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A GENERALIZATION OF THE BATEMAN'S POLYNOMIAL $F_n(x)$

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ABSTRACT. We give a generalization of the Bateman's polynomial $F_n(x)$ and also give two generating functions for the generalized polynomial.

As noted in [3, p. 23], the polynomial $F_n(z)$ satisfies the two operational equations

$$F_n(D) \text{ sech } x = \text{ sech } x \; P_n(\tanh x),$$
$$F_n(D) x \text{ sech } x = \text{ sech } x \; Q_n(\tanh x),$$

in which D denotes the operator d/dx and $P_n(t)$, $Q_n(t)$ are the two standard solutions of Legendre's differential equation.

Bateman [3] obtained the generating relation

$$(1) \qquad (1-t)^{-1}{}_2F_1\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2}+\frac{1}{2}z; \\ & 1; \end{matrix} \; \frac{-4t}{(1-t)^2}\right] = \sum_{n=0}^{\infty} F_n(z)t^n$$

and the pure recurrence relation

$$(2) \qquad n^2 F_n(z) = -(2n-1)zF_{n-1}(z) + (n-1)^2 F_{n-2}(z)$$

together with numerous mixed relations involving a shift in argument as well as in index.

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From (1), Bateman [3, p. 24] deduced the following polynomial in an elementary way and studied it quite extensively

$$F_n(z) = {}_3F_2(-n, n+1, \frac{1}{2}(1+z); 1, 1; 1), \tag{3}$$

where ${}_pF_q$ denotes a generalized hypergeometric function with p numerators and q denominators, defined by

$${}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q;\end{matrix} z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_p)_n}{(\beta_1)_n \ldots (\beta_q)_n} \frac{z^n}{n!}, \tag{4}$$

where $(\lambda)_n = \lambda(\lambda+1)\ldots(\lambda+n-1) = \Gamma(\lambda+n)/\Gamma(\lambda)$. In (3), note that the variable z is contained in a parameter of the ${}_3F_2$, not in the argument. That the $F_n(z)$ form a simple set of polynomials should be apparent upon consideration of the nature of the terms in ${}_3F_2$.

In 1956 Touchard [6] introduced polynomials for which he did not give either an explicit formula or a generating relation. Later that year Wyman and Moser [7] obtained for Touchard's polynomials a finite sum formula and a generating function. Their generating function was equivalent to Bateman's (1) above. Carlitz [5] pointed out that Touchard's polynomials and Bateman's $F_n(z)$ are essentially the same, the former being

$$\frac{(-1)^n (n!)^2 F_n(1+2x)}{2^n (1/2)_n}.$$

Also in 1957, Brafman [4] obtained two generating functions for Touchard polynomials, and one of these is equivalent to that of Wyman and Moser and therefore to Bateman's (1) above. Brafman's other generating relation is useful contribution to the study of Bateman's $F_n(z)$:

$${}_1F_1\left(\frac{1}{2} - \frac{1}{2}z; 1; t\right) {}_1F_1\left(\frac{1}{2} + \frac{1}{2}z; 1; -t\right) = \sum_{n=0}^{\infty} \frac{F_n(z) t^n}{n!}. \tag{5}$$

In this note we are aiming at giving a generalization of the Bateman's polynomial $F_n(x)$ and also providing two generating functions for the generalized polynomial. For our purpose we first introduce some basic facts involved in $(\lambda)_n$ and ${}_pF_q$: For k and n being integers with $0 \le k \le n$,

$$(\alpha)_{n-k} = \frac{(-1)^k(\alpha)_n}{(1-\alpha-n)_k}, \tag{6}$$

in which setting $\alpha = 1$ yields

$$(n-k)! = \frac{(-1)^k n!}{(-n)_k}. \tag{7}$$

For $|z| < 1$, we have the binomial expansion:

$$(1-z)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n. \tag{8}$$

It is also well-known (see [1, p. 284, Ex. 23]) that, for $|z| < 1$ and $|z/(1-z)| < 1$, we have

$${}_2F_1\left[\begin{matrix} a, b; \\ c; \end{matrix} z\right] = (1-z)^{-a}{}_2F_1\left[\begin{matrix} a, c-b; \\ c; \end{matrix} \frac{-z}{1-z}\right]. \tag{9}$$

First, before we go to our main concern, we will deduce the polynomial (3) more systematically. From (1), using (8), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} t^n F_n(z) &= \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_k(\frac{1+z}{2})_k(-4)^k t^k (1-t)^{-(2k+1)}}{(k!)^2} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k(\frac{1+z}{2})_k(-4)^k (2k+1)_n}{(k!)^2 n!} t^{n+k} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k(\frac{1+z}{2})_k(-4)^k}{(k!)^2} \frac{(2k+1)_{n-k}}{(n-k)!} t^n,
\end{aligned}$$

from which equating the coefficients of t^n yields

$$F_n(z) = \sum_{k=0}^{n} \frac{(\frac{1}{2})_k(\frac{1+z}{2})_k(-4)^k}{(k!)^2} \frac{(2k+1)_{n-k}}{(n-k)!}. \tag{10}$$

