Comm. Korean Math. Soc. **15** (2000), No. 2, pp. 331–338

ON SPECIAL FINSLER SPACES WITH COMMON GEODESICS

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ABSTRACT. In the present paper, we investigate a problem in a symmetric Finsler space, which is a special space. First we prove that if a symmetric space remains to be a symmetric one under the $\mathcal{Z}$-projective change, then the space is of zero curvature. Further we will study $\mathcal{W}$-recurrent space and $\mathcal{D}$-recurrent space under the projective change.

0. Introduction

If any geodesic on F^n is also a geodesic on $\bar{F}^n$ and the inverse is true, the change $\sigma : L \to \bar{L}$ of the metric is called *projective*. It is known that the Douglas tensor and the Weyl tensor are invariant under any projective change. Moreover, h-curvature tensor in the Berwald connection $B\Gamma$ is also invariant under a special projective change ($\mathcal{Z}$-projective change). In the paper [4], M. Fukui and T. Yamada dealt with it and had some results. A Finsler space of zero curvature remains a space of zero curvature by the $\mathcal{Z}$-projective change which is characterized as $Q_i = 0$.

The purpose of the present paper is to consider the condition that a symmetric space remains to be a symmetric space. Especially, in section 4, we treat a $\mathcal{W}$-recurrent space and a $\mathcal{D}$-recurrent space.

1. Berwald connection

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space, where M^n is a connected differential manifold of dimension n and $L(x, y)$ is the fun-

Received December 7, 1999. Revised February 17, 2000.
2000 Mathematics Subject Classification: 53B40.
Key words and phrases: projective change, symmetric space, Douglas tensor.
* This paper was supported by the Kyungil University Research Grants in 1999.

damental function defined on the manifold $T(M)/0$ of non-zero tangent vectors. We assume that L is positive and the fundamental metric tensor $g_{ij} = (1/2)\dot{\partial}_j\dot{\partial}_i L^2$ is positive definite, where $\dot{\partial}_i = \partial/\partial y^i$.

A geodesic on F^n is given by the differential equation

$$d^2x^i/ds^2 + 2G^i(x, dx/ds) = 0,$$

where s is the arc-length of the curve. In the present paper, we are mainly concerned with the Berwald connection $B\Gamma = (G_j{}^i{}_k, G^i{}_j, 0)$, which is defined as: $G^i{}_j = \dot{\partial}_j G^i$, $G_j{}^i{}_k = \dot{\partial}_k G^i{}_j$. For a Finsler tensor field X^i, the h-covariant derivative with respect to $B\Gamma$ is given by

$$X^h{}_{;i} = \partial_i X^h - G^r{}_i(\dot{\partial}_r X^h) + X^r G_r{}^h{}_i, \tag{1.1}$$

where $\partial_i = \partial/\partial x^i$.

For $B\Gamma$ we consider the torsion and the curvature. According to the theory of Finsler connection ([1],[5]), the $h(v)$-torsion R^1 is the same with that of Cartan connection $C\Gamma$, because $B\Gamma$ and $C\Gamma$ have the common spray connection $(G^i{}_j)$. And the h-curvature tensor R^2 and the hv-curvature tensor P^2 are usually written as $H = (H_h{}^i{}_{jk})$ and $G = (G_h{}^i{}_{jk})$ respectively. These tensors are written as

$$\begin{aligned} H_j{}^i{}_k &= \mathcal{U}_{(jk)}\{\partial_k G^i{}_j - G_j{}^i{}_r G^r{}_k\}, \\ H_h{}^i{}_{jk} &= \mathcal{U}_{(jk)}\{\partial_k G_h{}^i{}_j - G^r{}_k(\dot{\partial}_r G_h{}^i{}_j) + G_h{}^r{}_j G_r{}^i{}_k\}, \\ G_h{}^i{}_{jk} &= \dot{\partial}_h G_j{}^i{}_k, \end{aligned} \tag{1.2}$$

where $\mathcal{U}_{(jk)}$ means the interchange of indices j, k and subtraction.

