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INJECTIVE PROPERTY OF GENERALIZED INVERSE POLYNOMIAL MODULE

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ABSTRACT. Northcott and McKerrow proved that if R is a left noetherian ring and E is an injective left R-module, then $E[x^{-1}]$ is an injective left $R[x]$-module. In this paper we generalize Northcott and McKerrow's result so that if R is a left noetherian ring and E is an injective left R-module, then $E[x^{-S}]$ is an injective left $R[x^S]$-module, where S is a submonoid of N (N is the set of all natural numbers).

1. Introduction

Northcott ([3]) considered the module $K[x^{-1}]$ of inverse polynomial over the polynomial ring $K[x]$ (with K a field), and Northcott and McKerrow ([1]) proved that if R is a left noetherian ring and E is an injective left R-module, then $E[x^{-1}]$ is an injective left $R[x]$-module. In this paper we generalize Northcott and McKerrow's result so that if R is a left noetherian ring and E is an injective left R-module, then $E[x^{-S}]$ is an injective left $R[x^S]$-module, where S is a submonoid of N (N is the set of all natural numbers). Inverse polynomial modules were developed in [4] [5] and recently in [2].

DEFINITION 1.1. Let R be a ring and M be a left R-module. Then $M[x^{-1}]$ is a left $R[x]$-module such that

$$x(m_0 + m_1x^{-1} + \cdots + m_nx^{-n}) = m_1 + m_2x^{-1} + \cdots + m_nx^{-n+1}$$

and such that

$$r(m_0 + m_1x^{-1} + \cdots + m_nx^{-n}) = rm_0 + rm_1x^{-1} + \cdots + rm_nx^{-n}$$

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where $r \in R$.

Similarly, we also can define $M[[x^{-1}]], M[x, x^{-1}], M[x, x^{-1}]]$, and also $M[[x, x^{-1}]$ as left $R[x]$-modules where, for example, $M[[x, x^{-1}]$ is the set of Laurent series in x with coefficients in M, i.e., the set of all formal sums $\sum_{k \geq n_0} m_k x^k$ with n_0 any element of Z (Z is the set of all integers).

DEFINITION 1.2. Let R be a ring and M be a left R-module, and $S = \{0, k_1, k_2, \cdots\}$ be a submonoid of N (N is the set of all natural numbers. Then $M[x^{-S}]$ is a left $R[x^S]$-module such that

$$\begin{aligned} &x^{k_i}(m_0 + m_1 x^{-k_1} + m_2 x^{-k_2} \cdots + m_n x^{-k_n}) \\ &= m_1 x^{-k_1+k_i} + m_2 x^{-k_2+k_i} + \cdots + m_n x^{-k_n+k_i} \end{aligned}$$

$$\text{where} \qquad \begin{aligned} x^{-k_j+k_i} &= x^{-k_j+k_i}, \quad \text{if} \quad -k_j + k_i \in S \\ &= 0, \quad \text{if} \quad -k_j + k_i \notin S. \end{aligned}$$

For example, if $S = \{0, 2, 3, 4, 5, \cdots\}$, then $m_0 + m_2 x^{-2} + m_3 x^{-3} + \cdots + m_i x^{-i} \in M[x^{-S}]$ and if $S = \{0, 1, 2, 3, 4, \cdots\}$, then $M[x^{-S}] = M[x^{-1}]$.

Similarly, we can define $M[[x^{-S}]]$ as a left $R[x^S]$-module.

LEMMA 1.3. *Let M be a left R-module and $S = \{0, k_1, k_2, \cdots\}$ be a submonoid of N (N is the set of all natural numbers). Then*

$$Hom_R(R[x^S],\ M) \cong M[[x^{-S}]]$$

as left $R[x^S]$-modules.

PROOF. Define $\phi : \mathrm{Hom}_R(R[x^S],\ M) \to M[[x^{-S}]]$ by

$$\phi(f) = f(1) + f(x^{k_1})x^{-k_1} + f(x^{k_2})x^{-k_2} + \cdots .$$

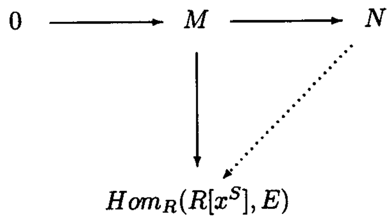
Then ϕ is an isomorphism. □

THEOREM 1.4. *If E is an injective left R-module, then $E[[x^{-S}]]$ is an injective left $R[x^S]$-module.*

PROOF. Let $0 \to M \to N$ be an exact sequence of left $R[x^S]$-modules. Since,

$$\mathrm{Hom}_R(R[x^S], E) \cong E[[x^{-S}]],$$

equivalently we want to prove that the following diagram



can be completed to a commutative diagram. Note that

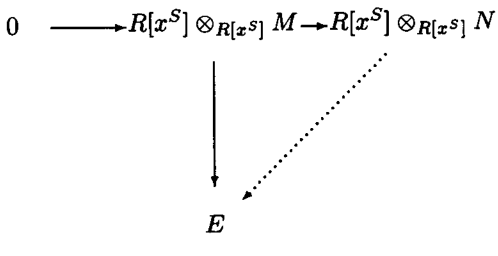
$$\mathrm{Hom}_{R[x^S]}(N, \mathrm{Hom}_R(R[x^S], E)) \cong \mathrm{Hom}_R(R[x^S] \otimes_{R[x^S]} N, E)$$

$$\mathrm{Hom}_{R[x^S]}(M, \mathrm{Hom}_R(R[x^S], E)) \cong \mathrm{Hom}_R(R[x^S] \otimes_{R[x^S]} M, E).$$

