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ON FINSLER SPACES WITH (G, N)-STRUCTURES

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1. Introduction

Let M be a differentiable manifold and $T(M)$ its tangent bundle. We assume the zero-vector to be excluded from $T(M)$. A coordinate system (x^i) in M induces a canonical coordinate system (x^i, y^i) in $T(M)$. We put $\partial_k = \partial/\partial x^k$, $\dot{\partial}_k = \partial/\partial y^k$.

A positive-valued differentiable function $L(x, y)$ defined on a domain D of $T(M)$ is called a *Finsler metric* in M, if it satisfies

$$\det(g_{ij}) \neq 0, \quad \text{where} \quad g_{ij} = \dot{\partial}_j \dot{\partial}_i (L^2/2),$$
$$L(x, \lambda y) = \lambda L(x, y) \quad \text{for} \quad \lambda > 0.$$

According to Miron([9]), a Finsler tensor G_{ij} defined on a domain D of $T(M)$ is called a *generalized Finsler metric* in M, if it satisfies

$$G_{ij} = G_{ji}, \quad \det(G_{ij}) \neq 0, \quad G_{ij}(x, \lambda y) = G_{ij}(x, y) \quad \text{for} \quad \lambda > 0.$$

Now we introduce some concepts as follows: A differentiable manifold M is said to admit a (G, N)-*structure*, or simply called a (G, N)-*manifold*, if M admits a generalized Finsler metric G_{ij} and a non-linear connection $N^i{}_j$. A Finsler space $F^n = (M^n, L(x, y))$ is said to be conformal to another Finsler space $\overline{F}^n = (M^n, \overline{L}(x, y))$ if there exists a

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scalar field $\sigma(x)$ satisfying $\overline{L}(x,y) = e^{\sigma(x)}L(x,y)$, where L and $\overline{L}$ are positive-valued.

Here we deal with the conformal changes of a (G, N)-structure.

In the present paper, we are manily concerned with the (G, N)-structure where G_{ij} are a generalized Finsler metric. First, we find a condition that a Finsler space with a (G, N)-structure to be a Berwald space. Next, we obtain a condition for a (G, N)-structure to be a Minkowski space. Finally, we investigate a conformally invariant Finsler connection and several conformally invariant tensors of a generalized Finsler metric.

2. Preliminaries

Let M be a differentiable manifold admitting a (G, N)-structure, that is to say, M admits a generalized Finsler metric $G_{ij}(x,y)$ and a non-linear connection $N^i{}_j(x,y)$. We assume here that $G_{ij}(x,y)$ and $N^i{}_j(x,y)$ are positively homogeneous of degree 0 and 1 for y, respectively. Moreover, we put

$$L^*(x,y) = \sqrt{G_{ij}(x,y)y^iy^j}, \tag{2.1}$$

$$g^*{}_{ij} = \dot{\partial}_i\dot{\partial}_j(L^{*2}(x,y)/2). \tag{2.2}$$

And we assume

$$\det(g^*{}_{ij}) \neq 0. \tag{2.3}$$

A (G, N)-structure admitting (2.3) is called a regular (G, N)-structure.

Now we put

$$\begin{aligned} X_k &= \partial_k - N^m{}_k\dot{\partial}_m, \\ F_j{}^i{}_k &= G^{im}(X_jG_{mk} + X_kG_{jm} - X_mG_{jk})/2, \\ C_j{}^i{}_k &= G^{jm}(\dot{\partial}_jG_{mk} + \dot{\partial}_kG_{jm} - \dot{\partial}_mG_{jk})/2. \end{aligned} \tag{2.4}$$

The Finsler connection $N\Gamma$ given by (2.4) is called the (G, N)-*connection associated with the* (G, N)-*structure.*

We denote by ∇ and $\dot{\nabla}$ respectively the h- and v-covariant derivatives with respect to this (G, N)-connection. According to Matsumoto [7], we write the h-torsion and hv-torsion of the (G, N)-connection as

$$R^i{}_{jk} = X_k N^i{}_j - X_j N^i{}_k, \quad P^i{}_{jk} = \dot{\partial}_k N^i{}_j - F_k{}^i{}_j, \tag{2.5}$$

and the three kinds of curvatures as

$$\begin{cases} R_h{}^i{}_{jk} = X_k F_h{}^i{}_j - X_j F_h{}^i{}_k + F_m{}^i{}_k F_h{}^m{}_j \\ \qquad -F_m{}^i{}_j F_h{}^m{}_k + C_h{}^i{}_m R^m{}_{jk}, \\ P_h{}^i{}_{jk} = \dot{\partial}_k F_h{}^i{}_j - \nabla_j C_h{}^i{}_k + C_h{}^i{}_m P^m{}_{jk}, \\ S_h{}^i{}_{jk} = \dot{\partial}_k C_h{}^i{}_j - \dot{\partial}_j C_h{}^i{}_k + C_m{}^i{}_k C_h{}^m{}_j - C_m{}^i{}_j C_h{}^m{}_k. \end{cases} \tag{2.6}$$

