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ARGUMENT ESTIMATES OF CERTAIN MEROMORPHIC FUNCTIONS

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ABSTRACT. The object of the present paper is to obtain some argument properties of certain meromorphic functions in the punctured open unit disk. Furthermore, we investigate their integral preserving properties in a sector.

1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k,$$

which are analytic in the punctured open unit disk $\mathcal{D} = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\}$. We denote by $\Sigma^*(\beta)$ the subclass of Σ consisting of all functions which are meromorphic starlike of order β in $\mathcal{U} = \mathcal{D} \cup \{0\} (0 \leq \beta < 1)$. The Hadamard product or convolution of two analytic functions f and g in Σ will be denoted by $f * g$.

Let

$$\begin{aligned} D^n f(z) &= \frac{1}{z(1-z)^{n+1}} * f(z) \quad (n \in \mathbb{N}_0 = \{0, 1, 2, \cdots\} \\ &= \frac{1}{z}\left(\frac{z^{n+1} f(z)}{n!}\right)^{(n)} \\ &= \frac{1}{z} + \sum_{k=0}^{\infty} c(n,k) a_k z^k, \end{aligned} \tag{1.1}$$

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where

$$c(n,k) = \frac{(n+1)(n+2)\cdots(n+k+1)}{(k+1)!} \quad (k \in \mathbb{N}_0).$$

For various interesting developments involving the operators D^n for functions belonging to Σ, the reader may be refereed to the recent works of Uralegaddi et al. ([3,10,11]) and others ([1,9]).

For analytic functions g and h with $g(0) = h(0)$, g is said to be subordinate to h if there exists an analytic function w such that $w(0) = 0, |w(z)| < 1 \ (z \in \mathcal{U})$, and $g(z) = h(w(z))$. We denote this subordination by $g \prec h$ or $g(z) \prec h(z)$.

Let

$$(1.2) \qquad \Sigma^*[n;A,B] = \left\{ f \in \Sigma : -\frac{z(D^n f(z))'(z)}{D^n f(z)} \prec \frac{1+Az}{1+Bz},\ z \in \mathcal{U} \right\},$$

where $-1 \leq B < A \leq 1$. In particular, we note that $\Sigma^*[0; 1-2\beta, -1] (0 \leq \beta < 1)$ is the well known class of meromorphic starlike functions of order β. From (1.2), we observe ([7]) that a function f is in $\Sigma^*[n;A,B]$ if and only if

(1.3)

$$\left| \frac{z(D^n f(z))'(z)}{D^n f(z)} + \frac{1-AB}{1-B^2} \right| < \frac{A-B}{1-B^2} \ (-1 < B < A \leq 1\ ;\ z \in \mathcal{U}).$$

A function $f \in \Sigma$ is said to be in the class $\Sigma_c(\beta,\gamma)$ if there is a meromorphic function $g \in \Sigma^*(\beta)$ such that

$$-\text{Re}\left\{ \frac{zf'(z)}{g(z)} \right\} > \gamma \quad (0 \leq \gamma < 1\ ;\ z \in \mathcal{U}).$$

Libera and Robertson ([4]) showed that $\Sigma_c(0,0)$, the class of meromorphic close-to-convex functions, is not univalent. Also, $\Sigma_c(\beta,\gamma)$ provides an interesting generalization of the class of meromorphic close-to-convex functions ([8]).

The object of the present paper is to give some argument estimates of meromorphic functions belonging to Σ and the integral preserving properties in connection with the differential operators D^n defined by (1.1). Furthermore, we investigate some applications of meromorphic close-to-convex functions as special cases.

2. Main results

To establish our main results, we need the following lemmas.

LEMMA 2.1 ([2]). *Let h be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and* Re $(\beta h(z) + \gamma) > 0 (\beta, \gamma \in \mathbb{C})$. *If q is analytic in $\mathcal{U}$ with $q(0) = 1$, then*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec h(z) \quad (z \in \mathcal{U})$$

implies

$$q(z) \prec h(z) \quad (z \in \mathcal{U}).$$

LEMMA 2.2 ([5]). *Let h be convex univalent in $\mathcal{U}$ and λ be analytic in $\mathcal{U}$ with* Re $\lambda(z) \geq 0$. *If q is analytic in $\mathcal{U}$ and $q(0) = h(0)$, then*

$$q(z) + \lambda(z)zq'(z) \prec h(z) \quad (z \in \mathcal{U})$$

implies

$$q(z) \prec h(z) \quad (z \in \mathcal{U}).$$

LEMMA 2.3 ([6]). *Let q be analytic in $\mathcal{U}$ with $q(0) = 1$ and $q(z) \neq 0$ in $\mathcal{U}$. Suppose that there exists a point $z_0 \in \mathcal{U}$ such that*

