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BOUNDARY DISTORTION OF CERTAIN DOMAINS IN $\overline{\mathbb{R}}^n$

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1. Introduction

Suppose that D is a domain in the extended complex plane $\overline{\mathbb{R}}^n$. For each $z_0 \in \mathbb{R}^n$ and $0 < r < \infty$, we let $B(z_0, r) = \{z \in \mathbb{R}^n : |z - z_0| < r\}$, $S(z_0, r) = \partial B(z_0, r)$, and $B^*(z_0, r) = \{z \in \overline{\mathbb{R}}^n : |z - z_0| > r\}$.

For non-empty sets A, $B \subset \overline{\mathbb{R}}^n$, $diam(A)$ is the diameter of A and $d(A, B)$ is the distance of A and B.

A domain D in $\overline{\mathbb{R}}^n$ is a K-*quasiball*, $1 \leq K < \infty$, if it is the image of the unit ball B(0,1) under a K- quasiconformal self mapping f of $\overline{\mathbb{R}}^n$. The boundary S of a K-quasiball D is called a K-*quasisphere*. For $n = 2$ we call the domain a quasidisk, and call the boundary of a K-quasidisk D a K-*quasicircle*([1, 6]).

Next we say that a Jordan curve C in $\overline{\mathbb{R}}^2$ has *circular distortion* c, $1 \leq c < \infty$, if for each Möbius transformation ϕ, either $\phi(C)$ separates the boundary circles of an annulus

$$A = A(z_0; r, s) = \{z \in \mathbb{R}^2 : r \leq |z - z_0| \leq s\}$$

with radii ratio $\frac{s}{r} = c$ or $\phi(C)$ contains the point ∞. The circular distortion is a Möbius invariant which measures how far a Jordan curve differs from being a circle or line. In particular, C has circular distortion 1 if and only if it is a circle or line.

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R. Kühnau established the following relation between these two concepts.

LEMMA 1.1. [5] *If C is a K-quasicircle in $\overline{\mathbb{C}}$, then C has circular distortion c, where c depends only on K.*

R. Kühnau found sharp bounds for the constant c in terms of K and then asked if the converse of lemma 1.1 is true, that is, if C is a curve with circular distortion c, then it is a K-quasicircle where K depends only on c. F. W. Gehring and C. Pommerenke [3] answered this question as follows.

LEMMA 1.2. [3] *If C is a Jordan curve in $\overline{\mathbb{R}}^2$ with circular distortion $c < \sqrt{2}$, then C is a K-quasicircle where K depends only on c.*

Their proof was based on elementary classical properties of the exterior conformal mapping $g : B^*(0,1) \to \operatorname{ext}(C)$, defined by

$$g(z) = z + \sum_0^\infty b_j z^{-j}.$$

Next lemma plays an important role in proving Lemma 1.2.

LEMMA 1.3. [3] *If C is a Jordan curve in $\overline{\mathbb{R}}^2$ which separates the boundary circles of an annulus A with radii ratio c and if g maps $B^*(0,1)$ onto $\operatorname{ext}(C)$, then*

$$|b_1| \leq \frac{c^2 - 1}{c^2 + 1}. \tag{1.4}$$

REMARK 1.5. [3] The mapping

$$g(z) = z + \frac{c-1}{c+1}\frac{1}{z}$$

shows that one can not replace the upper bound in (1.4) by anything less than $\frac{c-1}{c+1}$.

One of the main purposes of this paper is to give some other extreme examples for the constant $|b_1|$ in Lemma 1.3(see Section 2).

The bound $c < \sqrt{2}$ in Lemma 1.2 is not sharp [3]. While we looked for the sharp bound of c and another example of a Jordan curve C in $\overline{\mathbb{C}}$ with finite circular distortion c which is not a quasicircle, K. Kim found one more geometric condition, so-called the double disk property, which is in between a quasicircle and circular distortion [4].