Moreover we put

$$\begin{cases} K_h{}^i{}_{jk} = R_h{}^i{}_{jk} - C_h{}^i{}_m R^m{}_{jk} \\ \qquad = X_k F_h{}^i{}_j - X_j F_h{}^i{}_k + F_m{}^i{}_k F_h{}^m{}_j - F_m{}^i{}_j F_h{}^m{}_k, \\ Q_h{}^i{}_{jk} = \nabla_j C_h{}^i{}_k - C_h{}^i{}_m P^m{}_{jk} = \dot{\partial}_k F_h{}^i{}_j - P_h{}^i{}_{jk}. \end{cases} \tag{2.7}$$

It is obvious that the relations

$$\nabla_k G_{ij} = 0, \quad \dot{\nabla}_k G_{ij} = 0 \tag{2.8}$$

hold. Applying the Ricci identities to (2.8), we have

$$R_{ijkh} = -R_{jikh}, \quad P_{ijkh} = -P_{jikh}, \quad S_{ijkh} = -S_{jikh}. \tag{2.9}$$

Due to the second Bianchi identity, we have

$$\nabla_j C_{khi} - \nabla_k C_{jhi} + C_{jhr} P^r{}_{ki} - C_{khr} P^r{}_{ji} - P_{jhki} + P_{khji} = 0.$$

By virtue of (2.7), this identity can be rewritten as

$$Q_{khji} - Q_{jhki} - P_{jhki} + P_{khji} = 0. \tag{2.10}$$

Now applying the so-called Christoffel process [7] with respect to j, h, k to the above, we have

$$P_{hkji} = \nabla_h C_{kij} - \nabla_k C_{hij} + C_{hjr} P^r{}_{ki} - C_{kjr} P^r{}_{hi}.$$

Hence we obtain

$$P_{hkji} = Q_{kjhi} - Q_{jhki}, \tag{2.11}$$

where we put $Q_{jhki} = G_{hm}Q_j{}^m{}_{ki}$. The relations (2.11) and (2.8) lead us to

$$\dot{\partial}_k F_h{}^i{}_j = G^{im}(Q_{mjhk} - Q_{jhmk}) + Q_h{}^i{}_{jk}. \tag{2.12}$$

Moreover, by using the relation $Q_{hijk} = \nabla_j C_{khi} - C_{ihm}P^m{}_{jk}$ and (2.12), we can show easily

$$Q_{hijk} = Q_{ihjk}, \tag{2.13}$$

$$Q_h{}^k{}_{ji} = \frac{1}{2}(\dot{\partial}_i F_h{}^k{}_j + G^{km}G_{hr}\dot{\partial}_i F_m{}^r{}_j). \tag{2.14}$$

REMARK. We put $C_{hij} = G_{im}C_h{}^m{}_j$ for the (G, N)-connection $N\Gamma$;

$$C_{hij} = (\dot{\partial}_h G_{ij} + \dot{\partial}_j G_{hi} - \dot{\partial}_i G_{hj})/2. \tag{2.15}$$

Paying attention to $C_{hij} + C_{ihj} = \dot{\partial}_j G_{hi}$, we put

$$\overset{\circ}{C}_{hij} = \dot{\partial}_j G_{hi}/2, \quad \overset{\circ}{Q}_{hijk} = \nabla_j \overset{\circ}{C}_{hik} - \overset{\circ}{C}_{him}P^m{}_{jk}, \tag{2.16}$$

where $\overset{\circ}{C}_{hij} = (C_{hij} + C_{ihj})/2$.

3. Berwald space of a (G, N)-structure

Among Finsler manifolds, there exists such a special one as a Berwald space, which brings us fruitful results.

