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ON NULL SCROLLS SATISFYING THE CONDITION $\Delta H = AH$

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ABSTRACT. In the present paper, we study a non-degenerate ruled surface along a null curve in a 3-dimensional Minkowski space $\mathbb{E}^3_1$, which is called a null scroll, and investigate some characterizations of null scrolls satisfying the condition $\Delta H = AH, A \in Mat(3,\mathbb{R})$, where Δ denotes the Laplacian of the surface with respect to the induced metric, H the mean curvature vector and $Mat(3,\mathbb{R})$ the set of 3×3-real matrices.

1. Introduction

The theory of Gauss map concerning with mean curvature vector plays an important role in the study of immersed submanifolds in Euclidean space and pseudo-Euclidean space, and it has been investigated from the various viewpoints by many differential geometers [1, 2, 3, 4, 7, 11].

On the other hand, Baikoussis and Blair [2] studied ruled surfaces in Euclidean 3-space $\mathbb{E}^3$ such that its Gauss map G satisfies a special condition,

$$\Delta G = AG, \quad A \in Mat(3,\mathbb{R}), \tag{1.1}$$

where Δ denotes the Laplacian of the surface with respect to the induced metric and $Mat(3,\mathbb{R})$ the set of 3×3-real matrices. Also, for the Lorentz version Choi [5] investigated the ruled surfaces with non-null base curve in a 3-dimensional Minkowski space $\mathbb{E}^3_1$ satisfying the condition (1.1). Furthermore, for the mean curvature vector H, Chen [3] completely classified ruled surfaces in $\mathbb{E}^m$ satisfying the condition $\Delta H = AH$ for some $A \in Mat(m,\mathbb{R})$.

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Let γ be a null curve with null frame $F = \{X, Y, Z\}$, which will be briefly denoted by (γ, F). Then (γ, F) is called a *(proper) framed null curve with null frame* F. A non-degenerate ruled surface M in a 3-dimensional Minkowski space $\mathbb{E}^3_1$ along γ parameterized by

$$x(s,t) = \gamma(s) + tY(s)$$

is called a *null scroll*. It is a time-like surface. Furthermore, for a Cartan framed null curve γ with Cartan frame $F = \{X, Y, Z\}$ the ruled surface is called a *B-scroll* [6]. On the other hand, Kim and the second author [9] studied the ruled surfaces with pointwise 1-type Gauss map in $\mathbb{E}^3_1$ and obtained a new characterization of minimal ruled surfaces. In [7] Kim, Kim and the second author obtained the complete classification theorem of ruled surfaces with 1-type Gauss map in $\mathbb{E}^m_1$ and also characterized the extended B-scroll with Gauss map. Recently Choi, Ki and Suh [4] obtained the following result: *Let M be a null scroll along the framed null curve with (proper) frame field. Then the Gauss map satisfies the condition* (1.1) *if and only if the mean curvature is constant.*

In the present paper, as an improvement of those situations in [4], we investigate null scrolls and B- scrolls satisfying the condition

$$\Delta H = AH, \quad A \in Mat(3, \mathbb{R}), \tag{1.2}$$

and prove the following theorem :

THEOREM. *Let M be a null scroll along the framed null curve with (proper) frame field. Then the mean curvature vector H satisfies the condition* (1.2) *if and only if the mean curvature is constant.*

COROLLARY. *Let M be a B-scroll along the framed null curve with (proper) frame field. Then the mean curvature vector H satisfies the condition* (1.2) *if and only if the third curvature is constant.*

2. Preliminaries

Let $\mathbb{E}^m_s$ be an m-dimensional pseudo-Euclidean space with signature $(s, m-s)$. Then the metric tensor g in $\mathbb{E}^m_s$ has the form

$$g = -\sum_{i=1}^{s} dx_i^2 + \sum_{i=s+1}^{m} dx_i^2,$$

where $(x_1, x_2, \ldots, x_m)$ is a standard rectangular coordinate system in $\mathbb{E}_s^m$. In particular, for $m \geq 2$, $\mathbb{E}_1^m$ is called *Minkowski m - space.* A vector X in $\mathbb{E}_s^m$ is said to be *space-like* if $g(X,X) > 0$ or $X = 0$, *time-like* if $g(X,X) < 0$ and *null* if $g(X,X) = 0$ and $X \neq 0$. A curve in $\mathbb{E}_s^m$ is called *null* if its tangent vector field is null along it.

