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SOME PROPERTIES ON DERIVATION IN NEAR-RINGS

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1. Introduction

Throughout this paper, N will denote a zero-symmetric left near-ring. A near-ring N is called a *prime near-ring* if N has the property that for $a,\ b \in N$, $aNb = \{0\}$ implies $a = 0$ or $b = 0$. N is called a *semiprime near-ring* if N has the property that for $a \in N$, $aNa = \{0\}$ implies $a = 0$. A nonempty subset U of N is called a *right N-subset* (resp. *left N-subset*) if $UN \subset U$(resp. $NU \subset U$), and if U is both a right N-subset and a left N-subset, it is said to be an *N-subset* of N. Every right ideal and right semigroup ideal of N are right N-subsets of N, and symmetrically, we can apply for left case. A *derivation D* on N is an additive endomorphism of N with the property that for all $a,\ b \in N$, $D(ab) = aD(b) + D(a)b$.

All other basic properties, terminologies and concepts are appeared in the book of G. Pilz [7].

2. Properties on derivation in near-rings

LEMMA 2.1. *Let D be an arbitrary additive endomorphism of N. Then $D(ab) = aD(b) + D(a)b$ if and only if $D(ab) = D(a)b + aD(b)$ for all $a,\ b \in N$.*

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PROOF. Suppose that $D(ab) = aD(b) + D(a)b$, for all $a, b \in N$. Since N satisfies left distributive law and $a(b+b) = ab+ab$, we have

$$\begin{aligned} D(a(b+b)) &= aD(b+b) + D(a)(b+b) \\ &= a(D(b) + D(b)) + D(a)b + D(a)b \\ &= aD(b) + aD(b) + D(a)b + D(a)b \end{aligned}$$

and

$$D(ab + ab) = D(ab) + D(ab) = aD(b) + D(a)b + aD(b) + D(a)b.$$

Comparing these two equalities, we have $aD(b) + D(a)b = D(a)b + aD(b)$. Hence $D(ab) = D(a)b + aD(b)$.

A similar argument proves that the converse holds.

LEMMA 2.2. *Let D be a derivation on N. Then N satisfies the following right distributive law, that is, for all a, b, c in N*

$$\{aD(b) + D(a)b\}c = aD(b)c + D(a)bc,$$

$$\{D(a)b + aD(b)\}c = D(a)bc + aD(b)c.$$

PROOF. From the calculation for $D((ab)c) = D(a(bc))$ and Lemma 2.1, we can induce our result.

LEMMA 2.3. *Let N be a prime near-ring and let U be a nonzero N-subset of N. If x is an element of N such that $Ux = \{0\}$ (or $xU = \{0\}$), then $x = 0$.*

PROOF. Since $U \neq \{0\}$, there exist an element $u \in U$ such that $u \neq 0$. Consider that $uNx \subset Ux = \{0\}$. Since $u \neq 0$ and N is a prime near-ring, we have $x = 0$.

COROLLARY 2.4. *Let N be a semiprime near-ring and let U be a nonzero N-subset of N. If x is an element of $N(U)$, the normalizer of U, such that $Ux^2 = \{0\}$ (or $x^2U = \{0\}$), then $x = 0$.*

THEOREM 2.5. *Let N be prime and U a nonzero N-subset of N. If D is a nonzero derivation on N. Then*

(i) If $a, b \in N$ and $aUb = \{0\}$, then $a = 0$ or $b = 0$.

(ii) If $a \in N$ and $D(U)a = \{0\}$, then $a = 0$.

(iii) If $a \in N$ and $aD(U) = \{0\}$, then $a = 0$.

PROOF. (i) Let $a, b \in N$ and $aUb = \{0\}$. Then $aUNb \subset aUb = \{0\}$. Since N is a prime near-ring, $aU = 0$ *or* $b = 0$.
If $b = 0$, then we are done. So if $b \neq 0$, then $aU = 0$. Applying Lemma 2.3 $a = 0$.