A Finsler manifold (M, L) is called a *Berwald space* or said to be *Berwald*, if the Berwald connection $B\Gamma = (\Gamma_j{}^i{}_k, N^i{}_j, 0)$ is linear. Since $B\Gamma$ satisfies $Q_h{}^i{}_{jk} = 0$, we have $P_h{}^i{}_{jk} = \dot{\partial}_k \Gamma_h{}^i{}_j$. Thus a Berwald space is characterized by $P_h{}^i{}_{jk} = 0$ with respect to $B\Gamma$.

For a Cartan connection $C\Gamma = (\Gamma_j{}^i{}_k, N^i{}_j, C_j{}^i{}_k)$, a Berwald space is defined as a Finsler manifold whose $C\Gamma$ is linear, and is characterized by

$$\nabla_k C_{hij} = 0, \quad (\text{or} \quad \nabla_k C_h{}^i{}_j = 0), \tag{3.1}$$

where $C_{hij} = g_{im} C_h{}^m{}_j = \dot\partial_j g_{hi}/2$. It is noted that the condition (3.1) is equivalent to

$$Q_{hijk} = 0, \quad (\text{or} \quad Q_h{}^i{}_{jk} = 0). \tag{3.2}$$

In fact, we have $P^i{}_{jk} = (\nabla_l C_j{}^i{}_k) y^l$ and $P^i{}_{jk} y^j = 0$ with respect to $C\Gamma$. (3.2) directly follows from (3.1). Conversely, contracting (3.2) with y^j we have $P^i{}_{jk} = 0$, which yields (3.1).

For (G,N)-structures a Berwlad space is defined as follows :

DEFINITION. In a (G,N)-structure, a Berwald space is defined as a generalized Finsler manifold whose $N\Gamma$ is linear, and is characterized by

$$\nabla_k \overset{\circ}{C}_{hij} = 0, \quad (\text{or} \quad \nabla_k \overset{\circ}{C}_h{}^i{}_j = 0). \tag{3.3}$$

Then we have the following theorem due to Ichijō[4].

THEOREM 3.1. *A (G,N)-structure is Berwald if and only if*

$$Q_{hijk} + Q_{ihjk} = 0. \tag{3.4}$$

PROOF. From (2.15) and (2.16) it is noted that the condition (3.4) is equivalent to

$$\overset{\circ}{Q}_{hijk} = 0. \tag{3.5}$$

For the (G,N)-connection $N\Gamma = (F_j{}^i{}_k, N^i{}_j, C_j{}^i{}_k)$, we associate the corresponding C-zero connection $N\Gamma' = (F_j{}^i{}_k, N^i{}_j, 0)$, and apply to G_{hi} one of the Ricci identities:

$$\begin{aligned} \nabla_j(\dot\nabla_k G_{hi}) - \dot\nabla_k(\nabla_j G_{hi}) &= G_{mi} P_h{}^m{}_{jk} + G_{hm} P_i{}^m{}_{jk} \\ &\quad + (\nabla_m G_{hi}) C_j{}^m{}_k + (\dot\nabla_m G_{hi}) P^m{}_{jk}. \end{aligned} \tag{3.6}$$

For $N\Gamma'$ we have $\dot\nabla_h = \dot\partial_h$ and $P_h{}^i{}_{jk} = \dot\partial_k F_h{}^i{}_j$, whereas ∇_h and $P^i{}_{jk}$ are unchanged, so we have from $\nabla_j G_{hi} = 0$ and (2.16) that

$$\mathring{Q}_{hijk} = (G_{mi}\dot\partial_k F_h{}^m{}_j + G_{hm}\dot\partial_k F_i{}^m{}_j)/2. \tag{3.7}$$

If $N\Gamma$ is linear, then $\dot\partial_h F_j{}^i{}_k = 0$. So we have (3.5), i.e., (3.4). If we apply to (3.7) the so-called Christoffel process with respect to the indices h, i, j, the converse follows from

$$G_{mj}\dot\partial_k F_i{}^m{}_h = \mathring{Q}_{hijk} + \mathring{Q}_{ijhk} - \mathring{Q}_{jhik}. \tag{3.8}$$

Thus we have proved Theorem 3.1. This proof also shows that a Finsler manifold (M, L) is Berwlad if and only if (3.2), i.e., (3.1) holds.

