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EXISTENCE AND CONVERGENCE FOR FIXED POINTS OF NON-LIPSCHITZIAN MAPPINGS IN BANACH SPACES WITHOUT UNIFORM CONVEXITY

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ABSTRACT. The existence and convergence theorems for fixed points of mappings of asymptotically nonexpansive type are established in Banach spaces with a weakly continuous duality mapping but without uniform convexity.

1. Introduction and preliminaries

Let D be a nonempty subset of a Banach space X and T be a nonlinear mapping from D into itself. Then T is said to be asymptotically nonexpansive ([5]) if there exists a sequence $\{k_n\}$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \le k_n \|x - y\|$$

for all $x, y \in D$ and $n = 0, 1, 2, \cdots$. There appear in the literature two definitions of non-Lipschitzian asymptotically nonexpansive mappings. One is due to Kirk ([9]): T is called a mapping of asymptotically nonexpansive type if

$$\limsup_{n\to\infty} \sup_{y\in D} (\|T^n x - T^n y\| - \|x - y\|) \le 0$$

for all $x \in D$. Another non-Lipschitzian mapping between these two was introduced by Bruck, Kuczumovw and Reich ([3]): T is said to be

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asymptotically nonexpansive in the intermediate sense if T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in D} (\|T^n x - T^n y\| - \|x-y\|) \leq 0.$$

In 1974, Kirk ([9]) introduced the class of mappings of asymptotically nonexpansive type and proved that, if D is closed bounded convex of a Banach space with it characteristic of convexity is less than one and T^N is continuous, then every mapping T of asymptotically nonexpansive type has a fixed point. Xu ([14]) extended the Kirk's result to nearly uniformly convex (NUC) Banach space. Recently, Jung and Sahu ([8]) obtained a fixed point theorem without convexity in a Banach space which improves the corresponding results of Kirk ([9]) and Xu ([14]).

On the other hand, the asymptotic behavior of the iterates $\{T^n x\}$ for an asymptotically nonexpansive mappings T has been studied by many authers (see [1], [6], [11], [13], [14]) in uniformly convex Banach spaces.

The purpose of the present paper is to prove the existence and convergence for fixed points of mappings of asymptotically nonexpansive type in Banach spaces without uniform convexity. Our results generalize and improve the results of Bose ([1]), Górnick ([6]), Lim and Xu ([10]), and Cho, Shurma and Thakur ([4]).

A Banach space X is said to satisfy Opial condition ([12]) if, for each sequence $\{x_n\}$ in X, the condition $x_n \to x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all y in X with $y \neq x$. Spaces possessing that property include the Hilbert spaces and the l^p spaces for $1 \leq p < \infty$. However, L^p, $p \neq 2$, do not satisfy that property. By a gauge, we mean a continuous strictly increasing function ϕ defined on $\mathbb{R}^+ := [0,\infty)$ such that $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. To a gauge ϕ, we associate a (generally multivalued) duality mapping $J_\phi : X \to X^*$, the dual space of X, defined by

$$J_\phi(x) = \{x^* \in X^* : \langle x, x^*\rangle = \|x\|\phi(\|x\|) \;\text{ and }\; \|x^*\| = \phi(\|x\|)\},$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing between x and x^*. If $\phi(t) = t$, then we write J instead of J_ϕ. Define

$$\Phi(t) = \int_0^t \phi(t)dt.$$

Then it is known that $J_\phi(x)$ is the subdifferential of the convex function $\phi(\|\cdot\|)$ at $x \in X$. X is said to have a weakly continuous duality mapping if there exists a gauge ϕ such that the duality mapping J_ϕ is single valued and (sequentially) continuous from X with the weak topology to X^* with the weak* topology. It is known that each Banach space with a weakly continuous duality mapping satisfies Opial's condition (see [2]). Every l^p $(1 < p < \infty)$ has a weakly continuous duality mapping with the gauge $\phi(t) = t^{p-1}$.