(2.1) $$\left|\arg\ q(z)\right| < \frac{\pi}{2}\alpha \text{ for } |z| < |z_0|$$

and

(2.2) $$\left|\arg\ q(z_0)\right| = \frac{\pi}{2}\alpha \quad (0 < \alpha \leq 1).$$

Then we have

(2.3) $$\frac{z_0 q'(z_0)}{q(z_0)} = ik\alpha,$$

where

(2.4) $$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \text{ when } \arg\ q(z_0) = \frac{\pi}{2}\alpha$$

(2.5) $$k \leq -\frac{1}{2}\left(a + \frac{1}{a}\right) \text{ when } \arg\ q(z_0) = -\frac{\pi}{2}\alpha$$

and

(2.6) $$q(z_0)^{\frac{1}{\alpha}} = \pm ia \ (a > 0).$$

At first, with the help of Lemma 2.1, we obtain the following

PROPOSITION 2.1. *Let h be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and* Re h *be bounded in $\mathcal{U}$. If $f \in \Sigma$ satisfies the condition*

$$-\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} \prec h(z) \quad (z \in \mathcal{U}),$$

then

$$-\frac{z(D^n f(z))'}{D^n f(z)} \prec h(z) \quad (z \in \mathcal{U})$$

for $\max_{z\in\mathcal{U}} \operatorname{Re} h(z) < n+2$ *(provided $D^n f(z) \neq 0$ in $\mathcal{U}$).*

PROOF. Let

$$q(z) = -\frac{z(D^n f(z))'}{D^n f(z)}.$$

By using the equation

$$z(D^n f(z))' = (n+1)D^{n+1}f(z) - (n+2)D^n f(z), \tag{2.7}$$

we get

$$q(z) - n + 2 = -\frac{(n+1)D^{n+1}f(z)}{D^n f(z)}. \tag{2.8}$$

Taking logarithemic derivatives in both sides of (2.8) and multiplying by z, we have

$$\frac{zq'(z)}{-q(z)+n+2} + q(z) = -\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} \prec h(z) \quad (z \in \mathcal{U}).$$

From Lemma 2.1, it follows that $q(z) \prec h(z)$ for Re $(-h(z)+n+2) > 0$ $(z \in \mathcal{U})$, which means

$$-\frac{z(D^n f(z))'}{D^n f(z)} \prec h(z) \quad (z \in \mathcal{U})$$

for $\max_{z\in\mathcal{U}} \operatorname{Re} h(z) < n+2$. □

PROPOSITION 2.2. *Let h be convex univalent in $\mathcal{U}$ with $h(0) = 1$ and* Re *h be bounded in $\mathcal{U}$. Let F be the integral operator defined by*

$$F(z) = \frac{c}{z^{c+1}} \int_0^z t^c f(t)dt \quad (c > 0). \tag{2.9}$$

If $f \in \Sigma$ satisfies the condition

$$-\frac{z(D^n f(z))'}{D^n f(z)} \prec h(z) \quad (z \in \mathcal{U}),$$

then

$$-\frac{z(D^n F(z))'}{D^n F(z)} \prec h(z) \quad (z \in \mathcal{U})$$

for $\max_{z\in\mathcal{U}}$ Re $h(z) < c+1$ *(provided $D^n F(z) \neq 0$ in $\mathcal{U}$).*

PROOF. From (2.9), we have

$$z(D^n F(z))' = cD^n f(z) - (c+1)D^n F(z). \tag{2.10}$$

Let

$$q(z) = -\frac{z(D^n F(z))'}{D^n F(z)}.$$

Then, by using (2.10), we get

$$q(z) - (c+1) = -c\frac{D^n f(z)}{D^n F(z)}. \tag{2.11}$$

Taking logarithemic derivatives in both sides of (2.11) and multiplying by z, we have

$$\frac{zq'(z)}{-q(z)+(c+1)} + q(z) = -\frac{z(D^n f(z))'}{D^n f(z)} \prec h(z) \quad (z \in \mathcal{U}).$$

Therefore, by Lemma 2.1, we have

$$-\frac{z(D^n F(z))'}{D^n F(z)} \prec h(z) \quad (z \in \mathcal{U})$$

for $\max_{z\in\mathcal{U}}$Re $h(z) < c+1$ (provided $D^n F(z) \neq 0$ in $\mathcal{U}$). □

REMARK. Taking $h(z) = \frac{1+z}{1-z}$ in Proposition 2.1 and Proposition 2.2, we have the results obtained by Ganigi and Uralegaddi ([3]).

