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EQUIVALENCE RELATIONS OF DOMAINS OF HOLOMORPHY AND PSEUDOCONVEX DOMAINS

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1. Introduction

F. Hartogs discovered an example exhibiting the remarkable extension properties of holomorphic functions in more than one variable for the first time. Let $n \geq 2$, let $D \subset \mathbb{C}^n$ be an open set, and let K be a compact subset of D such that $D \setminus K$ is connected. Then, for every $h \in \mathcal{O}(D \setminus K)$ there exists $H \in \mathcal{O}(D)$ such that $H = h$ in $D \setminus K$. That is, for $n \geq 2$, there are examples of open set $D \subsetneq \tilde{D} \subset \mathbb{C}^n$ such that every holomorphic function in D admits a holomorphic extension to $\tilde{D}$.

In this paper, we introduce some of the elementary phenomena of domain of holomorphy in Section 2, and the pseudoconvexity followed by the early work of F. Hartogs and E. E. Levi in Section 3. We also discuss holomorphic convexity, an intrinsic global characterization of domains of holomorphy which was introduced in 1932 by H. Cartan and P Thullen. By constructing a real analytic strictly plurisubharmonic exhaustion function on a holomorphically convex domain, one can see directly that such a domain is pseudoconvex. The converse of this, the so called Levi Problem, is much harder. We prove it in Section 3 and for general case in Section 4.

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In Section 4, we introduce a class of complex analytic manifolds and analytic spaces, whose definition is modeled on the properties of domains of holomorphy in $\mathbb{C}^n$.

2. Domains of holomorphy

DEFINITION 2.1. (1) A holomorphic function f on D is *completely singular* at $p \in \partial D$ if for every connected neighborhood V of p there dose not exist $h \in \mathcal{O}(V)$ which agrees with f on some connected component of $V \cap D$.

(2) D is called a *domain of existence* (or sometimes it is called a *weak domain of holomorphy*) if for every $p \in \partial D$ there is $f_p \in \mathcal{O}(D)$ which is completely singular at p.

(3) D is called a *domain of holomorphy* if there exist $f \in \mathcal{O}(D)$ which is completely singular at every boundary point $p \in \partial D$.

DEFINITION 2.2. A function $f \in \mathcal{O}(D)$ is called *holomorphically extendible* (from a point $a \in D$) to a polydisk $P(a, \rho)$ if its Taylor series,

$$\sum_{\alpha \in \mathbb{N}^n} \frac{(D^\alpha f)(a)}{\alpha!}(z-a)^\alpha,$$

converges on $P(a, \rho)$; it is called holomorphically extendible at a point $b \in \mathbb{C}^n \setminus D$ if for some $a \in D$ the point b lies in a polydisk $P(a, \rho)$ to which f is holomorphically extendible.

DEFINITION 2.3. A domain $D \subset \mathbb{C}^n$ is called *holomorphically convex* if for each compact subset $K \subset D$, the holomorphically convex hull of K in D,

$$\hat{K}_{\mathcal{O}(D)} = \{z \in D : |f(z)| \leq \sup_K |f| = \|f\|_K, \forall f \in \mathcal{O}(D)\},$$

is compact in D.

DEFINITION 2.4. Let $D \subset \mathbb{C}^n$ be a domain and μ a distance function. Define

$$\mu_D(z) = \mu(z, \complement_D) = \inf_{w \in \complement_D} \mu(z - w).$$

If $X \subset D$ is a set, we write $\mu_D(X) = \inf_{x \in X} \mu_D(x)$.

THEOREM 2.5. *If $D \subset \mathbb{C}^n$ be an open set and $f \in \mathcal{O}(D)$ then the following are equivalent:*

(1) *D is holomorphically convex;*

(2) *For each sequence of points $(a_j)_{j\in\mathbb{N}}$ with no limit point in D, there exists an $f \in \mathcal{O}(D)$ such that $\sup_{j\in\mathbb{N}} |f(a_j)| = \infty$;*

(3) *D is a domain of holomorphy (or the point $p \in \partial D$ is essential);*

(4) *For each $z \in \mathbb{C}^n \setminus D$, there exists an $f \in \mathcal{O}(D)$ that is not holomorphically extendible at z;*