(ii) Suppose $D(U)a = \{0\}$, for $a \in N$. Then for all $u \in U$and $b \in N$, it follows from Lemma 2.2, that

$$0 = D(bu)a = (bD(u) + D(b)u)a = bD(u)a + D(b)ua = D(b)ua.$$

Hence $D(b)Ua = \{0\}$ for all $b \in N$.
Since D is a nonzero derivation on N, we have $a = 0$ by the statement (i).

(iii) Suppose $aD(U) = \{0\}$ for $a \in N$. Then for all $u \in U$ and $b \in N$,

$$0 = aD(ub) = a\{uD(b) + D(u)b\} = auD(b) + aD(u)b = auD(b).$$

Hence $aUD(b) = \{0\}$ for all $b \in N$.
From the statement (i) and the fact $D \neq 0$ on N, we get $a = 0$.

Any statement in Theorem 2.4 may not hold if we assume U is a right N-subset, even in the case when N is a ring.
Consider the following example:

EXAMPLE 2.6. *Let R be the prime ring $Mat_2(F)$, where F is an arbitrary field. Let $U = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} R$ and let D be the inner derivation of R given by*

$$D(w) = w \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} w.$$

Then $D(U) = \{\begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} R \mid a \in F\}$, and

$$xUy = xD(u) = D(U)x = \{0\}, \text{ where } x = y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

THEOREM 2.7. *Let N be a prime near-ring and U a right N-subset of N. If D is a nonzero derivation on N such that $D^2(U) = 0$, then $D^2 = 0$.*

PROOF. For all $u, v \in U$, we have $D^2(uv) = 0$. Thus

$$\begin{aligned} 0 = D^2(uv) &= D(D(uv)) = D\{D(u)v + uD(v)\} \\ &= D^2(u)v + D(u)D(v) + D(u)D(v) + uD^2(v) \\ &= D^2(u)v + 2D(u)D(v) + uD^2(v), \end{aligned}$$

and $2D(u)D(U) = \{0\}$ for all $u \in U$. From Lemma 2.5 (iii), we have $2D(U) = \{0\}$.

Now for all $b \in N$ and $u \in U$, $D^2(ub) = uD^2(b) + 2D(u)D(b) + D^2(u)b$. Hence $UD^2(b) = \{0\}$ for all $b \in N$. By Lemma 2.3, we have $D^2(b) = 0$ for all $b \in N$. Consequently $D^2 = 0$.

LEMMA 2.8. *Let D be a derivation of a prime near-ring N and a be an element of N. If $aD(x) = 0$ for all $x \in N$, then either $a = 0$ or D is zero.*

PROOF. Suppose that $aD(x) = 0$ for all $x \in N$. Replacing x by xy, we have that $aD(xy) = 0 = aD(x)y + axD(y)$ by Lemma 2.2. Then $axD(y) = 0$ for all $x, y \in N$. If D is not zero, that is, if $D(y) \neq 0$ for some $y \in N$, then, since N is a prime near-ring, $aN_D(y)$ implies that $a = 0$.

Now we prove our main result, which extends a famous theorem on rings of E.C. Posner [8] to near-rings with some condition.

THEOREM 2.9. *Let N be a prime near-ring of 2-torsion free and let D_1 and D_2 be derivations on N such that D_1D_2 is also a derivation on N with the condition $D_1(a)D_2(b) = D_2(b)D_1(a)$ for all $a, b \in N$. Then either $D_1 = 0$ or $D_2 = 0$.*

PROOF. Since D_1D_2 is a derivation, we have

$$D_1D_2(ab) = aD_1D_2(b) + D_1D_2(a)b. \tag{1}$$

Also, we get

$$\begin{aligned} D_1D_2(ab) &= D_1(D_2(ab)) = D_1(aD_2(b) + D_2(a)b) \\ &= D_1(aD_2(b)) + D_1(D_2(a)b) \\ &= aD_1D_2(b) + D_1(a)D_2(b) + D_2(a)D_1(b) \\ &\quad + D_1D_2(a)b. \end{aligned} \tag{2}$$