We say that a topological n-sphere S in $\overline{\mathbb{R}}^n$ has *the double ball property* if there exists a constant b, $1 \leq b < \infty$, such that for each $z_0 \in S$ and $0 < r \leq \operatorname{diam}(S)$, there exist open balls B_i and B_e in $\mathbb{R}^n$ with

$$B_i \subset \operatorname{int}(S), \quad B_e \subset \operatorname{ext}(S), \quad B_i \cup B_e \subset B(z_0, r),$$

$$b \operatorname{diam}(B_i) \geq r, \quad b \operatorname{diam}(B_e) \geq r, \tag{1.6}$$

where int(S) and ext(S) are interior and exterior of S, respectively.

If we replace a topological n-sphere S in $\overline{\mathbb{R}}^n$ by a Jordan curve C in $\mathbb{R}^2$ and replace open balls by open disks, then we say that a Jordan curve C in $\mathbb{R}^2$ has the *the double disk property.*

It is also equivalent to asking that for $0 < r \leq \operatorname{diam}(S)$ (or $\operatorname{diam}(C)$), each point z of S (or C) should subtend balls (or disks) of a fixed visual angle in each complementary domain of S (or C) within distance r of z. Again the constant b in (1.6) measures how far S (or C) differs from being a n-sphere (or circle), respectively. In particular, $b = 1$ if and only if S (or C) is n-sphere (or circle), respectively.

In [4] K. Kim established relations between the double disk property, quasicircle and circular distortion.

LEMMA 1.7. [4] *If C is a K-quasicircle in $\mathbb{R}^2$, then C has the double disk property with constant b, where b depends only on K. If C has the double disk property with constant b, then C has circular distortion $c = 16\,b$.*

We say that a topological n-sphere S in $\overline{\mathbb{R}}^n$ has *spherical distortion c*, $1 \leq c < \infty$, if for each Möbius transformation ϕ, either $\phi(S)$ separates the boundary spheres of a spherical annulus

$$A = A(z_0; r, s) = \{z \in \mathbb{R}^n : r \leq |z - z_0| \leq s\}$$

with radii ratio $\frac{s}{r} = c$ or $\phi(S)$ contains the point ∞. It is well known fact that S is a 1-quasisphere if and only if S is a plane or n-sphere if and only if S has spherical distortion 1.

In Section 3, we give higher dimensional analogues of Lemmas 1.1 and 1.7 to the case of quasisphere as follows. If S is a K-quasisphere in $\overline{\mathbb{R}}^n$, then S has spherical distortion c, where c depends only on K. If S is a K-quasisphere in $\overline{\mathbb{R}}^n$ then S has the double ball property with constant b, where b depends only on K.

Next we say that a domain D in $\mathbb{R}^n$ is called an (α, β)*-John domain*, $0 < \alpha \leq \beta < \infty$, if there is $z_0 \in D$ such that for each $z \in D$, z has a rectifiable curve $\gamma : [0, \ell] \to D$, with arc length as parameter $\gamma(0) = z$, $\gamma(\ell) = z_0$, $\ell \leq \beta$, and $d(\gamma(t), \partial D) \geq \frac{\alpha}{\ell} t$, for all $t \in [0, \ell]$. We call z_0 a *John center* (see [7]).

A simply connected John domain D in $\mathbb{R}^2$ is called an (α, β)*-John disk*. John disks can be thought of "one-sided quasidisk" [9, 10]. For example, a Jordan domain in the plane is a quasidisk if and onlt if D and $D^* = \overline{\mathbb{R}}^2 \setminus \overline{D}$ are John disks [9]. In [4] K. Kim showed that a John disk satisfies the one-sided analogue of the double disks property of quasidisks.

LEMMA 1.8. [4] *If a Jordan curve C in $\mathbb{R}^2$ is the boundary of a (α, β)-John disk D, then there exists a constant b, $1 \leq b < \infty$, such that for each $w \in C$ and $0 < r \leq \operatorname{diam}(C)$, there is an open disk B with*

$$B \subset \operatorname{int}(C), \quad B \subset B(w, r), \quad b \operatorname{diam}(B) \geq r,$$

where b depends only on α and β.

In Section 4, we also give higher dimensional analogue of Lemma 1.8 to the case of John domain in $\mathbb{R}^n$.