Throughout the following the index 0 denotes the transvection by y^i, for example, $y^i F^h{}_i = F^h{}_0$. For later use, we introduce the following relations ([8]):

$$\begin{aligned} &(a)\ H_0{}^i{}_{jk} = H_j{}^i{}_k,\ (b)\ H_0{}^i{}_k = H^i{}_k,\ (c)\ H_j{}^i{}_k = -H_k{}^i{}_j, \\ &(d)\ H_j{}^i{}_k = (1/3)\ \mathcal{U}_{(jk)}\{\dot{\partial}_j H^i{}_k\},\ (e)\ H_h{}^i{}_{jk} = \dot{\partial}_h H_j{}^i{}_k. \end{aligned} \tag{1.3}$$

2. Projective changes of metrics

We consider two Finsler spaces $F^n = (M^n, L)$ and $\bar{F}^n = (M^n, \bar{L})$ on a common underlying manifold M^n. Let the change $\sigma : L \to \bar{L}$ be projective. It is well known that σ is projective if and only if there exists a positively homogeneous degree 1 Finsler scalar field $p(x, y)$ on M^n, satisfying

$$\bar{G}^i(x,y) = G^i(x,y) + p(x,y)y^i, \ \ p \neq 0. \tag{2.1}$$

This $p(x, y)$ is called the *projective factor* of the projective change under consideration.

We shall show how the torsion and curvature tensors are changed by a projective change. Let $B\bar{\Gamma} = (\bar{G}^i_{jk}, \bar{G}^i{}_j, 0)$ be the Berwald connection on the space $\bar{F}^n = (M^n, \bar{L})$, obtained from $F^n = (M^n, L)$ by a projective change σ. Then, (2.1) immediately gives

$$\begin{aligned} \bar{G}^i{}_j &= G^i{}_j + y^i p_j + \delta^i_j p, \\ \bar{G}_j{}^i{}_k &= G_j{}^i{}_k + y^i p_{jk} + \delta^i_j p_k + \delta^i_k p_j, \end{aligned} \tag{2.2}$$

where we put $p_i = \dot{\partial}_i p$ and $p_{ij} = \dot{\partial}_j p_i$.

On the other hand, the hv-curvature tensor $\bar{G}_i{}^h{}_{jk}$ and the h-curvature tensor $\bar{H}_k{}^h{}_{ij}$ of $B\bar{\Gamma}$ are given by

$$\bar{G}_i{}^h{}_{jk} = G_i{}^h{}_{jk} + y^h p_{ijk} + \mathcal{A}_{(ijk)}\{\delta^h_i p_{jk}\}, \tag{2.3}$$

$$\bar{H}_k{}^h{}_{ij} = H_k{}^h{}_{ij} + y^h Q_{ij\cdot k} + \delta^h_k Q_{ij} + \mathcal{U}_{(ij)}\{\delta^h_i Q_{j\cdot k}\}, \tag{2.4}$$

where we put $\cdot k = \dot{\partial}_k$, $Q_i = p_{;i} - pp_i$, $Q_{ij} = \mathcal{U}_{(ij)} p_{i;j}$ and $\mathcal{A}_{(ijk)}$ means cyclic permutation of the indices i, j, k and summation.

We have two essential projective invariants, one is the Weyl curvature tensor W and the other is the Douglas tensor D. If $Q_i = 0$, from (2.4) the h-curvature tensor H is also invariant under the projective change. In the paper [7], S. C. Rastogi discussed the properties of the projective factor $p(x, y)$ satisfying the condition $Q_i = 0$. A projective change of a Finsler space of zero curvature is also a Finsler space of zero curvature if and only if the projective factor p satisfies the equation $Q_i = 0$.

DEFINITION 1. ([4]) A projective change σ is called a $\mathcal{Z}$-projective change if $Q_i = 0$.

S. C. Rastogi ([7]) proved the following.

THEOREM A. *If $Q_i = 0$, then the scalar $p(x, y)$ and its derivative satisfy the equations:*

$$(a)\ p_r H_j{}^r{}_i = 0,\ (b)\ \mathcal{A}_{(ijk)}\{p_{rk} H_j{}^r{}_i\} = 0. \tag{2.5}$$

3. Symmetric spaces

DEFINITION 2. A Finsler space is called a *symmetric space* if its h-curvature tensor R^2 satisfies the relation $H_h{}^i{}_{jk;m} = 0$.

Meher's paper ([6]) concerned with the existence of projective motion in a symmetric Finsler space and obtained a relation of the Berwald's scalar curvature. Moreover he discussed a scalar function, which gives rise to the projective motion.

We can see that every Finsler space of zero curvature is a symmetric space. But the converse is not true.