Recall that a nonempty subset D of a Banach space X is said to satisfy the property (P) ([10]) if the following holds:

$$\text{(P)} \qquad x \in D \;\text{ implies }\; \omega_w(x) \subset D,$$

where $w_w(x)$ is the weak ω-limit set of T of x, i.e., the set

$$\{y \in X : y = weak - \lim_i T^{n_i}x \;\text{ for some }\; n_i \to \infty\}.$$

2. Existence Theorems

Before presenting main result, we need the following:

LEMMA 1. *Let X be a Banach space with a weakly continuous duality mapping J_ϕ, D be a weakly compact convex subset of X and $T : D \to D$ be a mapping of asymptotically nonexpansive type. Suppose that there is a closed bounded convex subset C of D with the property (P) such that, for each $x \in C$,*

$$\limsup_{n\to\infty} \|T^n x - y\| = \text{a constant}$$

for all $y \in C$. If the subsequence $\{T^{n_i}x\}$ of $\{T^n x\}$ converges weakly to a point z in X, then $\{T^{n_i}x\}$ converges strongly to z in C.

PROOF. Let r be a constant such that

$$\limsup_{n\to\infty} \|T^n x - y\| = r$$

for all $x,\ y \in C$. By hypothesis, for $x \in C$, the subsequence $\{T^{n_i}x\}$ of $\{T^n x\}$ converges weakly to some z in X. By the property (P), we know that z actually lies in C. Now, we set

$$\lim_{i\to\infty} \|T^{n_i}x - z\| = s.$$

Since $J_\phi(x)$ is the Gâteaux derivative of the convex functional $\Phi(\|x\|)$, it follows ([13]) that

$$\Phi(\|x+y\|) = \Phi(\|x\|) + \int_0^1 \langle y, J_\phi(x+ty)\rangle dt. \tag{1}$$

For any integers $n_i,\ m \geq 1$, we have

$$\begin{aligned}&\Phi(\|T^{n_i}x - T^m x\|)\\ &= \Phi(\|T^{n_i}x - z\|) + \int_0^1 \langle z - T^m x, J_\phi(T^{n_i}x - z + t(z - T^m x))\rangle dt.\end{aligned}$$

Taking the limit as $i \to \infty$ each sides, we have

$$\lim_{i\to\infty} \Phi(\|T^{n_i}x - T^m x\|) = \Phi(s) + \int_0^1 \langle z - T^m x, J_\phi(t(z - T^m x))\rangle dt.$$

Taking the limit superior as $m \to \infty$ each sides, we have

$$\begin{aligned}\Phi(r) + \Phi(s) &= \limsup_{m\to\infty}(\lim_{i\to\infty} \Phi(\|T^{n_i}x - T^m x\|))\\ &\leq \limsup_{m\to\infty}(\lim_{i\to\infty} \Phi([\|T^{n_i}x - T^m(T^{n_i-m}x)\|\\ &\quad - \|x - T^{n_i-m}x\|] + \|x - T^{n_i-m}x\|))\\ &\leq \limsup_{m\to\infty}(\lim_{i\to\infty} \Phi(\sup_{y\in D}[\|T^m x - T^m y\| - \|x-y\|]\\ &\quad + \|x - T^{n_i-m}x\|))\\ &\leq \limsup_{n\to\infty} \Phi(\|x - T^n x\|) = \Phi(r),\end{aligned}$$

which implies that $\{T^{n_i}x\}$ converges strongly to z. □

THEOREM 1. *Let X be a Banach space with a weakly continuous duality mapping J_ϕ, D be a weakly compact convex subset of X and $T: D \to D$ be a mapping of asymptotically nonexpansive type. Then T has a fixed point in D.*

PROOF. Let F be the family of subsets C of D which are nonempty closed convex and satisfy the property (P). F is then ordered by inclusion. The weak compactness of D allows one to use Zorn's lemma to obtain a minimal element (say) K in F. For each $x \in D$, define the functional r_x by

$$r_x(y) = \limsup_{i\to\infty} \|T^n x - y\|$$

for all $y \in D$. Let $x \in K$. Then, by Lemma 1 of [14], r_x is a constant functional over $y \in K$, i.e.,

$$\limsup_{n\to\infty} \|T^n x - y\| = r \quad \text{(a constant)}$$

for all $x,\ y \in K$.