2. Some extreme examples for the constant $|b_1|$ in lemma 1.3

EXAMPLE 2.1. *Suppose that C is a boundary curve of a simply connected domain $D = B(0,1) \setminus \{[-1, -\frac{1}{c}] \cup [\frac{1}{c}, 1]\}$, $1 \leq c < \infty$. Suppose also that f maps $B(0,1)$ conformally onto D with $f(0) = 0$, $f'(0) > 0$.*

Then the mapping

$$g(\zeta) = \frac{f'(0)}{f(\frac{1}{\zeta})}$$

shows that one can not replace the upper bound in (1.4) by anything less than $(\frac{c^2-1}{c^2+1})^2$.

PROOF. Let

$$h : \overline{\mathbb{R}}^2 \setminus [-1,1] \to \overline{\mathbb{R}}^2 \setminus [-\frac{1}{2}(c+\frac{1}{c}), \frac{1}{2}(c+\frac{1}{c})], \ \ h(w) = \frac{1}{2}(c+\frac{1}{c})w,$$

and let

$$k_1 : B(0,1) \to \overline{\mathbb{R}}^2 \setminus [-1,1], \ \ k_2 : D \to \overline{\mathbb{R}}^2 \setminus [-\frac{1}{2}(c+\frac{1}{c}), \frac{1}{2}(c+\frac{1}{c})],$$

$$k_i(w) = \frac{1}{2}(w+\frac{1}{w}), \ \ i = 1, 2.$$

Then g maps $B^*(0,1)$ onto $\mathrm{ext}(\frac{f'(0)}{f(C)})$ and $h \circ k_1 = k_2 \circ f$. Therefore

$$S_{h \circ k_1}(z) = S_{k_2 \circ f}(z)$$

for each $z \in B(0,1)$, where $S_f(z) = (\frac{f''}{f'})' - \frac{1}{2}(\frac{f''}{f'})^2$ is the Schwarzian derivative of f. Thus

$$S_h(k_1(z))(k_1'(z))^2 + S_{k_1}(z) = S_{k_2}(f(z))(f'(z))^2 + S_f(z).$$

Since $S_h(k_1(z)) = 0$, we get

$$\frac{-6}{(z^2-1)^2} = \frac{-6}{(f(z)^2-1)^2}(f'(z))^2 + S_f(z).$$

Hence

$$S_f(z) = \frac{-6}{(z^2-1)^2} + \frac{6}{(f(z)^2-1)^2}(f'(z))^2. \tag{2.2}$$

Since

$$f(z) = k_2^{-1} \circ h \circ k_1(z)$$
$$= \frac{1}{4}(c + \frac{1}{c})(z + \frac{1}{z}) - \sqrt{\frac{1}{4^2}(c + \frac{1}{c})^2 (z + \frac{1}{z})^2 - 1},$$

by L'Hospital's rule we have

$$\lim_{z \to 0} (\frac{f'(z)}{(f(z))^2 - 1})^2 = 4(\frac{c}{c^2 + 1})^2.$$

Thus by (2.2)

$$S_f(0) = -6 + 24(\frac{c}{c^2 + 1})^2.$$

Let $g(\zeta) = \zeta + \sum_0^\infty b_j \zeta^{-j}$. Then by [2] and [3],

$$b_1 = -\frac{1}{6} S_f(0) = (\frac{c^2 - 1}{c^2 + 1})^2.$$

Therefore one can not replace the upper bound in (1.4) by anything less than $(\frac{c^2-1}{c^2+1})^2$.

EXAMPLE 2.3. *Suppose that C is a boundary curve of a simply connected domain $D = B(0,1) \setminus [\frac{1}{c}, 1]$, $1 \le c < \infty$. Suppose also that f maps $B(0,1)$ conformally onto D with $f(0) = 0$, $f'(0) > 0$. Then the mapping*

$$g(\zeta) = \frac{f'(0)}{f(\frac{1}{\zeta})}$$

also shows that one can not replace the upper bound in (1.4) by anything less than $(\frac{c-1}{c+1})^2(1 + \frac{4c}{(c+1)^2})$.