Applying Lemma 2.2, Lemma 2.3 and Proposition 2.1, we now derive:

THEOREM 2.1. *Let $f \in \Sigma$. Choose an integer n such that*

$$n \geq \frac{1+A}{1+B} - 2,$$

where $-1 < B < A \leq 1$. If

$$\left|\arg\left(-\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta \;\; (0 \leq \gamma < 1\,;\; 0 < \delta \leq 1)$$

for some $g \in \Sigma^*[n+1; A, B]$, then

$$\left|\arg\left(-\frac{z(D^n f(z))'}{D^n g(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where α $(0 < \alpha \leq 1)$ is the solution of the equation

$$\delta = \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha \sin\frac{\pi}{2}(1 - t(A,B))}{\frac{(n+2)(1-B)+A-1}{1-B} + \alpha\cos\frac{\pi}{2}(1-t(A,B))}\right) \tag{2.12}$$

when

$$t(A,B) = \frac{2}{\pi}\sin^{-1}\left(\frac{A-B}{(n+2)(1-B^2) - (1-AB)}\right). \tag{2.13}$$

PROOF. Let

$$q(z) = -\frac{1}{1-\gamma}\left(\frac{z(D^n f(z))'}{D^n g(z)} + \gamma\right).$$

By (2.7), we have

(2.14)

$$\begin{aligned}&(1-\gamma)zq'(z)D^n g(z) + (1-\gamma)q(z)z(D^n g(z))' - (n+2)z(D^n f(z))' \\ &= -(n+1)z(D^{n+1}f(z))' - \gamma z(D^n g(z))'(z).\end{aligned}$$

Dividing (2.14) by $D^n g(z)$ and simplifying, we get

$$q(z)+\frac{zq'(z)}{-r(z)+n+2}=-\frac{1}{1-\gamma}\left(\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)}+\gamma\right), \tag{2.15}$$

where

$$r(z)=-\frac{z(D^n g(z))'}{D^n g(z)}.$$

Since $g\in\Sigma^*[n+1;A,B]$, from Proposition 2.1, we have

$$r(z)\ \prec\ \frac{1+Az}{1+Bz}.$$

From (1.3), we have

$$-r(z)+n+2=\rho e^{i\frac{\pi}{2}}\phi,$$

where

$$\begin{cases}\frac{(n+2)(1+B)-(1+A)}{1+B}\ <\ \rho\ <\ \frac{(n+2)(1-B)+A-1}{1-B}\\ -t(A,B)\ <\ \phi\ <\ t(A,B)\end{cases}$$

when $t(A,B)$ is given by (2.13). Let h be a function which maps $\mathcal{U}$ onto the angular domain $\{w\ :\ |\arg w|\ <\ \frac{\pi}{2}\delta\}$ with $h(0)=1$. Applying Lemma 2.2 for this h with $\lambda(z)=\frac{1}{-r(z)+n+2}$, we see that $\operatorname{Re} q(z)>0$ in $\mathcal{U}$ and hence $q(z)\neq 0$ in $\mathcal{U}$.

If there exists a point $z_0\in\mathcal{U}$ such that the conditions (2.1) and (2.2) are satisfied, then(by Lemma 2.3) we obtain (2.3) under the restrictions (2.4), (2.5) and (2.6).

At first, suppose that $q(z_0)^{\frac{1}{\alpha}}=ia$ $(a>0)$. Then we obtain

$$\begin{aligned}
&\arg\left[-\frac{1}{1-\gamma}\left(\frac{z_0(D^{n+1}f(z_0)'}{D^{n+1}g(z_0)}+\gamma\right)\right]=\arg\left(q(z_0)+\frac{z_0q'(z_0)}{-r(z_0)+n+2}\right)\\
&=\frac{\pi}{2}\alpha+\arg\left(1+i\alpha k(\rho e^{i\frac{\pi}{2}\phi})^{-1}\right)\\
&=\frac{\pi}{2}\alpha+\tan^{-1}\left(\frac{\eta k\sin\frac{\pi}{2}(1-\phi)}{\rho+\alpha k\cos\frac{\pi}{2}(1-\phi)}\right)\\
&\geq\frac{\pi}{2}\alpha+\tan^{-1}\left(\frac{\alpha\sin\frac{\pi}{2}(1-t(A,B))}{\frac{(n+2)(1-B)+A-1}{1-B}+\alpha\cos\frac{\pi}{2}(1-t(A,B))}\right)\\
&=\frac{\pi}{2}\delta,
\end{aligned}$$

where δ and $t(A,B)$ are given by (2.12) and (2.13), respectively. This is a contradiction to the assumption of our theorem.