From (1) and (2),

$$D_1(a)D_2(b) + D_2(a)D_1(b) = 0 \text{ for all } a, b \in N. \tag{3}$$

Replacing a by $aD_2(c)$ in (3), and using Lemma 2.1 and Lemma 2.2, we obtain that

$$\begin{aligned} 0 &= D_1(aD_2(c))D_2(b) + D_2(aD_2(c))D_1(b) \\ &= \{D_1(a)D_2(c) + aD_1D_2(c)\}D_2(b) + \{aD_2^2(c) + D_2(a)D_2(c)\}D_1(b) \\ &= D_1(a)D_2(c)D_2(b) + aD_1D_2(c)D_2(b) + aD_2^2(c)D_1(b) + D_2(a)D_2(c)D_1(b) \\ &= D_1(a)D_2(c)D_2(b) + a\{D_1D_2(c)D_2(b) + D_2^2(c)D_1(b)\} \\ &\quad + D_2(a)D_2(c)D_1(b). \end{aligned}$$

From the last equality, we get

$$a\{D_1D_2(c)D_2(b) + D_2^2(c)D_1(b)\} = 0.$$

So we obtain that $D_1D_2(c)D_2(b) + D_2^2(c)D_1(b) = 0$ by replacing a by $D_2(c)$ in (3). Hence we have the following equality : for all $a, b, c \in N$,

$$D_1(a)D_2(c)D_2(b) + D_2(a)D_2(c)D_1(b) = 0. \tag{4}$$

Replacing a and b by c in (3) respectively, we see that

$$\begin{aligned} D_2(c)D_1(b) &= -D_1(c)D_2(b), \\ D_1(a)D_2(c) &= -D_2(a)D_1(c). \end{aligned}$$

So that (4) becomes

$$\begin{aligned} 0 &= \{-D_2(a)D_1(c)\}D_2(b) + D_2(a)\{-D_1(c)D_2(b)\} \\ &= D_2(a)(-D_1(c))D_2(b) + D_2(a)(-D_1(c))D_2(b) \\ &= D_2(a)\{(-D_1(c))D_2(b) - D_1(c)D_2(b)\} \text{ for all } a,b,c \in N. \end{aligned}$$

If $D_2 \neq 0$, then by Lemma 2.8, we have the equality :

$$(5) \qquad \begin{aligned} &(-D_1(c))D_2(b) - D_1(c)D_2(b) = 0, \text{ that is,} \\ &D_1(c)D_2(b) = (-D_1(c))D_2(b) \text{ for all } b,c \in N. \end{aligned}$$

On the other hand, using the given condition of our theorem,

$$(6) \qquad \begin{aligned} (-D_1(c))D_2(b) &= D_1(-c)D_2(b) = D_2(b)D_1(-c) \\ &= D_2(b)(-D_1(c)) = -D_2(b)D_1(c) \\ &= -D_1(c)D_2(b). \end{aligned}$$

From (5) and (6) we have that for all $b,c \in N$, $2D_1(c)D_2(b) = 0$. Since N is of 2-torsion free, $D_1(c)D_2(b) = 0$. Also, since D_2 is not zero, by Lemma 2.8, we see that $D_1(c) = 0$ for all $c \in N$.
Therefore $D_1 = 0$. Consequently, either $D_1 = 0$ or $D_2 = 0$. Thus our proof is complete.

As a consequence of Theorem 2.9, we get the following important statement :

COROLLARY 2.10. *Let N be a prime near-ring of 2-torsion free, and let D be a derivation on N such that $D^2 = 0$. Then $D = 0$.*

THEOREM 2.11. *Let N be a near-ring with derivation D such that $D^2 \neq 0$. Then every subnear-ring generated by D(N) contains a non-zero two sided N-subgroup of N.*

PROOF. This proof is immediately obtained by Y. U. Cho [5].

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