PROOF. Let

$$h : \overline{\mathbb{R}}^2 \setminus [\frac{1}{4}, \infty) \to \overline{\mathbb{R}}^2 \setminus [\frac{c}{(c+1)^2}, \infty), \quad h(w) = \frac{4c}{(c+1)^2} w.$$

Let

$$k_1 : B(0,1) \to \overline{\mathbb{R}}^2 \setminus [\frac{1}{4}, \infty) \quad k_2 : D \to \overline{\mathbb{R}}^2 \setminus [\frac{c}{(c+1)^2}, \infty),$$

and

$$k_i(w) = \frac{w}{(1+w)^2}, \quad i = 1, 2.$$

Then g maps $B^*(0,1)$ onto $\mathrm{ext}(\frac{f'(0)}{f(C)})$ and $h \circ k_1 = k_2 \circ f$. Therefore

$$S_{h \circ k_1}(z) = S_{k_2 \circ f}(z)$$

for each $z \in B(0,1)$. With the same procedure as we have done for the identity (2.2) and the following equality,

$$\frac{f(z)}{(1+f(z))^2} = \frac{4c}{(c+1)^2} \frac{z}{(1+z)^2},$$

we obtain

$$f(z)(1+z)^2 = \frac{4c}{(c+1)^2} z(1+f(z))^2. \tag{2.4}$$

By (2.4) and by $f(0) = 0$, we get

$$\frac{f'(0)}{1-(f(0))^2} = \frac{4c}{(c+1)^2}.$$

Thus

$$S_f(0) = -6 + 6(\frac{4c}{(c+1)^2})^2.$$

Let $g(\zeta) = \zeta + \sum_0^\infty b_j \zeta^{-j}$. Then

$$\begin{aligned} b_1 &= -\frac{1}{6} S_f(0) \\ &= 1 - (\frac{4c}{(c+1)^2})^2 \\ &= (\frac{c-1}{c+1})^2 (1 + \frac{4c}{(c+1)^2}). \end{aligned}$$

Therefore one can not replace the upper bound in (1.4) by anything less than $(\frac{c-1}{c+1})^2(1+\frac{4c}{(c+1)^2})$.

REMARK 2.5. By Example 2.1, Example 2.3 and Remark 1.5 one can not replace the upper bound in (1.4) by anything less than $(\frac{c^2-1}{c^2+1})^2$, $(\frac{c-1}{c+1})^2(1+\frac{4c}{(c+1)^2})$ and $\frac{c-1}{c+1}$.

3. Quasisphere, the double ball property and spherical distortion

THEOREM 3.1. *If a topological n-sphere S is a K-quasisphere in $\overline{\mathbb{R}}^n$, then S has spherical distortion c, where c depends only on K.*

PROOF. Suppose that S is a K-quasisphere in $\overline{\mathbb{R}}^n$. Then there is a K-quaisiconformal self mapping f of $\overline{\mathbb{R}}^n$ which maps $S(0,1)$ onto S and $f(\infty)=\infty$. Let $w_0=f(0)$ and let

$$s=\max_{|z|=1}|f(z)-w_0|, \qquad r=\min_{|z|=1}|f(z)-w_0|.$$

Then $\frac{s}{r}\leq\lambda(K)$, where $\lambda(K)=\frac{1}{16}e^{\pi K}-\frac{1}{2}+O(e^{-\pi K})$[6]. Thus S separates boundary spheres of $A(w_0;r,cr)$, $c=\lambda(K)$. If ϕ is any Möbius transformation with $\phi(S)\subset\mathbb{R}^n$, then $g=\phi\circ f$ is also a K-quasiconformal mapping. Let $\zeta_0=\phi(w_0)$. We have an annulus $A(\zeta_0;s,cs)$ whose boundary spheres are separated by $\phi(S)$. Therefore S has spherical distortion c, where c depends only on K.

THEOREM 3.2. *If S is a K-quasisphere in $\overline{\mathbb{R}}^n$, then S has the double ball property with constant b, where b depends only on K.*

However the proof is similar to that of Lemma 1.7 in [4], but for the completeness we give the proof.