Next, let us consider a regular Berwaldian (G, N)-structure. In this case, we have $\dot\partial_k F_j{}^i{}_k = 0$. Since $X_k G_{ir} y^i y^r = \partial_k G_{ir} y^i y^r$, we have $G^i (= \gamma_0{}^i{}_0) = F_0{}^i{}_0$. From this, we have $G_j{}^i{}_k = \frac{1}{2}\dot\partial_j \dot\partial_k G^i = F_j{}^i{}_k(x)$. Thus L^* is a Berwald metric. Moreover $\mathring{Q}_h{}^i{}_{jk} = 0$, that is, $\nabla_k \mathring{C}_{ijr} = \mathring{C}_{ijm} P^m{}_{kr}$. Hence from (3.3) we see that

$$\mathring{C}_{ijm} P^m{}_{k0} = \nabla_k \mathring{C}_{ijr} y^r = 0.$$

Conversely, we assume that L^* is a Berwald metric and $\mathring{C}_{ijm} P^m{}_{k0} = 0$ holds. Since $P^i{}_{k0} = N^i{}_k - F_0{}^i{}_k$, we have $\mathring{C}_{ijm} N^m{}_k = \mathring{C}_{ijm} F_0{}^m{}_k$, from which $X_k G_{ij} = \partial_k G_{ij} - 2\mathring{C}_{ijm} F_0{}^m{}_k$. Thus we have

$$F_j{}^i{}_k = \gamma_j{}^i{}_k - G^{ir}(\mathring{C}_{mkr} F_0{}^r{}_j + \mathring{C}_{jmr} F_0{}^r{}_k - \mathring{C}_{jkr} F_0{}^r{}_m),$$

from which we see

$$F_0{}^i{}_k = \gamma_0{}^i{}_k - \mathring{C}_k{}^i{}_r F_0{}^r{}_0,\ F_0{}^i{}_0 = \gamma_0{}^i{}_0 \text{ and } F_0{}^i{}_k = \gamma_0{}^i{}_k - \mathring{C}_k{}^i{}_r \gamma_0{}^r{}_0.$$

The last relation leads us to $F_j{}^i{}_k = \Gamma^{*i}_j{}_k$ (Cartan). Form our assumption, we have $\Gamma^{*i}_j{}_k = \Gamma^{*i}_j{}_k(x)$. Therefore we have $F_j{}^i{}_k = F_j{}^i{}_k(x)$. Consequently, we obtain the following;

THEOREM 3.2. *A regular (G,N)-structure is a Berwaldian (G,N)-structure if and only if the generalized Finsler metric G_{ij} is a Berwald metric and the (G,N)-connection satisfies $\overset{\circ}{C}_{ijm}P^m{}_{k0}=0$.*

4. Minkowski space of a (G,N)-structure

In this section, we shall deal with the notion of a locally Minkowski space, or a local flatness on a manifold admitting a (G,N)-structure.

Now we consider the case of $\dot{\partial}_k F_h{}^i{}_j = 0$, that is, $\Gamma_h{}^i{}_j = \Gamma_h{}^i{}_j(x)$. Thus, from (2.6) and (2.7), we have $P_{hijk} = -Q_{hijk}$. Further, from (2.9), we obtain

$$Q_{hijk}+Q_{ihjk}=0. \tag{4.1}$$

Conversely, we suppose that (4.1) holds. Applying the so-called Christoffel process with respect to k, h and j to (2.10), and using (2.9), we have

$$2P_{jhki}=Q_{khji}+Q_{hjki}+Q_{hkji}-Q_{jhki}-Q_{kjhi}-Q_{jkhi}.$$

From our assumption (4.1), this equation is reduced to $P_{jhki}=-Q_{jhki}$. Hence, from (2.6), we obtain $\dot{\partial}_k F_h{}^i{}_j = 0$.

Thus we have the following:

THEOREM 4.1. *With respect to the Finsler connection associated with a (G,N)-structure, $\dot{\partial}_h F_j{}^i{}_k = 0$ holds good if and only if $\overset{\circ}{Q}_{ihjk}=0$, where $\overset{\circ}{Q}_{ihjk}=Q_{ihjk}+Q_{hijk}$.*

DEFINITION. Let M be a manifold admitting a (G,N)-structure such that for any point p of M, there exists a coordinate neighborhood (U,x^i) containing p.

We call the (G,N)-structure flat if it satisfied the condition

(1) $X_k G_{ij} = 0$, and strongly flat if it satisfies the conditions (2) $\partial_k G_{ij}=0$, $N^m{}_k \overset{\circ}{C}_{ijm}=0$ on U.