Next, suppose that $p(z_0)^{\frac{1}{\alpha}} = -ia$ $(a > 0)$. Applying the same method as the above, we have

$$\begin{aligned}&\arg\left[-\frac{1}{1-\gamma}\left(\frac{z_0(D^{n+1}f(z_0))'}{D^{n+1}g(z_0)}+\gamma\right)\right]\\ &\le -\frac{\pi}{2}\alpha - \tan^{-1}\left(\frac{\alpha\sin\frac{\pi}{2}(1-t(A,B))}{\frac{(n+2)(1-B)+A-1}{1-B}+\alpha\cos\frac{\pi}{2}(1-t(A,B))}\right)\\ &= -\frac{\pi}{2}\delta,\end{aligned}$$

where δ and $t(A,B)$ are given by (2.12) and (2.13), respectively, which contradicts the assumption. Therefore we complete the proof of our theorem. □

Letting $n=0,\ A=1,\ B=0$ and $\delta=1$ in Theorem 2.1, we have:

Corollary 2.1. *Let $f \in \Sigma$. If*

$$-\mathrm{Re}\left\{\frac{z(zf''(z)+3f'(z))}{zg'(z)+2g(z)}\right\} > \gamma \quad (0 \le \gamma < 1)$$

for some $g \in \Sigma$ satisfying the condition

$$\left|\frac{z(zf''(z)+3f'(z))}{zg'(z)+2g(z)}+1\right| < 1,$$

then

$$-\mathrm{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > \gamma.$$

Taking $n=0,\ A=1,\ B=0$ and $g(z)=\frac{1}{z}$ in Theorem 2.1, we have:

Corollary 2.2. *Let $f \in \Sigma$. If*

$$\arg\{-z^2(zf''(z)+3f'(z)-\gamma\} < \frac{\pi}{2}\delta \quad (0 \le \gamma < 1\ ;\ 0 < \delta \le 1),$$

then

$$\arg\{-z^2f'(z)-\gamma\} < \frac{\pi}{2}\delta.$$

By the same techniques as in the proof of Theorem 2.1, we obtain:

THEOREM 2.2. *Let $f \in \Sigma$. Choose an integer n such that*

$$n \geq \frac{1+A}{1+B} - 2,$$

where $-1 < B < A \leq 1$. If

$$\left|\arg\left(\frac{z(D^{n+1}f(z))'}{(D^{n+1}g(z))} + \gamma\right)\right| < \frac{\pi}{2}\delta \quad (\gamma > 1\ ;\ 0 < \delta \leq 1)$$

for some $g \in \Sigma^*[n+1; A, B]$, then

$$\left|\arg\left(\frac{z(D^{n}f(z))'}{D^{n}g(z)} + \gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where α $(0 < \alpha \leq 1)$ is the solution of the equation given by (2.12).

Next, we prove

THEOREM 2.3. *Let $f \in \Sigma$ and choose a positive number c such that*

$$c \geq \frac{1+A}{1+B} - 1,$$

where $-1 < B < A \leq 1$. If

$$\left|\arg\left(-\frac{z(D^{n}f(z))'}{D^{n}g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta \quad (0 \leq \gamma < 1\ ;\ 0 < \delta \leq 1)$$

for some $g \in \Sigma^*[n; A, B]$, then

$$\left|\arg\left(-\frac{z(D^{n}F(z))'}{D^{n}G(z)} - \gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where F is the integral operator given by (2.9),

$$G(z) = \frac{c}{z^{c+1}} \int_0^z t^c g(t)dt, \quad (c > 0), \tag{2.16}$$

and α $(0 < \alpha \leq 1)$ is the solution of the equation

$$\delta = \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha \sin\frac{\pi}{2}(1 - t(A,B,c))}{\frac{(c+1)(1-B)+A-1}{1-B} + \alpha\cos\frac{\pi}{2}(1 - t(A,B,c))}\right) \tag{2.17}$$

when

$$t(A,B,c) = \frac{2}{\pi}\sin^{-1}\left(\frac{A-B}{(c+1)(1-B^2) - (1-AB)}\right).$$