We will recall the notion of cross product in a 3-dimensional Minkowski space $\mathbb{E}_1^3$. There is a natural orientation in $\mathbb{E}_1^3$ defined as follows : an ordered basis $F = \{X, Y, Z\}$ in $\mathbb{E}_1^3$ is positively oriented if $\det[XYZ] > 0$, where $[XYZ]$ is the matrix with X, Y, Z as column vectors. Let ω be the volume element on $\mathbb{E}_1^3$ defined by $\omega(X,Y,Z) = \det[XYZ]$. Then given $X, Y \in \mathbb{E}_1^3$, the cross product $X \times Y$ is the unique vector in $\mathbb{E}_1^3$ such that $g(X \times Y, Z) = \omega(X,Y,Z)$ for any $Z \in \mathbb{E}_1^3$.

We shall consider a ruled surface along a null curve in a 3-dimensional Minkowski space $\mathbb{E}_1^3$. A basis $F = \{X, Y, Z\}$ of $\mathbb{E}_1^3$ is called a (*proper*) *null frame* if it satisfies the following conditions :

$$
\begin{aligned}
&g(X,X) = g(Y,Y) = 0, \qquad g(X,Y) = -1, \\
&g(X,Z) = g(Y,Z) = 0, \qquad g(Z,Z) = 1,
\end{aligned}
\tag{2.1}
$$

where $Z = X \times Y$.

Let $\gamma = \gamma(s)$ be a null curve in $\mathbb{E}_1^3$. For a given smooth positive function $k_0 = k_0(s)$ let us put $X = X(s) = {k_0}^{-1}\gamma'$. Then X is a null vector field along γ. Furthermore, there exists a null vector field $Y = Y(s)$ along γ satisfying $g(X,Y) = -1$. Here, if we put $Z = X \times Y$, then we can obtain a (proper) null frame field $F = \{X, Y, Z\}$ along γ. In the case, the pair (γ, F) is said to be a (*proper*) *framed null curve.* Then, a framed null curve (γ, F) satisfies the following Frenet equations:

$$
\begin{cases}
X'(s) = k_1(s)X(s) + k_2(s)Z(s), \\
Y'(s) = -k_1(s)Y(s) + k_3(s)Z(s), \\
Z'(s) = k_3(s)X(s) + k_2(s)Y(s),
\end{cases}
\tag{2.2}
$$

where k_i $(i = 1, 2, 3)$ is an i-th curvature for γ. It follows from the fundamental theorem of ordinary differential equations that a framed null curve (γ, F) is uniquely determined by the functions $k_0(> 0), k_1, k_2, k_3$ and the initial condition.

Let (γ, F) be a null curve with null frame $F = \{X, Y, Z\}$. A non-degenerate ruled surface M along γ parametrized by

$$x(s,t) = \gamma(s) + tY(s)$$

is called a *null scroll.* It is a time-like surface.

A framed null curve (γ, F) with $k_0 = 1$ and $k_1 = 0$ is called a *Cartan framed null curve* and the frame F is called a *Cartan* one. Furthermore, for a Cartan framed null curve γ with Cartan frame $F = \{X, Y, Z\}$ the ruled surface is called a *B-scroll.*

It is well known that in terms of local coordinates $\{x_i\}$ of M the Laplacian can be written as :

$$\Delta = -\frac{1}{\sqrt{|\mathcal{G}|}} \sum_{i,j} \frac{\partial}{\partial x^i}\left(\sqrt{|\mathcal{G}|}\, g^{ij} \frac{\partial}{\partial x^j}\right), \tag{2.3}$$

where $\mathcal{G} = \det(g_{ij}), (g^{ij}) = (g_{ij})^{-1}$ and (g_{ij}) are the components of the matrix on M with respect to $\{x_i\}$.

