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DUAL MIXED VOLUMES OF EQUICHORDAL CONVEX BODIES AND VOLUME PRODUCTS OF CENTROID BODIES

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ABSTRACT. We evaluate the dual quermassintegrals of r-equichordal bodies and obtain the lower bound for the volume product of p-centroid body in E^2.

1. Introduction

Equichordal points were first studied many years ago. Fujiwara[2] and Yanagihara[9] noted that there are noncircular planar convex bodies containing one equichordal point. Kelly[5] constructed a whole family of such examples with one equichordal point. The natural generalization, the i-equichordal points for arbitrary i, are first defined explicitly in [3], but are implicit in many earlier papers.

We evaluate the dual quermassintegrals of i-equichordal bodies.

The p-centroid body of a star body was defined by Lutwak and Zhang[7]. They proved that if K is a star body (about the origin) in E^n, then for $1 \leq p \leq \infty$,

$$V(K)V(\Gamma_p^* K) \leq \omega_n^2, \tag{1}$$

with equality if and only if K is an ellipsoid centered at the origin.

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We obtain the lower bound for the volume product of p-centroid body, not necessarily centered convex body, in E^2.

Our results is:
(i) Let K be an i-equichordal body with origin in E^n with constant c. Then

$$\tilde{W}_{n-i}(K) = \frac{c}{2}\omega_n, \quad i \in \{1, 2, \cdots, n\}.$$

(ii) Let K be a convex body in E^2 such that the center of approximating ellipsoid pair E_i and E_o is the origin. Then, for each real $p \geq 2$,

$$V(K)V(\Gamma_p^* K) \geq (\sqrt{2}-1)^4 \omega_2^2.$$

2. Preliminaries

A set E is said to be *centered* if $-x \in E$ whenever $x \in E$, and *centrally symmetric* if there is a vector c such that the translate $E - c$ of E by $-c$ is centered. In the latter case c is called a *center* of E.

By a *convex body* in E^n, $n \geq 2$, we mean a compact convex subset of E^n with nonempty interior. Let μ be a measure in E^n and E a bounded set in E^n of finite positive μ-measure. The *centroid* of E with respect to μ is the point

$$c = \frac{1}{\mu(E)} \int_E x d\mu(x).$$

Let S^{n-1} denote the unit sphere centered at the origin in E^n, and write O_{n-1} for the $(n-1)$-dimensional volume of S^{n-1}. Let B be the closed unit ball in E^n, write ω_n for the n-dimensional volume of B. Note that,

$$\omega_n = \pi^{n/2}/\Gamma(1 + \frac{n}{2}), \quad \text{and } O_{n-1} = n\omega_n.$$

For real $p \geq 1$, define $c_{n,p}$ by

$$c_{n,p} = \frac{\omega_{n+p}}{\omega_2 \omega_n \omega_{p-1}}.$$

For each direction $u \in S^{n-1}$, we define the *support function* $h(K,u)$ on S^{n-1} of the convex body K by

$$h(K,u) = \sup\{u \cdot x \mid x \in K\},$$

and the *radial function* $\rho(K,u)$ on S^{n-1} of the convex body K by

$$\rho(K,u) = \sup\{\lambda > 0 \mid \lambda u \in K\}.$$

If $\rho(K,\cdot)$ is positive and continuous, call K a *star body* (about the origin), and write $\mathcal{S}$ for the set of star bodies (about the origin) of E^n. Two star bodies $K, L \in \mathcal{S}$ are said to be dilated (of one another) if $\rho(K,u)/\rho(L,u)$ is independent of $u \in S^{n-1}$.