First, let M be a manifold admitting a flat (G,N)-structure. Then M is covered by a system of local coordinate neighborhoods $\{(U,x^i)\}$ such that, in each U, $\partial_k G_{ij} - N^m{}_k \dot{\partial}_m G_{ij} = 0$ holds good. Hence we know that $F_j{}^i{}_k = 0$ holds in each U. From (2.7) and Theorem 4.1, we obtain $K_h{}^i{}_{jk} = 0$ and $\overset{\circ}{Q}_{ihjk} = 0$.

Conversely, we assume that $K_h{}^i{}_{jk} = 0$ and $\overset{\circ}{Q}_{ihjk} = 0$ hold good on M. Due to Theorem 4.1, $F_j{}^i{}_k = F_j{}^i{}_k(x)$ on M. And $K_h{}^i{}_{jk}$ gives us that M is covered by a system of local coordinate neighborhoods such that $F_j{}^i{}_k = 0$ holds on each U. Thus the facts that $F_j{}^i{}_k = 0$ and $\nabla_k G_{ij} = 0$ lead us to $X_k G_{ij} = 0$ on each U, that is, the given (G,N)-structure is flat. Thus we have the following

THEOREM 4.2. *A (G,N)-structure is flat if and only if*

$$K_h{}^i{}_{jk} = 0, \quad \overset{\circ}{Q}_{ihjk} = 0 \tag{4.2}$$

hold good. In this case, $F_j{}^i{}_k$ is a symmetric, flat linear connection on the manifold M.

Next, assume the (G,N)-structure be strongly flat. Then the (G,N)-structure is obviously flat. By virtue of Theorem 4.2, we see that $K_h{}^i{}_{jk} = 0$, $\overset{\circ}{Q}_{ihjk} = 0$. With respect to the each coordinate neighborhood (U, X^i) which assigns the strongly flatness of the given (G,N)-structure, we obtain $N^m{}_i \dot{\partial}_m G_{jk} = 0$. On the other hand, we find from (2.4) that $\dot{\partial}_m G_{jk} = C_{jkm} + C_{kjm}$, that is, $\overset{\circ}{C}_{jkm} = \dot{\partial}_m G_{jk}/2$. From (2.6), we find also that $P^i{}_{jr} y^r = N^i{}_j - F_r{}^i{}_j y^r$. Since $F_j{}^i{}_k = 0$ in U, we have $P^i{}_{j0} = N^i{}_j$. Therefore we get $\overset{\circ}{C}_{jkm} P^m{}_{i0} = 0$.

Conversely, we suppose that $K_h{}^i{}_{jk} = 0$, $\overset{\circ}{Q}_{hijk} = 0$, $\overset{\circ}{C}_{jkm} P^m{}_{i0} = 0$ hold good. By virtue of Theorem 4.2, we see that the (G,N)-structure is flat. Hence, with respect to the assigned coordinate neighborhood U of the flatness, $X_k G_{ij} = 0$, from which $F_j{}^i{}_k = 0$. Thus $P^i{}_{j0} = N^i{}_j$ holds in this U. From $\dot{\partial}_m G_{jk} = 2\overset{\circ}{C}_{jkm}$ and $\overset{\circ}{C}_{jkm} P^m{}_{i0} = 0$, we obtain $N^m{}_i \dot{\partial}_m G_{jk} = 0$ in U. Since $X_k G_{ij} = 0$, $\partial_k G_{ij} = 0$ is also true in this U, that is, the given (G,N)-structure is strongly flat. Thus we have the following

THEOREM 4.3. *A regular (G, N)-structure is strongly flat if and only if*

$$(4.7) \qquad K_h{}^i{}_{jk} = 0, \quad \overset{\circ}{Q}_{ihjk} = 0, \quad \overset{\circ}{C}_{jkm} P^m{}_{i0} = 0$$

hold good.