3. Proof of Theorem

Let $\gamma = \gamma(s)$ be a null curve in $\mathbb{E}^3_1$ and $Y = Y(s)$ be a null vector field along γ. Then, the null scroll M is parametrized by

$$x = x(s,t) = \gamma(s) + tY(s).$$

Therefore, from the Frenet equation (2.2) we have the natural frame $\{x_s, x_t\}$ given by

$$x_s := \frac{\partial x}{\partial s} = k_0 X - k_1 tY + k_3 tZ, \quad x_t := \frac{\partial x}{\partial t} = Y.$$

Accordingly, the induced pseudo-Riemannian metric on M is obtained by

$$g(x_s, x_s) = 2k_0k_1t + (k_3t)^2, \quad g(x_s, x_t) = -k_0 \quad \text{and} \quad g(x_t, x_t) = 0.$$

Thus, using (2.3) we show that the Laplacian Δ of M can be expressed as ([4])

$$\Delta = \frac{2}{k_0}\frac{\partial^2}{\partial s \partial t} + \frac{2}{k_0^2}(k_3^2 t + k_0 k_1)\frac{\partial}{\partial t} + \frac{1}{k_0^2}(k_3^2 t^2 + 2k_0 k_1 t)\frac{\partial}{\partial t^2}. \tag{3.1}$$

The Beltrami equation $\Delta x = -2H$ gives

$$H(s,t) = -\Big(\frac{k_3}{k_0}\Big)^2 tY - \Big(\frac{k_3}{k_0}\Big)Z,$$

where $H(s,t)$ denotes the mean curvature vector field of M. To compute the Laplacian ΔH of the mean curvature vector field H, we need the following :

$$\frac{\partial H}{\partial t} = -\Big(\frac{k_3}{k_0}\Big)^2 Y,$$

$$\frac{\partial^2 H}{\partial s \partial t} = \Big\{\frac{-2k_0 k_3 k_3' + 2k_0' k_3^2}{k_0^3} + k_1\Big(\frac{k_3}{k_0}\Big)^2\Big\} Y - k_3\Big(\frac{k_3}{k_0}\Big)^2 Z,$$

$$\frac{\partial^2 H}{\partial t^2} = 0.$$

It follows from (3.1) that

$$\Delta H = -\frac{2}{k_0}\Big(\frac{k_3^2}{k_0^2}\Big)' Y + 2\Big(\frac{k_3}{k_0}\Big)^2 H. \tag{3.2}$$

We assume that the mean curvature vector field H of the null scroll M satisfies (1.2). Then, it is clear from (1.2) and (3.2) that

$$\Big(\frac{k_3}{k_0}\Big)^2 tAY + \Big(\frac{k_3}{k_0}\Big)AZ = 2\Big\{\frac{1}{k_0}\Big(\frac{k_3^2}{k_0^2}\Big)' + \Big(\frac{k_3}{k_0}\Big)^4 t\Big\} Y + 2\Big(\frac{k_3}{k_0}\Big)^3 Z$$

for the parameter t. Hence we can obtain

$$\Big(\frac{k_3}{k_0}\Big)^2 AY = 2\Big(\frac{k_3}{k_0}\Big)^4 Y, \tag{3.3}$$

$$\Big(\frac{k_3}{k_0}\Big)AZ = \frac{2}{k_0}\Big(\frac{k_3^2}{k_0^2}\Big)' Y + 2\Big(\frac{k_3}{k_0}\Big)^3 Z. \tag{3.4}$$