Now suppose that, for $x \in K$, the subsequence $\{T^{n_i}x\}$ of a sequence $\{T^n x\}$ converges weakly to some z in K. Then, by Lemma 1, $\{T^{n_i}x\}$ converges strongly to z in K. This shows that if $x \in K$, then $w(x) \subset K$, where

$$w(x) = \{y \in X : \text{y} = \text{strong} \lim_{i\to\infty} T^{n_i}x \quad \text{for some} \quad n_i \to \infty\}.$$

Clearly $w(x)$ is nonempty closed. Now we claim that $w(x)$ is norm compact. For this, let $\{u_j\}$ be a sequence in $w(x)$. Then we can construct a subsequence $\{T^{m_j}x\}$ of $\{T^n x\}$ such that $\|T^{m_j}x - u_j\| < j^{-1}$ for all $j \geq 1$. Repeating the above argument, we obtain a subsequence $\{T^{m_{j'}}x\}$ of $\{T^{m_j}x\}$ converging strongly to some $z \in w(x)$. Hence, $u_j \to z$ strongly which indicates the norm-compactness of $w(x)$. Therefore, the result follows from Lemma 2 of [14]. □

3. Convergence Theorems

In this section, we prove the weak convergence of trajectories of non-Lipschitzian asymptotically mappings in a Banach space with a weakly continuous duality mapping.

THEOREM 2. *Let X be a Banach space with a weakly continuous duality mapping J_ϕ, D be a weakly compact convex subset of X, $x \in D$*

and $T : D \to D$ be a mapping of asymptotically nonexpansive type. Then $\{T^n x\}$ converges weakly to a fixed point of T if and only if T is weakly asymptotically regular at x, i.e.,

$$w - \lim_{n\to\infty}(T^n x - T^{n+1}x) = 0.$$

To prove the theorem, we need the following lemma.

LEMMA 2. *Let X be a Banach space with a weakly continuous duality mapping J_ϕ, D be a weakly compact convex subset of X, $x \in X$ and $T : D \to D$ be a mapping of asymptotically nonexpansive type and weakly asymptotically regular at x. Then $w_w(x) \subset F(T)$.*

PROOF. Let $z = w - \lim_{i\to\infty} T^{n_i}x$ be an arbitrary element of $w_w(x)$. Since T is weakly asymptotically regular at x,

$$w - \lim_{i\to\infty} T^{n_i+m}x = z$$

for all $m \geq 0$. Set

$$r_m = \limsup_{i\to\infty}\|T^{n_i+m}x - z\|.$$

Then, by Opial's condition of X, we have

$$\begin{aligned} r_{m+l} &= \limsup_{i\to\infty}\|T^{n_i+m+l}x - z\| \\ &\leq \limsup_{i\to\infty}\|T^{n_i+m+l}x - T^l z\| \\ &\leq \limsup_{i\to\infty}(\|T^l(T^{n_i+m}x) - T^l z\| - \|T^{n_i+m}x - z\|) \\ &\quad + \limsup_{i\to\infty}\|T^{n_i+m}x - z\| \\ &\leq r_m + \sup_{y\in D}(\|T^l y - T^l z\| - \|y - z\|), \end{aligned}$$

which implies

$$\limsup_{l\to\infty} r_{m+l} \leq r_m,$$

i.e., $\lim_{m\to\infty} r_m = r$ exists. Using (1), we have

$$\begin{aligned}
&\Phi(\|T^{n_i+2m}x - z\|)\\
&= \Phi(\|T^{n_i+2m}x - T^m z\|)\\
&\quad + \int_0^1 \langle T^m z - z, J_\phi(T^{n_i+2m}x - T^m z + t(T^m z - z))\rangle dt\\
&\le \Phi(\sup_{y\in D}(\|T^m y - T^m z\| - \|y - z\|) + \|T^{n_i+m}x - z\|)\\
&\quad + \int_0^1 \langle T^m z - z, J_\phi(T^{n_i+2m}x - T^m z + t(T^m z - z))\rangle dt.
\end{aligned}$$

