G_{ij}). \tag{4.12}$$

Differentiating (4.11) partially with respect to y^h, we find

$$G_m{}^i{}_{kh} f^m{}_l f^l{}_j = f^i{}_l f^l{}_m G_j{}^m{}_{kh}.$$

Thus we have

$$\begin{aligned} &f^i{}_l f^l{}_m (y^m \dot{\partial}_h G_{jk} + \delta_j^m G_{kh} + \delta_k^m G_{hj} + \delta_h^m G_{jk}) \\ &\quad = (y^i \dot{\partial}_h G_{mk} + \delta_m^i G_{kh} + \delta_k^i G_{hm} + \delta_h^i G_{mk}) f^m{}_l f^l{}_j. \end{aligned}$$

Contracting this with respect to i and h, we have

$$f^r{}_l f^l{}_m y^m \dot{\partial}_r G_{jk} + f^r{}_l f^l{}_k G_{rj} = n G_{mk} f^m{}_l f^l{}_j. \tag{4.13}$$

Transvecting (4.13) by y^j, we find, from (2) of (4.9) and (4.10)

$$-f^r{}_l f^l{}_m y^m G_{rk} = n G_{mk} f^m{}_l f^l{}_j y^j,$$

that is, $f^r{}_l f^l{}_m y^m G_{rk} = 0$. Differentiating this with respect to y^j, we find

$$f^r{}_l f^l{}_j G_{rk} + f^r{}_l f^l{}_m y^m \dot{\partial}_r G_{jk} = 0$$

by virtue of $\dot{\partial}_k G_{rj} = \dot{\partial}_r G_{jk}$. Therefore (4.13) leads to $G_{mk} f^m{}_l f^l{}_j = 0$. Since $f^m{}_l f^l{}_j$ is an almost complex structure on $\mathcal{L}$, $G_{ij} = 0$ on $\mathcal{L}$. Thus we obtain $G_i{}^h{}_{jk} = 0$ on $\mathcal{L}$ from (4.12). Consequently we have

THEOREM 4.3. *If a Finsler manifold F^n with the vanishing h-covariant derivative of a structure $f^i{}_j$ satisfying (2.1) with respect to Cartan connection has a vanishing Douglas tensor, then the distribution $\mathcal{L}$ is a Berwald space.*

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Hong-Suh Park
Department of Mathematics
Yeungnam University
Gyongsan 712-749, Korea
E-mail: hspark@ynuco.yeungnam.ac.kr

Ha-Yong Park and Byung-Doo Kim
Department of Mathematics
Kyungil University
Gyongsan 712-701, Korea
E-mail: hypark@bear.kyungil.ac.kr
bdkim@bear.kyungil.ac.kr