For simplicity, we put $k = \frac{k_3}{k_0}$. Differentiating (3.4), we get

(3.5)

$$k'AZ + k(AZ)' = 2k^3k_3X + \frac{1}{k_0^2}(4k_0k'^2 + 4kk_0k'' - 4kk_0'k' - 4k_0k_1kk' + 2k_0^2k_2k^3)Y + 10k^2k'Z.$$

On the other hand, the Frenet equation (2.2) gives

$$(AZ)' = AZ' = k_3AX + k_2AY,$$

from which together with (3.3), (3.4) and (3.5), we have

$$k_3k^2AX = BY + CZ + DX, \tag{3.6}$$

where we have put

$$B = \frac{1}{k_0^2}(4k^2k_0k'' - 4k^2k_0'k' - 4k_0k_1k^2k'), \quad C = 8k^3k', \quad D = 2k^4k_3.$$

Differentiating (3.6) and multiplying k_3kk' to the equation thus obtained, we have

(3.7)

$$\begin{aligned} k'k_3'k_3k^3AX + 2k_3^2k^2k'^2AX + k_3^2k^3k'(AX)' &= (kk'k_3^2C + kk'k_1k_3D + kk'k_3D')X \\ &\quad + (-kk'k_1k_3B + kk'k_2k_3C + kk'k_3B')Y \\ &\quad + (kk'k_3^2B + kk'k_2k_3D + kk'k_3C')Z. \end{aligned}$$

Multiplying $kk'k_3'$ and $2k_3k'^2$ to (3.6) respectively, we have the following equations

$$k'k_3'k_3k^3AX = kk'k_3'DX + kk'k_3'BY + kk'k_3'CZ, \tag{3.8}$$

and

$$2k_3^2k^2k'^2AX = 2k_3k'^2DX + 2k_3k'^2BY + 2k_3k'^2CZ. \tag{3.9}$$

Furthermore, from the Frenet equation (2.2) we have $(AX)' = AX' = k_1AX + k_2AZ$, which implies using (3.4) and (3.6)

(3.10)
$$k_3^2k^3k'(AX)' = k_1k_3kk'DX + (k_1k_3kk'B + 4k_2k_3k^4k'^2)Y + (k_1k_3kk'C + 2k_3^2k_2k^5k')Z.$$

Thus, by combining (3.7)-(3.10), we have
(3.11)
$$\begin{aligned}&\{kk'k_3^2C - (kk'k_3' + 2k_3k'^2)D + kk'k_3D'\}X\\ &+ \{(-2kk'k_1k_3 - kk'k_3' - 2k_3k'^2)B + kk'k_3B' + kk'k_2k_3C - 4k_3^2k_2k^4k'^2\}Y\\ &+ \{kk'k_3^2B - (kk'k_3' + 2k_3k'^2 + k_1k_3kk')C + kk'k_3C' - 2k_3^2k_2k^5k'\}Z = 0.\end{aligned}$$

Since X, Y and Z are linearly independent, (3.11) yields

$$kk'k_3^2C - (kk'k_3' + 2k_3k'^2)D + kk'k_3D' = 0, \tag{3.12}$$

(3.13)
$$(2kk'k_1k_3 + kk'k_3' + 2k_3k'^2)B - kk'k_2k_3C - kk'k_3B' + 4k_3^2k_2k^4k'^2 = 0,$$

$$kk'k_3^2B - (kk'k_3' + 2k_3k'^2 + k_1k_3kk')C + kk'k_3C' - 2k_3^2k_2k^5k' = 0. \tag{3.14}$$

From the equation (3.12) we can easily find $k^6k'^2 = 0$, that is, k is constant. Of course, in this case the equations (3.13) and (3.14) are automatically satisfied. On the other hand, the mean curvature $\alpha = \sqrt{|g(H,H)|}$ is $\frac{k_3}{k_0}$. Thus if the surface satisfies the condition (1.2), then the mean curvature α is constant. The converse is quite straightforward. Hence, we complete the proof.

Proof of Corollary. We know that the zero curvature k_0 of B-scroll is 1. By means of Theorem we can easily obtain the required result.

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