Using the well-known identity

$$(\alpha)_{2n} = 2^{2n}\left(\frac{\alpha}{2}\right)_n\left(\frac{\alpha+1}{2}\right)_n,$$

we can deduce the following identity

$$(2k+1)_{n-k} = \frac{(1)_{n+k}}{(1)_{2k}} = \frac{n!(n+1)_k}{2^{2k}(1/2)_k k!}. \tag{11}$$

Setting (7) and (11) in (10) leads to the desired result (3).

Now consider the polynomials

$$\psi_n(c,x,y) = \frac{(-1)^n(\frac{1}{2}+\frac{1}{2}x)_n}{(c)_n}\,{}_3F_2\left[\begin{matrix} -n,\ \frac{1}{2}-\frac{1}{2}x,\ 1-c-n; \\ c,\ \frac{1}{2}-\frac{1}{2}x-n; \end{matrix}\, y\right]. \tag{12}$$

In [2, p. 14], we find the following identity:

$${}_3F_2\left[\begin{matrix} a,\ b,\ c; \\ e,\ f; \end{matrix}\, 1\right] = \frac{\Gamma(e)\Gamma(f)\Gamma(s)}{\Gamma(a)\Gamma(s+b)\Gamma(s+c)}\,{}_3F_2\left[\begin{matrix} e-a,\ f-a,\ s; \\ s+b,\ s+c; \end{matrix}\, 1\right],$$

where $s = e+f-a-b-c$. Setting $a = 1/2 - 1/2x, b = c = -n, e = 1$ and $f = 1/2 - 1/2x - n$ in the identity just above with the aid of (3) shows us that $\psi_n(1,x,1) = F_n(x)$, which implies that $\psi_n(c,x,y)$ is a generalization of the Bateman's $F_n(x)$.

Using (6) and (7), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{\psi_n(c,x,y)t^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-n)_k(\frac{1}{2}-\frac{1}{2}x)_k(1-c-n)_k}{k!(c)_k(\frac{1}{2}-\frac{1}{2}x-n)_k}\,\frac{(-1)^n(\frac{1}{2}+\frac{1}{2}x)_n}{(c)_n}\,\frac{y^k t^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\frac{1}{2}-\frac{1}{2}x)_k(\frac{1}{2}+\frac{1}{2}x)_{n-k}(-1)^{n+k}(yt)^k t^{n-k}}{k!(c)_k(c)_{n-k}(n-k)!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\frac{1}{2}-\frac{1}{2}x)_k(\frac{1}{2}+\frac{1}{2}x)_n(yt)^k(-t)^n}{k!(c)_k(c)_n n!}
\end{aligned}$$

$$= \left\{ \sum_{k=0}^{\infty} \frac{(\frac{1}{2} - \frac{1}{2}x)_k (yt)^k}{k!(c)_k} \right\} \left\{ \sum_{n=0}^{\infty} \frac{(\frac{1}{2} + \frac{1}{2}x)_n (-t)^n}{n!(c)_n} \right\}$$
$$= {}_1F_1\left(\frac{1}{2} - \frac{1}{2}x; c; yt\right) {}_1F_1\left(\frac{1}{2} + \frac{1}{2}x; c; -t\right),$$

from which we obtain the first generating function for $\psi_n(c, x, y)$:

$$(13) \quad {}_1F_1\left(\frac{1}{2} - \frac{1}{2}x; c; yt\right) {}_1F_1\left(\frac{1}{2} + \frac{1}{2}x; c; -t\right) = \sum_{n=0}^{\infty} \frac{\psi_n(c, x, y) t^n}{n!}.$$

For the next generating function for $\psi_n(c, x, y)$, making use of (6) and (7) again,

$$\sum_{n=0}^{\infty} \frac{(c)_n \psi_n(c, x, y) t^n}{n!}$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^n (\frac{1}{2} + \frac{1}{2}x)_n}{n!} \frac{(-n)_k (\frac{1}{2} - \frac{1}{2}x)_k (1 - c - n)_k}{k!(c)_k (\frac{1}{2} - \frac{1}{2}x - n)_k} y^k t^n$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n+k} (\frac{1}{2} - \frac{1}{2}x)_k (\frac{1}{2} + \frac{1}{2}x)_{n-k} (c)_n}{k!(c)_k (c)_{n-k} (n-k)!} y^k t^n$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^n (\frac{1}{2} - \frac{1}{2}x)_k (\frac{1}{2} + \frac{1}{2}x)_n (c)_{n+k}}{k!(c)_k (c)_n n!} y^k t^{n+k}$$
$$= \sum_{k=0}^{\infty} \frac{(\frac{1}{2} - \frac{1}{2}x)_k}{k!} (yt)^k \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2} + \frac{1}{2}x)_n (c+k)_n t^n}{n!(c)_n},$$