(5) *For each $a \in D$ and each polydisk $P(a,\rho) \not\subset D$, there exists an $f \in \mathcal{O}(D)$ that is not holomorphically extendible to $P(a,\rho)$;*

(6) *D is a domain of existence (or D is a weak domain of holomorphy);*

(7) *For any $f \in \mathcal{O}(D)$, $K \Subset D$ and any distance function μ, the inequality*

$$|f(z)| \le \mu_D(z),\ \forall z \in K \implies |f(z)| \le \mu_D(z),\ \forall z \in \hat{K}_{\mathcal{O}(D)};$$

(8) *For any $f \in \mathcal{O}(D)$, $K \Subset D$ and any distance function μ, we have*

$$\sup_{z\in K}\left\{\frac{|f(z)|}{\mu_D(z)}\right\} = \sup_{z\in \hat{K}_{\mathcal{O}(D)}}\left\{\frac{|f(z)|}{\mu_D(z)}\right\};$$

(9) *If $K \Subset D$, then for any distance function μ,*

$$\mu_D(K) = \mu_D(\hat{K}_{\mathcal{O}(D)});$$

(10) *Each $p \in \partial D$ has a neighborhood U_p such that $U_p \cap D$ is a domain of holomorphy;*

(11) *Each $p \in \partial D$ has a neighborhood U_p such that $U_p \cap D$ is holomorphically convex;*

(12) *For every infinite set $X \subset D$, which is discrete in D, there exists an $f \in \mathcal{O}(D)$ which is unbounded on X.*

PROOF. The detail proof of equivalent conditions may be found in G. M. Henkin [5], L. Kaup and B. Kaup [8] and S. G. Krantz [9].

3. Pseudoconvex domains

DEFINITION 3.1. Let $D \subset \mathbb{C}^n$, and let $f : D \to \mathbb{R} \cup \{-\infty\}$ be u.s.c.. We say that f is *plurisubharmonic* if for each complex line $l = \{a + b\zeta\} \subset \mathbb{C}^n$, the function $\zeta \longmapsto f(a + b\zeta)$ is subharmonic on $D_l = \{\zeta \in \mathbb{C} : a + b\zeta \in D\}$.

Let $PS(D)$ be the family of plurisubharmonic functions on D. Then $f \in PS(D)$ *if and only if* the complex Hessian of f is positive semi-definite at each point of D. That is,

$$\sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}(z) w_j \bar{w}_k \geq 0, {}^{\forall} z \in D, {}^{\forall} w \in \mathbb{C}^n.$$

A real-valued function $f \in C^2(D)$, $D \subset \mathbb{C}^n$ is called *strictly plurisubharmonic* if it has the Levi form:

$$\sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}(z) w_j \bar{w}_k > 0, {}^{\forall} z \in D, {}^{\forall} w (\neq 0) \in \mathbb{C}^n.$$

DEFINITION 3.2. (1) A function $\varphi : D \to \mathbb{R}$ on the open set D is called an *exhaustion function* for D if for every $c \in \mathbb{R}$ the set $D_c = \{z \in D : \varphi(z) < c\}$ is relatively compact in D.

(2) A domain $D \subset \mathbb{C}^n$ is called *L-pseudoconvex* (L originated with Lelong) if there is a C° plurisubharmonic exhaustion function.

(3) A domain $D \subset \mathbb{C}^n$ is called *G-pseudoconvex* (G originated with Grauert) if there is a C^∞ strictly plurisubharmonic exhaustion function.

DEFINITION 3.3. (1) Let $D \subset \mathbb{C}^n$ be a domain with C^2 boundary and let $p \in \partial D$. Let ρ be a C^2 defining function for D. We say that ∂D is *Levi pseudoconvex* at p, if

$$L_{\rho,p} = \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k \geq 0, {}^{\forall} w \in T_p(\partial D), \text{ and}$$

$$T_p(\partial D) = \{w \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(p) w_j = 0\}.$$

The expression on the left side is called the *Levi form*. The point p is said to be *strictly Levi pseudoconvex* if

$$L_{\rho,p} = \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p) w_j \bar{w}_k > 0, {}^{\forall} w (\neq 0) \in T_p(\partial D).$$

A domain $D \subset \mathbb{C}^n$ is called *Levi pseudoconvex* if all its boundary points are Levi pseudoconvex. We write it as *Le-pseudoconvex*.