The *polar body* of a convex body K, denoted by K^*, is another convex body defined by

$$K^* = \{y \mid x \cdot y \leq 1 \text{ for all } x \in K\}.$$

It is easily verified that for convex bodies K_1, K_2 in E^n there is an implication

$$K_1 \subset K_2 \Longrightarrow K_2^* \subset K_1^*. \tag{2}$$

The polar body has the well known property that

$$h(K^*,u) = 1/\rho(K,u) \text{ and } \rho(K^*,u) = 1/h(K,u).$$

Let K_j be a convex body in E^n with $o \in K_j$, $1 \leq j \leq n$. Then we define the *dual mixed volume* $\tilde{V}(K_1, \cdots, K_n)$ by

$$\tilde{V}(K_1, \cdots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho(K_1,u) \cdots \rho(K_n,u)du, \tag{3}$$

where du signifies the area element on S^{n-1}. Let

$$\tilde{V}_i(K_1, K_2) = \tilde{V}(\underbrace{K_1, \cdots, K_1}_{n-i}, \underbrace{K_2, \cdots, K_2}_{i}).$$

If K is a convex body in E^n with $o \in K$, the *dual volume* $\tilde{V}_i(K)$ and *dual quermassintegral* $\tilde{W}_{n-i}(K)$ of K are defined by

$$\tilde{V}_i(K) = \tilde{W}_{n-i}(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^i du.$$

Thus $\tilde{V}_i(K) = \tilde{V}_i(B,K)$. When $i=0$, we have $\tilde{V}_0(K) = \kappa_n$, and when $i=n$, we have

$$\tilde{V}_n(L) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^n du = V(K), \tag{4}$$

according to the formula for volume in polar coordinates.

Let K be a convex body in E^n. A pair of n-dimensional ellipsoids (E_i, E_o) is called an approximating pair for K, if $E_i \subset K \subset E_o$ and if E_i and E_o are homothetic, that is, they have parallel axes and have the same aspect ratio. We measure the quality $\lambda(E_i, E_o)$ of our approximating pair (E_i, E_o) as λ of an expansion $x \mapsto \lambda(x - x_o) + x_o$ (with center x_o and expansion factoı λ).

3. i-equichordal bodies and dual quermassintegrals

Let $i \in R^+ = \{x \mid x > 0\}$ and K convex body in E^n with the origin. The i-chord function $\rho_i(K,u)$ is defined for $u \in S^{n-1}$ as follows :

$$\rho_i(K,u) = \rho(K,u)^i + \rho(K,-u)^i.$$

Suppose that K is a convex body in E^n and that there is a $c > 0$ such that the i-chord function of K has the constant value c. Then K is called an *i-equichordal body* with constant c.

THEOREM 1. *Let K be an i-equichordal body with the origin in E^n with constant c. Then*

$$\tilde{W}_{n-i}(K) = \frac{c}{2}\omega_n, \quad i \in \{1,2,\cdots,n\}.$$

PROOF. From the definition of the dual mixed volume, it follows that

$$\tilde{W}_{n-i}(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^i du = \frac{1}{n}\int_{S^{n-1}} \rho(K,-u)^i du.$$

So

$$\begin{aligned} 2\tilde{W}_{n-i}(K) &= \frac{1}{n}\int_{S^{n-1}} \left(\rho(K,u)^i + \rho(K,-u)^i\right) du \\ &= \frac{c}{n}\int_{S^{n-1}} du \\ &= c\omega_n. \end{aligned}$$

The proof is complete.

4. p-centroid bodies and volume products

For $K \in \mathcal{S}$ and real $p \geq 1$, the *p-centroid body*, $\Gamma_p K$, of K is the body whose support function is given by

$$c_{n-2,p} h(\Gamma_p K, x)^p = \frac{\omega_n}{V(K)}\int_{S^{n-1}} |x\cdot v|^p \rho(K,v)^{n+p} dv,$$

for all $x \in E^n$.

The Minkowski integral inequality shows that $h_{\Gamma_p K}$ is the support function of a (centered) convex body. Note that the polar of $\Gamma_p K$ is denoted by $\Gamma_p^* K$.

From the definition of the p-centroid body we see that for $K \in \mathcal{S}$ and $\phi \in GL(n)$, so $\Gamma_p \phi(K) = \phi(\Gamma_p K)$. Thus if E is a centered ellipsoid, then

$$\Gamma_p E = E \tag{5}$$

and for a star body K in E^n and a positive real number r

$$\Gamma_p(rK) = r\Gamma_p K. \tag{6}$$

We obtain the lower bound for the volume product of p-centroid body, not necessarily centered convex body, in E^2.