PROOF. Let

$$q(z) = -\frac{1}{1-\gamma}\left(\frac{z(D^nF(z))'}{D^nG(z)} + \gamma\right).$$

Since $g \in \Sigma^*[n; A, B]$, from Proposition 2.2, $g \in \Sigma^*[n; A, B]$.
Using (2.10), we have

$$(1-\gamma)q(z)D^nG(z) - (c+1)D^nF(z) = -cD^nf(z) - \gamma D^nG(z).$$

Then, by a simple calculation, we get

$$(1-\gamma)(zq'(z)+q(z)(-r(z)+c+1))+\gamma(-r(z)+c+1) = -\frac{cz(D^nf(z))'}{D^nG(z)},$$

where

$$r(z) = -\frac{z(D^nG(z))'}{D^nG(z)}.$$

Hence we have

$$q(z) + \frac{zq'(z)}{-r(z)+c+1} = -\frac{1}{1-\gamma}\left(\frac{z(D^nf(z))'}{D^ng(z)} + \gamma\right).$$

The remaining part of the proof is similar to that of Theorem 2.1 and so we omit it. □

Letting $n = 0$, $A = 1$, $B = 0$ and $\delta = 1$ in Theorem 2.3, we have

COROLLARY 2.3. *Let $c > 0$ and $f \in \Sigma$. If*

$$-\text{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > \gamma \ \ (0 \le \gamma < 1)$$

for some $g \in \Sigma$ satisfying the condition

$$\left|\frac{zg'(z)}{g(z)} + 1\right| < 1,$$

then

$$-\text{Re}\left\{\frac{zF'(z)}{G(z)}\right\} > \gamma,$$

where F and G are given by (2.9) and (2.16), respectively.

Taking $n = 0$, $B \to A$ and $g(z) = \frac{1}{z}$ in Theorem 2.3, we have :

COROLLARY 2.4. *Let $c > 0$ and $f \in \Sigma$. If*

$$|\arg(-z^2 f'(z) - \gamma)| < \frac{\pi}{2}\delta, \quad (0 \leq \gamma < 1\ ;\ 0 < \delta \leq 1)$$

then

$$|\arg(-z^2 F'(z) - \gamma)| < \frac{\pi}{2}\alpha,$$

where F is the integral operator given by (2.9) and α $(0 < \alpha \leq 1)$ is the solution of the equation

$$\delta = \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha}{c}\right).$$

By using the same methods as in proving Theorem 2.3, we have :

THEOREM 2.4. *Let $f \in \Sigma$ and choose a positive number c such that*

$$c \geq \frac{1+A}{1+B} - 1,$$

where $-1 < B < A \leq 1$. If

$$\left|\arg\left(\frac{z(D^n f(z))'}{D^n g(z)} + \gamma\right)\right| < \frac{\pi}{2}\delta \ \ (\gamma > 1\ ;\ 0 < \delta \leq 1)$$

for some $g \in \Sigma^*[n; A, B]$, then

$$\left|\arg\left(\frac{z(D^n F(z))'}{D^n G(z)} + \gamma\right)\right| < \frac{\pi}{2}\alpha,$$

where F and G are given by (2.9) and (2.16), respectively, and $\alpha(0 < \alpha \leq 1)$ is the solution of the equation given by (2.17).

Finally, we derive :

THEOREM 2.5. *Let $f \in \Sigma$. Choose an integer n such that*

$$n \geq \frac{1+A}{1+B} - 2,$$

where $-1 < B < A \leq 1$. If

$$\left|\arg\left(-\frac{z(D^n f(z))'}{D^n g(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta \ \ (0 \leq \gamma < 1\ ;\ 0 < \delta \leq 1)$$

for some $g \in \Sigma^*[n+1; A, B]$, then

$$\left|\arg\left(-\frac{z(D^{n+1} F(z))'}{D^{n+1} G(z)} - \gamma\right)\right| < \frac{\pi}{2}\delta,$$

where F and G are given by (2.9) and (2.16) with $c = n+1$, respectively.

PROOF. From (2.7) and (2.8) with $c = n+1$, we have $D^n f(z) = D^{n+1}F(z)$.
Therefore

$$\frac{z(D^n f(z))'}{D^n g(z)} = \frac{z(D^{n+1}F(z))'}{D^{n+1}G(z)}$$

and the result follows. □

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