Moreover, if a regular (G, N)-structure is flat, M is covered by a system of local coordinate neighborhoods $\{(U, x^i)\}$ such that, in each U, $X_k G_{ij} = 0$, holds, that is, $\partial_k G_{ij} - N^m{}_k \dot{\partial}_m G_{ij} = 0$ holds. On transvecting this with $y^i y^j$, we have $\partial_k L^{*2} = 0$ where $L^{*2} = G_{ij} y^i y^j$, from which we find that L^* is a locally Minkowski metric and $\partial_k G_{ij} = 0$ in each U. Thus we have $\overset{\circ}{C}_{ijm} N^m{}_k = 0$. On the other hand, we see that $F_j{}^i{}_k = 0$ in each U. Hence, from $P^m{}_{jk} = \dot{\partial}_k N^m{}_j - F_k{}^m{}_j$; we see that $N^m{}_k = P^m{}_{k0}$ in each U. Consequently we find $\overset{\circ}{C}_{ijm} P^m{}_{k0} = 0$ holds on M.

Conversely, we assume that L^* is a locally Minkowski metric and $\overset{\circ}{C}_{ijm} P^m{}_{k0} = 0$ holds. Then, M is covered by a system of local coordinate neighborhoods $\{(U, x^i)\}$ such that $\partial_k G_{ij} = 0$ holds in each U. In this case, $X_k G_{ij} = -2\overset{\circ}{C}_{ijr} N^r{}_k$ holds. And we have $F_j{}^i{}_k = -\overset{\circ}{C}_r{}^i{}_k N^r{}_j - \overset{\circ}{C}_r{}^i{}_j N^r{}_k + G^{im} \overset{\circ}{C}_{rjk} N^r{}_m$. On the other hand, the condition $\overset{\circ}{C}_{ijm} P^m{}_{k0} = 0$ is equivalent to $\overset{\circ}{C}_{ijr} N^r{}_k = \overset{\circ}{C}_{ijr} F_0{}^r{}_k$. Hence, we have

$$F_j{}^i{}_k = -\overset{\circ}{C}_r{}^i{}_k F_0{}^r{}_j - \overset{\circ}{C}_r{}^i{}_j F_0{}^r{}_k + G^{im} \overset{\circ}{C}_{rjk} F_0{}^r{}_m.$$

By transvecting this with y^j, we have $F_0{}^i{}_k = -\overset{\circ}{C}_r{}^i{}_k F_0{}^r{}_0$, form which we have $F_0{}^i{}_0 = 0$ and $F_0{}^r{}_k = 0$. Thus we have $\overset{\circ}{C}_{ijr} N^r{}_k = 0$. Hence, in each (U, x^i) above, $X_k G_{ij} = 0$ holds. Namely, the given (G, N)-structure is flat. Thus we have the following

THEOREM 4.4 *A regular (G, N)-structure is strongly flat if and only if the generalized Finsler metirc L^* is a locally Minkowski metric and the (G, N)-connection satisfies $\overset{\circ}{C}_{ijm} P^m{}_{k0} = 0$, where we put $P^m{}_{k0} = P^m{}_{kr} y^r$.*

Moreover, from our assumption, the manifold is covered by a system of local coordinate neighborhoods $\{(U, x^i)\}$ such that, in each U, $\partial_k G_{ij} = 0$ and henceforth $\partial_h \overset{\circ}{C}_{ijk} = 0$ hold. In addition to this, from the proof of Theorem 4.3, we see that $\overset{\circ}{C}_{ijm} N^m{}_k = 0$ in each U. So, we have $\overset{\circ}{C}_{ijm} \partial_h N^m{}_k = 0$. Hence we see

$$\begin{aligned}\overset{\circ}{C}_{ijm} R^m{}_{hk} &= -\overset{\circ}{C}_{ijm} N^r{}_k \dot{\partial}_r N^m{}_h + \overset{\circ}{C}_{ijm} N^r{}_h \dot{\partial}_r N^m{}_k \\ &= N^r{}_k \dot{\partial}_r \overset{\circ}{C}_{ijm} N^r{}_h - N^r{}_h \dot{\partial}_r \overset{\circ}{C}_{ijm} N^m{}_k \\ &= 0.\end{aligned}$$

Thus we have the following:

THEOREM 4.5. *If a regular (G, N)-structure is strongly flat, then the (G, N)-connection always satisfies $\overset{\circ}{C}_j{}^i{}_m R^m{}_{hk} = 0$.*

5. Conformal changes of a (G, N)-structure

Let M be a differentiable manifold admitting a (G, N)-structure and let $\sigma(x)$ be a scalar field on M. Then $\overline{G}_{ij} = e^{2\sigma(x)} G_{ij}(x, y)$ is also a generalized Finsler metric. Here we consider the $(\overline{G}, N)$-structure defined on M. The $(\overline{G}, N)$-structure is called a *conformal changes of a (G, N)-structure.*