PROOF. Fix $z_0\in S$ and $0<r\leq\operatorname{diam}(S)$. By hypothesis, there exists a K- quasiconformal self mapping f of $\overline{\mathbb{R}}^n$ which maps S onto an n-sphere S'. By composing f with an auxiliary Möbius transformation we may further assume that $f(\infty)=\infty$ and hence

$$f(\operatorname{int}(S))=\operatorname{int}(S').$$

Let $w_0 = f(z_0)$, $B' = f(B(z_0, r))$ and let w_i, w_e and t_i, t_e denote the centers and radii of the largest balls in $B' \cap \operatorname{int}(S')$, $B' \cap \operatorname{ext}(S')$ which are tangent to S' at w_0, respectively Next set

$$z_i = g(w_i), \quad z_e = g(w_e),$$

where $g = f^{-1}$, and let

$$s_i = \max_{|w-w_i|=t_i} |g(w) - g(w_i)|, \quad r_i = \min_{|w-w_i|=t_i} |g(w) - g(w_i)|,$$

$$s_e = \max_{|w-w_e|=t_e} |g(w) - g(w_e)|, \quad r_e = \min_{|w-w_e|=t_e} |g(w) - g(w_e)|.$$

Then by [6, Theorem 9.3] we have

$$s_i \le \lambda(K)\, r_i, \quad s_e \le \lambda(K)\, r_e, \tag{3.3}$$

where $\lambda(K)$ is an increasing function of K with $\lambda(1) = 1$. Finally let $B_i = B(z_i, r_i)$ and $B_e = B(z_e, r_e)$. Then by (3.3)

$$\lambda(K)\operatorname{diam}(B_i) \ge 2\, s_i \ge r, \quad B_i \subset \operatorname{int}(S) \cap B(z_0, r),$$

$$\lambda(K)\operatorname{diam}(B_e) \ge 2\, s_e \ge r, \quad B_e \subset \operatorname{ext}(S) \cap B(z_0, r),$$

and hence B_i and B_e satisfy (1.6) with $b = \lambda(K)$.

4. John domains and the double ball property

LEMMA 4.1. [8] *Suppose that D in $\mathbb{R}^n$ is an (α, β)-John domain. If $0 < t \le \alpha$ and $z_0 \in \partial D$, then*

$$d(z, \partial D) \ge \frac{\alpha}{\beta} t$$

for some $z \in S(z_0, t) \cap D$.

THEOREM 4.2. *If a topological n-sphere S in $\mathbb{R}^n$ is the boundary of a (α,β)-John domain D, then there exists a constant b, $1 \le b < \infty$, and for each $w \in S$ and $0 < r \le \operatorname{diam}(S)$, there exists an open disk B with*

$$B \subset \operatorname{int}(S), \quad B \subset B(w,r), \quad b\operatorname{diam}(B) \ge r, \tag{4.3}$$

where b depends only on α and β.

However the proof is similar to that of Lemma 1.8 in [4], but for the completeness we give the proof.

PROOF. Let $z_0 \in S$. First we consider the case $0 < r \le \alpha \le \operatorname{diam}(S)$. Then by Lemma 4.1 with $t = \frac{r}{2}$,

$$d(z,S) \ge \frac{\alpha}{\beta}\cdot\frac{r}{2}$$

for some $z \in S(z_0, \frac{r}{2}) \cap D$. Hence there exists an open disk $B = B(z, \frac{\alpha}{2\beta}r)$ such that $B \subset D \cap B(z_0,r)$ and

$$\frac{2\beta}{\alpha}\operatorname{diam}(B) \ge r. \tag{4.4}$$

Secondly if $0 < \alpha \le r \le \operatorname{diam}(S)$, then by what we proved above, we can choose an open disk B_α for α such that $B_\alpha \subset D \cap B(z_0,\alpha)$ and $\frac{2\beta}{\alpha}\operatorname{diam}(B_\alpha) \ge \alpha$. Thus

$$B_\alpha \subset D \cap B(z_0,r), \qquad \operatorname{diam}(S)\,\frac{2\beta}{\alpha}\operatorname{diam}(B_\alpha) \ge r\alpha.$$

Since $\operatorname{diam}(S) \le 2\beta$, we have

$$(\frac{2\beta}{\alpha})^2 \operatorname{diam} B_\alpha \ge r. \tag{4.5}$$

Therefore by (4.4) and (4.5), we obtain (4.3) with $b = (\frac{2\beta}{\alpha})^2$.

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