$R[x^S]_{R[x^S]}$ is flat. So if $0 \to M \to N$ is exact, we have

$$0 \to R[x^S] \otimes_{R[x^S]} M \to R[x^S] \otimes_{R[x^S]} N$$

is exact. Since E is injective we can complete the following diagram:



$\mathrm{Hom}_R(R[x^S] \otimes_{R[x^S]} N, E) \to \mathrm{Hom}_R(R[x^S] \otimes_{R[x^S]} M, E) \to 0$ is exact. $\mathrm{Hom}_{R[x^S]}(N, \mathrm{Hom}_R(R[x^S], E)) \to \mathrm{Hom}_{R[x^S]}(M, \mathrm{Hom}_R(R[x^S], E)) \to 0$ is exact. Hence, $\mathrm{Hom}_R(R[x^S], E)$ is an injective left $R[x^S]$-module, i.e., $E[[x^{-S}]]$ is an injective left $R[x^S]$-module. □

2. Main Theorem

DEFINITION 2.1. Given any module M and $f \in End(M)$ we say f is locally nilpotent on M if for every $x \in M$, there exist $n \geq 1$ such that $f^n(x) = 0$.

The following Theorem 2.2 is originally due to Matlis and Gabriel.

THEOREM 2.2. *If R is a left noetherian ring, and E is an injective left R-module, and $f \in End(E)$ is such that E is an essential extension of $Ker(f)$, then f is locally nilpotent on E.*

PROOF. Let K be the kernel of f and E be an essential extension of K. Consider the direct sum $K \oplus K \oplus \cdots$ of countable number of $K's$. Choose $(a_1, a_2, \cdots) \in E \oplus E \oplus \cdots$. Then $a_i = 0$ for all $i \geq n$ for some n. Since E is an essential extension of K, we choose $r_1 \in R$ such that $r_1 a_1 \in K$. And choose $r_2 \in R$ such that $r_2(r_1 a_2) \in K$ and so on. We choose $r_k \in R$ such that $r_k(r_{k-1} \cdots r_2 r_1 a_k) \in K$. Then

$$(r_n r_{n-1} \cdots r_2 r_1)(a_1, a_2, \cdots, a_n, 0, 0, \cdots) \in K \oplus K \oplus \cdots .$$

Thus $E \oplus E \oplus \cdots$ is an essential extension of $K \oplus K \oplus \cdots$. Since R is left Noetherian, $E \oplus E \oplus \cdots$ is injective, so is an injective envelope of $K \oplus K \oplus \cdots$. If $M \subset E_1, M \subset E_2$ are injective envelopes of M and $\phi : E_1 \to E_2$ is the identity on M then ϕ is an isomorphism. So define a map

$$\begin{aligned} \phi : E \oplus E \oplus \cdots &\longrightarrow E \oplus E \oplus \cdots \\ (x_1, x_2, \cdots) &\longmapsto (x_1, x_2 - f(x_1), x_3 - f(x_2), \cdots). \end{aligned}$$

Then ϕ is a homomorphism, and $\phi|_{K \oplus K \oplus \cdots} = id_{K \oplus K \oplus \cdots}$. So ϕ is an automorphism of $E \oplus E \oplus \cdots$ and in particular ϕ is onto. Let $x \in E$ and consider $(x, 0, 0, \cdots)$. Then $\phi(x_1, x_2, x_3, \cdots) = (x, 0, 0, \cdots)$ for some $(x_1, x_2, x_3, \cdots) \in E \oplus E \oplus \cdots$. Then

$$\begin{aligned} x_1 &= x, \\ x_2 - f(x_1) &= 0, \\ x_3 - f(x_2) &= 0, \end{aligned}$$

and so on. So $x_n = f^{n-1}(x)$ for all $n \geq 2$. But for some n, $x_{n+1} = 0$, i.e., $f^n(x) = 0$. Therefore, f is locally nilpotent. $\square$

We now have our main Theorem.

THEOREM 2.3. *Let R be a commutative noetherian ring and S be a submonoid, and E be an injective left R-module. Then $E[x^{-S}]$ is an injective left $R[x^S]$-module.*

PROOF. Let $S = \{0, k_1, k_2, \cdots\}$ be a submonoid. Then

$$\mathrm{Hom}_R(R[x^S],\ E) \cong E[[x^{-S}]]$$

is an injective left $R[x^S]$-module. Define $\phi : E[[x^{-S}]] \to E[[x^{-S}]]$ by

$$\phi(f) = x^{k_1} f$$

for $f \in E[[x^{-S}]]$. Then ϕ is not locally nilpotent on $E[[x^{-S}]]$. So $E[[x^{-S}]]$ is not an essential extension of $Ker(\phi)$. Let $\bar{E}$ be an injective envelope of $Ker(\phi)$. Then

$$Ker(\phi) \subset \bar{E} \subset E[[x^{-S}]].$$

Then $\phi : \bar{E} \to \bar{E}$ defined by

$$\phi(f) = x^{k_1} f,$$

for $f \in \bar{E}$ is locally nilpotent on $\bar{E}$. So $\bar{E} \subset E[x^{-S}]$. But $E[x^{-S}]$ is an essential extension of $Ker(\phi)$, so that $E[x^{-S}]$ is an essential extension of $\bar{E}$. Therefore, $\bar{E} = E[x^{-S}]$. Hence, $E[x^{-S}]$ is an injective left $R[x^S]$-module. □

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