Take the limit superior as $i \to \infty$. Since we obtain

$$\begin{aligned}
\Phi(r_{2m}) &\le \Phi(\sup_{y\in D}(\|T^m y - T^m z\| - \|y - z\|) + r_m)\\
&\qquad + \int_0^1 \langle T^m z - z, J_\phi(z - T^m z + t(T^m z - z))\rangle dt\\
&= \Phi(\sup_{y\in D}(\|T^m y - T^m z\| - \|y - z\|) + r_m)\\
&\qquad - \int_0^1 \|T^m z - z\|\phi(t\|T^m z - z\|)dt\\
&= \Phi(\sup_{y\in D}(\|T^m y - T^m z\| - \|y - z\|) + r_m) - \Phi(\|T^m z - z\|),
\end{aligned}$$

it follows that

$$\begin{aligned}
\Phi(\|T^m z - z\|) &\le \Phi(\sup_{y\in D}(\|T^m y - T^m z\| - \|y - z\|) + r_m) - \Phi(r_{2m})\\
&\to 0 \ \text{ as } \ m \to \infty,
\end{aligned}$$

which implies $T^m y \to y$ strongly. Hence $y \in F(T)$. This completes the proof. □

PROOF OF THEOREM 2. Suppose that $\{T^n x\}$ converges weakly to p $(\in F(T))$. Then $\{T^n x\}$ is weakly asymptotically regular at x.

Conversely, assume that T is weakly asymptotically regular at x. Then, by Lemma 2, $w_w(x) \subset F(T)$. To complete the proof, we have to show that $w_w(x)$ is a singleton. First we observe that, for any $p \in F(T)$, $\lim_{n\to\infty} \|T^n x - p\|$ exists.

In fact, for all $m \geq 1$, we have

$$\begin{aligned}\limsup_{n\to\infty} \|T^n x - p\| &= \limsup_{n\to\infty} \|T^{n+m} x - p\| \\ &\leq \limsup_{n\to\infty} \sup_{y\in D}(\|T^n y - p\| - \|y - p\|) + \|T^m x - p\| \\ &\leq \liminf_{m\to\infty} \|T^m x - p\|\end{aligned}$$

and so it follows that $\lim_{n\to\infty} \|x_n - p\|$ exists. By a standard argument involving Opial's condition, $w_w(x)$ is a singleton. This completes the proof. □

From Theorem 2, we have the following corollaries:

Corollary 1. *Let X be a Banach space with a weakly continuous duality mapping J_ϕ, D be a weakly compact convex subset of X, $x \in D$ and $T : D \to D$ be an asymptotically nonexpansive mapping in the intermediate sense. Then $\{T^n x\}$ converges weakly to a fixed point of T if and only if T is weakly asymptotically regular at x.*

Corollary 2. *Let X be a Banach space with a weakly continuous duality mapping J_ϕ, D be a weakly compact convex subset of X, $x \in D$ and $T : D \to D$ be an asymptotically nonexpansive mapping. Then $\{T^n x\}$ converges weakly to a fixed point of T if and only if T is weakly asymptotically regular at x.*

Remark 1. Theorem 2 extends the result of Lim and Xu ([10]) from asymptotically nonexpansive mappings to much larger class of non-Lipschitzian mappings of asymptotically nonexpansive type. Theorem 2 is a good improvement of Theorem 6 of Cho, Sharma and Thakur ([4]) where the space X is assumed to be uniformly convex.

Remark 2. Corollary 2 is also a good improvement of Bose ([1]) and Górnicki ([6]).

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