Let $B\bar{\Gamma}$ be the Berwald connection on the space $\bar{F}^n$ obtained from F^n. Then, from (1.1) the covariant derivative of the h-curvature tensor in $\bar{F}^n$ is given by

$$\begin{aligned}\bar{H}_h{}^i{}_{jk\bar{;}m} =& \partial_m \bar{H}_h{}^i{}_{jk} - \dot{\partial}_a \bar{H}_h{}^i{}_{jk} \bar{G}^a{}_m + \bar{H}_h{}^a{}_{jk} \bar{G}_a{}^i{}_m \\ &- \bar{H}_a{}^i{}_{jk} \bar{G}_h{}^a{}_m - \bar{H}_h{}^i{}_{ak} \bar{G}_j{}^a{}_m - \bar{H}_h{}^i{}_{ja} \bar{G}_k{}^a{}_m,\end{aligned} \tag{3.1}$$

where $(\bar{;})$ denotes the h-covariant derivative with respect to $B\bar{\Gamma}$. The h-curvature tensor is invariant under the $\mathcal{Z}$-projective change. Paying attention to (2.2), we get

$$\begin{aligned}\bar{H}_h{}^i{}_{jk\bar{;}m} =& H_h{}^i{}_{jk;m} + \mathcal{A}_{(jkm)}\{H_h{}^i{}_{jk} p_m\} - p\dot{\partial}_m H_h{}^i{}_{jk} \\ &+ H_h{}^a{}_{jk} y^i p_{am} + H_h{}^a{}_{jk} \delta^i_m p_a - H_j{}^i{}_k p_{hm} \\ &- H_m{}^i{}_{jk} p_h - 3H_h{}^i{}_{jk} p_m + \mathcal{U}_{(jk)}\{H_h{}^i{}_{k0} p_{jm}\}.\end{aligned} \tag{3.2}$$

Since $p(x,y)$ and $R^2(x,y)$ are homogeneous function of degree 1 and 0 in y respectively, we find

$$(3.3) \qquad p_0 = p,\ p_{m0} = 0,\ \dot{\partial}_0 H_h{}^i{}_{jk} = 0.$$

We assume that a symmetric space F^n is transformed into another symmetric one $\bar{F}^n$ by the $\mathcal{Z}$-projective change. And transvecting (3.2) with y^m and y^h, from (1.3), (2.5) and (3.3) we have

$$(3.4) \qquad 3pH_j{}^i{}_k + \mathcal{U}_{(kj)}\{H^i{}_k p_j\} = 0.$$

Further, transvecting this with y^j, we obtain $pH^i{}_k = 0$, which implies $H_h{}^i{}_{jk} = 0$ by virtue of (1.3).

Summarizing up the above, we have:

THEOREM 3.1. *If a symmetric space remains to be a symmetric one by the $\mathcal{Z}$-projective change and the projective change is not trivial (i.e. $p \neq 0$), then the space is of zero curvature.*

4. Recurrent spaces

The Weyl projective deviation tensor W ([8]) is given by

$$(4.1) \qquad W^i{}_k = H^i{}_k - H\delta^i_k - y^i(\dot{\partial}_r H^r{}_k - \dot{\partial}_k H)/(n+1),$$

which is invariant under the projective change.

In the previous paper ([2]), S. Bacso defined an A-recurrent Finsler space, that is, for a tensor $A^i{}_k = H^i{}_k - Hh^i{}_k$,

$$(4.2) \qquad A^i{}_{k;0} = \psi(x,y)A^i{}_k,$$

where $\psi(x,y)$ is positively homogeneous function of degree 1 in y and $h^i{}_k$ is an angular metric tensor. Similarly we introduce $\mathcal{W}$-recurrent space as following.

DEFINITION 4.1. A Finsler space F^n is called $\mathcal{W}$-recurrent one if the deviation tensor W satisfies the following condition

$$(4.3) \qquad W^i{}_{k;0} = \psi(x,y)W^i{}_k,$$

where $\psi(x,y)$ is a positively homogeneous function of degree 1 in y.

Let's consider the projective change $\sigma : L \to \bar{L}$, where F^n is an arbitrary Finsler space but $\bar{F}^n$ is a $\mathcal{W}$-recurrent Finsler space, that is,

$$\bar{W}^i{}_{k\bar{;}0} = \bar{\psi}(x,y)\bar{W}^i{}_k, \tag{4.4}$$

where ($\bar{;}$) denotes the h-covariant derivative in $B\bar{\Gamma}$.

In $B\bar{\Gamma}$ of $\bar{F}^n$, from (1.1) and (2.2) we have

$$\begin{aligned} W^i{}_{k\bar{;}m} =& W^i{}_{k;m} - p\dot{\partial}_m W^i{}_k - 2W^i{}_k p_m + W^i{}_m p_k \\ &+ W^r{}_k p_{rm} y^i + W^r{}_k p_r \delta^i_m. \end{aligned} \tag{4.5}$$

Here is used the fact that the deviation tensor is invariant under the projective change.