(2) A domain $D \subset \mathbb{C}^n$ is called *Hartogs pseudoconvex* if there is a distance function μ such that $-\log \mu_D$ is plurisubharmonic on D. We write it as *H-pseudoconvex*.

(3) A domain $D \subset \mathbb{C}^n$ is called *D-pseudoconvex* (D originated with Distance) if $-\log \mu_D$ is plurisubharmonic on D for any distance function μ.

(4) A domain $D \subset \mathbb{C}^n$ is said to satisfy the *continuity principal* if for every family $\{S_\alpha : \alpha \in I\}$ of analytic discs in D the following implication holds;

$$\bigcup_{\alpha \in I} \partial S_\alpha \Subset D \implies \bigcup_{\alpha \in I} S_\alpha \Subset D.$$

Sometimes it is called *O-pseudoconvex* (O originated with Oka).

(5) A domain $D \subset \mathbb{C}^n$ is called *P-pseudoconvex* (P originated with plurisubharmonic) if for every compact set $K \subset D$, its plurisubharmonic convex hull,

$$\hat{K}_{PS(D)} = \{z \in D : u(z) \leq \sup_K u, {}^{\forall} u \in PS(D)\},$$

is relatively compact in D. Generally, we have $\hat{K}_{PS(D)} \subset \hat{K}_{\mathcal{O}(D)}$.

(6) A domain $D \subset \mathbb{C}^n$ is called *C-pseudoconvex* (C originated with Cartan) if each $p \in \partial D$ has an open neighborhood U such that $U \cap D$ is holomorphically convex.

DEFINITION 3.4. A domain $D \subset \mathbb{C}^n$ is called *pseudoconvex* if one of the equivalent condition in Theorem 3.5 is satisfied. A bounded domain $D \subset \mathbb{C}^n$ is called *strictly-pseudoconvex* if there are a neighborhood U of ∂D and a strictly.plurisubharmonic function $f \in C^2(U)$ such that $D \cap U = \{z \in U : f(z) < 0\}$.

THEOREM-3.5. *If $D \subset \mathbb{C}^n$ is an open set, then the following are equivalent:*

(1) *D is D-pseudoconvex, H-pseudoconvex, L-pseudoconvex, G-pseudoconvex, P-pseudoconvex, O-pseudoconvex (or D satisfy the continuity principle), Le-pseudoconvex and C-pseudoconvex;*
(2) *If μ is any distance function and if $d \subset D$ is any closed analytic disc, then $\mu_D(\partial d) = \mu_D(d)$;*
(3) *(2) is true for just one particular distance function;*
(4) *$D = \cup D_j$ where each D_j is H-pseudoconvex and $D_j \Subset D_{j+1}$;*
(5) *(4) is true except that each D_j is bounded strictly Le-pseudoconvex;*
(6) *The equation $\bar{\partial} u = f$ always has a solution*

$$u \in C^{\infty}_{(p,q)}(D), \quad {}^{\forall} f \in C^{\infty}_{(p,q+1)}(D), \bar{\partial} f = 0, q = 0, 1, \ldots, n-1.$$

PROOF. The detail proof of equivalent conditions may be found in L. Hörmander [7] and S. G. Kranz [9]

THEOREM 3.6 (LEVI PROBLEM). *Pseudoconvex is holomorphically convex.*

PROOF. By Theorem 3.5, D is G-pseudoconvex. That is, for every $c \in \mathbb{R}$, there exist $\varphi \in C^{\infty}(D)$ and strictly $PS(D)$ such that $D_c = \{z \in D : \varphi(z) < c\} \Subset D$. Let K be any compact set in D and let $c = \max \varphi(K) + 1$ then $K \subset D_c$. We only have to prove that $\hat{D}_{c\mathcal{O}(D)} \subset D_{c'}$. Then $\hat{K}_{\mathcal{O}(D)} \subset \hat{D}_{c\mathcal{O}(D)} \subset D_{c'} \Subset D$, ${}^{\forall} c'(> c) \in \mathbb{R}$. Hence, D is holomorphically convex.

4. Stein spaces

We introduce a class of complex analytic spaces (generally, manifolds $\subset$ reduced complex spaces $\subset$ complex spaces $\subset$ ringed spaces), whose definition is modeled on the properties of domains of a holomorphy in $\mathbb{C}^n$. We reprove the Levi problem in complex spaces.