LEMMA 1. ([6]) *For every centrally symmetric convex body M and every (not necessarily centrally symmetric) convex body C in E^2, there are two concentric affine images a and A of M with $a \subset C \subset A$ and quality $1+\sqrt{2}$.*

Therefore we obtain the following special case.

REMARK 1. *In the above lemma, for eillipsoid E and every (not necessarily centrally symmetric) convex body K in E^2, there are two concentric ellipsoids E_i and E_o of E with $E_i \subset K \subset E_o$ and quality $1+\sqrt{2}$.*

THEOREM 2. *Let K be a convex body in E^2 such that, in the above remark, the center of E_i and E_o is the origin. Then, for each real $p \geq 2$,*

$$V(K)V(\Gamma_p^* K) \geq (\sqrt{2}-1)^4 \omega_2^2.$$

PROOF. By the definition of the p-centroid body,

$$h(\Gamma_p K, x) = \left(\frac{\omega_2}{c_{0,p}}\right)^{\frac{1}{p}} \left(\frac{1}{V(K)}\right)^{\frac{1}{p}} \left(\int_{S^1} |x \cdot v| \rho(K,v)^{p+2} dv\right)^{\frac{1}{p}}.$$

Let E_i and E_o be the approximating ellipsoids of K with quality $1+\sqrt{2}$. Then $E_0 \subset (1+\sqrt{2})K$. Since $V(E_0) \leq (1+\sqrt{2})^2 V(K)$,

$$\left(\frac{1}{V(K)}\right)^{\frac{1}{p}} \leq (1+\sqrt{2})^{\frac{2}{p}} \left(\frac{1}{V(E_o)}\right)^{\frac{1}{p}}. \tag{7}$$

Using $\rho(K,x) \leq \rho(E_o,x)$ and (7), then

$$h(\Gamma_p K, x) \leq (1+\sqrt{2})^{\frac{2}{p}} h(\Gamma_p E_o, x),$$

and so

$$\Gamma_p K \subset (1+\sqrt{2})^{\frac{2}{p}} \Gamma_p E_o.$$

Using (2), we get

$$\left(\frac{1}{1+\sqrt{2}}\right)^{\frac{2}{p}} \Gamma_p^* E_o \subset \Gamma_p^* K,$$

and using the relation that $V(\lambda K) = \lambda^2 V(K)$, we get

$$\left(\frac{1}{1+\sqrt{2}}\right)^{\frac{4}{p}} V(\Gamma_p^* E_o) \leq V(\Gamma_p^* K). \tag{8}$$

Using (8) and $V(E_i) \leq V(K) \leq V(E_o)$, we get

$$V(K)V(\Gamma_p^* K) \geq \left(\frac{1}{1+\sqrt{2}}\right)^{\frac{4}{p}} V(E_i)V(\Gamma_p^* E_o). \tag{9}$$

From $E_o = (1+\sqrt{2})E_i$, using (5) we get

$$\Gamma_p E_o = (1+\sqrt{2})\Gamma_p E_i,$$

and so

$$\frac{1}{1+\sqrt{2}}\Gamma_p^* E_i = \Gamma_p^* E_o.$$

So

$$V(\Gamma_p^* E_o) = (\sqrt{2}-1)^2 V(\Gamma_p^* E_i). \tag{10}$$

By substituting (10) for (9), and by the case of the equality of (1), we get

$$V(K)V(\Gamma_p^* K) \geq \left[(\sqrt{2}-1)^{\frac{2}{p}+1}\right]^2 \omega_2^2.$$

Therefore, for $p \geq 2$, we obtain the desired result.

REMARK 2. *Theorem 2 is the result of the special case $n = 2$. Y. D. Chai and author(Y. S. Lee), in [1], obtain the general n and quality α for a convex body.*

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