Let's assume that the projective factor satisfies a condition $W^r{}_k p_r = 0$, which we denote by $\mathcal{W}$-condition. Since $W^i{}_k$ and p are positively homogeneous functions of degree 2 and 1 in y respectively, we get $\dot{\partial}_0 W^i{}_k = 2W^i{}_k$, $W^i{}_0 = 0$. Transvecting (4.5) with y^m and using (4.4), we obtain

$$W^i{}_{k;0} = (\bar{\psi} + 4p)W^i{}_k. \tag{4.6}$$

Putting $\psi = \bar{\psi} + 4p$, we find that F^n is also $\mathcal{W}$-recurrent.

Conversely, if $\bar{F}^n$ and F^n are $\mathcal{W}$-recurrent with the function $\psi = \bar{\psi} + 4p$, then from (4.5) we can find that projective factor satisfies the $\mathcal{W}$-condition. Thus we have:

THEOREM 4.1. *If a Finsler space F^n can be transformed into a $\mathcal{W}$-recurrent space $\bar{F}^n$ with the function $\bar{\psi}$ by the projective change, then F^n is also $\mathcal{W}$-recurrent one with the function $\psi = \bar{\psi} + 4p$ if and only if the projective factor p satisfies $\mathcal{W}$-condition.*

Next, we introduce the Douglas tensor ([1]):

$$D_h{}^i{}_{jk} = G_h{}^i{}_{jk} - (G_{hj\cdot k}y^i - \mathcal{A}_{(hjk)}\{G_{jk}\delta^i_h\})/(n+1), \tag{4.7}$$

where $G_{jk} = G_j{}^r{}_{kr}$. This tensor is invariant under the projective change.

On the other hand, a Finsler space is called a Berwald space, if the connection coefficients $G_j{}^i{}_k$ of $B\Gamma$ are function of position x alone, in any coordinate system. Therefore, from (4.7) if the hv-curvature tensor G vanishes, then the Douglas tensor vanishes identically. An n-dimensional Finsler space F^n is called a Douglas space ([3]) if the Douglas tensor vanishes identically. This fact has substantial importance in biology ([1]). Therefore we can state the following.

REMARK. If a Finsler space is projective to a Berwald space, then the space is the Douglas space.

Next, similar to the $\mathcal{W}$-recurrent case, we can define the $\mathcal{D}$-recurrent space.

DEFINITION 4.2. A Finsler space F^n is called $\mathcal{D}$-recurrent one if the Douglas tensor satisfies the following condition

$$D_h{}^i{}_{jk;0} = \varphi(x,y) D_h{}^i{}_{jk}, \tag{4.8}$$

where $\varphi(x,y)$ is a positively homogeneous function of degree 1 in y.

Since the Douglas tensor is positively homogeneous function of degree -1 in y, we find $\dot{\partial}_0 D_h{}^i{}_{jk} = -1$. And it satisfies the identities ([8]):

$$(a)\ D_0{}^i{}_{jk} = D_h{}^i{}_{0k} = D_h{}^i{}_{j0} = 0,\ (b)\ D_r{}^r{}_{jk} = 0. \tag{4.9}$$

We are concerned with the projective change $\sigma : L \to \bar{L}$, where F^n is arbitrary but $\bar{F}^n$ is $\mathcal{D}$-recurrent. From (1.1) and (2.2) we get

$$\begin{aligned} D_h{}^i{}_{jk\bar{;}m} =& D_h{}^i{}_{jk;m} - p\,\dot{\partial}_m D_h{}^i{}_{jk} + D_h{}^r{}_{jk} p_{rm} y^i \\ &+ D_h{}^r{}_{jk} p_r \delta^i_m - \mathcal{A}_{(hjm)} D_h{}^i{}_{jk} p_m - D_h{}^i{}_{jm} p_k. \end{aligned} \tag{4.10}$$

Suppose that the projective factor satisfies a condition $D_h{}^r{}_{jk} p_r = 0$, which we denote by $\mathcal{D}$-condition. Transvecting (4.10) with y^m and taking account of $\dot{\partial}_0 D_h{}^i{}_{jk} = -1$, we obtain

$$D_h{}^i{}_{jk;0} = \bar{\varphi} D_h{}^i{}_{jk}. \tag{4.11}$$

Putting $\varphi = \bar{\varphi}$, we find that F^n is also $\mathcal{D}$-recurrent.

Conversely, if $\bar{F}^n$ and F^n are $\mathcal{D}$-recurrent spaces with the function $\varphi = \bar{\varphi}$, then from (4.10) we get $D_h{}^r{}_{jk} p_r = 0$. Thus we have:

THEOREM 4.2. *If a Finsler space F^n can be transformed into a $\mathcal{D}$-recurrent space $\bar{F}^n$ with the function $\bar{\varphi}$ by the projective change, then F^n must be $\mathcal{D}$-recurrent one with the function $\varphi = \bar{\varphi}$ if and only if the projective factor p satisfies $\mathcal{D}$-condition.*

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