DEFINITION 4.1. Complex space D is called a *Stein space* (or holomorphically complete space) if it satisfies the conditions:

(1) D is holomorphically convex.

(2) D is holomorphically separable. That is, for $x \neq y \in D$ there exists an $f \in \mathcal{O}(D)$ such that $f(x) \neq f(y)$.

(3) Every connected component of D has a countable topology. That is, there exists a countable basis of open sets.

DEFINITION 4.2. We call L a B-set in D, if for every coherent analytic sheaf $\mathcal{F}$ defined near L, and every $q \geq 1$, $H^q(L, \mathcal{F}) = 0$. Open B-set in D are also called *B-spaces*.

THEOREM 4.3. *(Exhaustion theorem) [8] The following statements about a complex space D are equivalent:*

(1) D is a B-space;

(2) There exists an exhaustion $D = \bigcup_{j=1}^{\infty} D_j$ with open B-sets $D_j \Subset D_{j+1}$ in D such that each (D_{j+1}, D_j) is a Runge pair.

THEOREM 4.4. *(Characterization of Stein spaces) [8] If $D \subset \mathbb{C}^n$ be a domain then the following are equivalent:*

(1) D is holomorphically convex;

(2) D is a Stein space;

(3) D is weakly holomorphically convex;

(4) $H^1(D, \mathcal{I}) = 0$ for every coherent ideal $\mathcal{I}$ in ${}_D\mathcal{O}$ such that the zero set $N(\mathcal{I})$ is discrete;

(5) D is a B-space;

(6) Every $a \in \partial D$ admits a neighborhood U in $\mathbb{C}^n$ such that $D \cap U$ is Stein;

(7) There exists an exhaustion $D = \bigcup_{j=1}^{\infty} D_j$ by open subsets $D_j \Subset D_{j+1} \Subset D$ that are Stein;

(8) Every hyperplane section H of D in $\mathbb{C}^n$ is Stein, and the induced restriction-homomorphism ${}_D\mathcal{O}(D) \to {}_H\mathcal{O}(H)$ is surjective;

(9) Every hyperplane section H of D in $\mathbb{C}^n$ is Stein, and every additive Cousin problem on D has a solution;

(10) For every complex line E in $\mathbb{C}^n$, the restriction-homomorphism ${}_D\mathcal{O}(D) \to {}_{D\cap E}\mathcal{O}(D \cap E)$ is surjective;

(11) The sequence

$$0 \to \mathcal{O}(D) \to \mathcal{E}^{0,0}(D) @ > \bar{\partial} >> \dots @ > \bar{\partial} >> \mathcal{E}^{0,n}(D) \to 0$$

is exact;

(12) $H^q(D, \mathcal{O}) = 0$ for $q = 1, \dots, n-1$;

(13) For $f_1, \dots, f_m \in \mathcal{O}(D)$ without common zeros, there exist $g_1, \dots, g_m \in \mathcal{O}(D)$ such that $1 = \sum_{j=1}^{m} f_j g_j$;

(14) The mapping $e : D \to S_p(D)$, $z \longmapsto \epsilon_z$, is surjective.

Now we reprove the Levi problem in complex spaces.

THEOREM 4.5. *Every pseudoconvex domain in $\mathbb{C}^n$ is holomorpically convex.*

PROOF. Fix a strictly pseudoconvex exhaustion $D = \cup D_j$ satisfying the following two conditions for each j:

(1) $D_j \Subset D_{j+1}$, and

(2) There exists a strictly plurisubharmonic function $\varphi_j \in C^\infty(D_{j+1}, \mathbb{R})$ such that the intersection of D_j with each connected component of D is a connected component of $\{\varphi_j < 0\}$.

Then $D_{j-1} = \{x \in D_j : \varphi_{j-1}(x) < 0\} \Subset D_j$, hence D_{j-1} is Stein. And if a strictly pseudoconvex domain in a Stein spsce D is of the form $U = \{x \in D : \varphi(x) < 0\}$ for an appropriate $\varphi \in \mathcal{C}(D, \mathbb{R})$ which is strictly subharmonic on D, then (D, U) is a Runge pair of Stein spaces. So, (D_j, D_{j+1}) is a Runge pair of Stein spaces. Thus, Theorem 4.3 implies that D is $B-$space and by Theorem 4.4, we have the result

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