from which we get

$$(14) \quad \sum_{n=0}^{\infty} \frac{(c)_n \psi_n(c, x, y) t^n}{n!} = \sum_{n=0}^{\infty} \frac{(\frac{1}{2} - \frac{1}{2}x)_n}{n!} (yt)^n {}_2F_1\left[\begin{array}{r} \frac{1}{2} + \frac{1}{2}x,\ c+n; \\ c; \end{array} -t \right].$$

Using (9) we easily deduce the following:

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix}\frac{1}{2}+\frac{1}{2}x,\ c+n; \\ c;\end{matrix}\ -t\right] \\
&= (1+t)^{-\frac{1}{2}-\frac{1}{2}x}\,{}_2F_1\left[\begin{matrix}\frac{1}{2}+\frac{1}{2}x,\ -n; \\ c;\end{matrix}\ \frac{t}{1+t}\right].
\end{aligned} \tag{15}
$$

From (14) and (15), using (7) and (8), we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(c)_n\psi_n(c,x,y)t^n}{n!} \\
&= f(t)\sum_{n=0}^{\infty}\frac{(\frac{1}{2}-\frac{1}{2}x)_n}{n!}(yt)^n\,{}_2F_1\left[\begin{matrix}\frac{1}{2}+\frac{1}{2}x,\ -n; \\ c;\end{matrix}\ \frac{t}{1+t}\right] \\
&= f(t)\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\frac{1}{2}-\frac{1}{2}x)_n(\frac{1}{2}+\frac{1}{2}x)_k(yt)^n}{(n-k)!k!(c)_k}\left(-\frac{t}{1+t}\right)^k \\
&= f(t)\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\frac{1}{2}-\frac{1}{2}x)_{n+k}(\frac{1}{2}+\frac{1}{2}x)_k(yt)^{n+k}}{n!k!(c)_k}\left(-\frac{t}{1+t}\right)^k \\
&= f(t)\sum_{k=0}^{\infty}\frac{(\frac{1}{2}-\frac{1}{2}x)_k(\frac{1}{2}+\frac{1}{2}x)_k}{k!(c)_k}\left(-\frac{yt^2}{1+t}\right)^k\sum_{n=0}^{\infty}\frac{(\frac{1}{2}-\frac{1}{2}x+k)_n}{n!}(yt)^n \\
&= f(t)(1-yt)^{-\frac{1}{2}+\frac{1}{2}x}\sum_{k=0}^{\infty}\frac{(\frac{1}{2}-\frac{1}{2}x)_k(\frac{1}{2}+\frac{1}{2}x)_k}{k!(c)_k}\left(-\frac{yt^2}{(1-yt)(1+t)}\right)^k,
\end{aligned}
$$

where $f(t)=(1+t)^{-\frac{1}{2}-\frac{1}{2}x}$, from which we obtain the second generating function for $\psi_n(c,x,y)$:

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{(c)_n\psi_n(c,x,y)t^n}{n!} &= (1+t)^{-\frac{1}{2}-\frac{1}{2}x}(1-yt)^{-\frac{1}{2}+\frac{1}{2}x} \\
&\times {}_2F_1\left[\begin{matrix}\frac{1}{2}-\frac{1}{2}x,\ \frac{1}{2}+\frac{1}{2}x; \\ c;\end{matrix}\ \frac{-yt}{(1-yt)(1+t)}\right].
\end{aligned} \tag{16}
$$

References

[1] L. C. Andrews, *Special Functions for engineers and applied Mathematicians*, Macmillan Company, New York, 1985.

[2] W. N. Bailey, *Generalized Hypergeometric Series*, Cambridge at the University Press, 1935.

[3] H. Bateman, *Some properties of a certain set of polynomials*, Tôhoku Math. J. **37** (1933), 23-38.

[4] F. Brafman, *On Touchard polynomials*, Canadian J. Math. **9** (1957), 191-193.

[5] L. Carlitz, *Some polynomials of Touchard connected with the Bernoulli numbers*, Canadian J. Math. **9** (1957), 188-191

[6] J Touchard, *Nombres exponentiels et nombres de Bernoulli*, Canadian J. Math. **8** (1956), 305-320

[7] M. Wyman and L. Moser, *On some polynomials of Touchard*, Canadian J. Math. **8** (1956), 321-322.

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