In this section, we deal with the conformal changes of the (G, N)-structure where G_{ij} is a generalized Finsler metric in M. Paying attention to well-known Deicke's theorem, we assume more strictly that $C = \sqrt{C_m C^m} \neq 0$ where $C_i = C_i{}^m{}_m$. We have easily obtained that, as for the $(\overline{G}, N)$-structure where $\overline{G}_{ij} = e^{2\sigma(x)} G_{ij}$, the following relations hold:

$$
(5.1)\quad
\begin{aligned}
&(a)\quad \overline{G}_{ij} = e^{2\sigma(x)}G_{ij}, \quad \overline{G}^{ij} = e^{-2\sigma(x)}G^{ij},\\
&(b)\quad \overline{R}^i{}_{jk} = R^i{}_{jk},\\
&(c)\quad \overline{C}_j{}^i{}_k = C_j{}^i{}_k,\\
&(d)\quad \overline{F}_j{}^i{}_k = F_j{}^i{}_k + \sigma_j\delta^i_k + \sigma_k\delta^i_j - \sigma^i G_{jk},\\
&(e)\quad \overline{P}^i{}_{jk} = P^i{}_{jk} - \sigma_j\delta^i_k - \sigma_k\delta^i_j + \sigma^i G_{jk},\\
&(f)\quad \overline{P}_h{}^i{}_{jk} = P_h{}^i{}_{jk} + \sigma_r C_m{}^r{}_k G_{hj}G^{im} - G^{ir}\sigma_r G_{hm}C_j{}^m{}_k\\
&\qquad\qquad + \sigma_h C_j{}^i{}_k - \sigma_m C_h{}^m{}_k\delta^i_j,\\
&(g)\quad \overline{K}_h{}^i{}_{jk} = K_h{}^i{}_{jk} + \delta^i_j\sigma_{kh} - \delta^i_k\sigma_{jk} - G_{hj}\sigma^i{}_k + G_{hk}\sigma^i{}_j,\\
&(h)\quad \overline{R}_h{}^i{}_{jk} = R_h{}^i{}_{jk} + \delta^i_j\sigma_{kh} - \delta^i_k\sigma_{jh} - G_{hj}\sigma^i{}_k + G_{hk}\sigma^i{}_j,
\end{aligned}
$$

where $\sigma_i = \partial_i\sigma$, $\sigma^i = G^{im}\sigma_m$, $\sigma_{hk} = \nabla_k\sigma_h - \sigma_k\sigma_h + \frac{1}{2}\sigma_r\sigma^r G_{hk}$ and $\sigma^h{}_k = G^{im}\sigma_{mk}$.

Now, from (5.1(d)) and (5.1(c)) we have

$$\overline{C}_m\overline{F}^m{}_{k0} = C_m F^m{}_{k0} + \sigma_0 C_k - C_m\sigma^m y_k,$$

where $C_m y^m = 0$ and $\sigma_0 = \sigma_m y^m$. On the other hand, we get

$$\overline{C}_k = C_k, \quad \overline{C}^k = e^{-2\sigma}C^k, \quad \overline{C}^2 = e^{-2\sigma}C^2.$$

Hence we get

$$\overline{C}_m\overline{F}^m{}_{r0}\overline{C}^r = e^{-2\sigma}(C_m F^m{}_{r0}C^r + \sigma_0 C^2).$$

Since $C^2 \neq 0$, we have

$$\sigma_0 = \overline{C}_m\overline{F}^m{}_{r0}\overline{C}^r/\overline{C}^2 - C_m F^m{}_{r0}C^r/C^2.$$

If we put

$$(5.2)\qquad M = C_m F^m{}_{r0}C^r/C^2, \quad M_k = \dot{\partial}_k M, \quad M^k = G^{km}M_m,$$

then we have

$$\sigma_0 = \overline{M} - M, \quad \sigma_k(x) = \overline{M}_k - M_k. \tag{5.3}$$

Using (5.1(e)), (5.1(c)) and (5.3), we obtain

$$\overline{C}_j{}^i{}_m \overline{P}^m{}_{k0} = C_j{}^i{}_m P^m{}_{k0} - (\overline{M} - M) C_j{}^i{}_k + C_j{}^i{}_m G^{mr}(\overline{M}_r - M_r) y_k.$$

The equation just above is rewritten in the form

$$\overline{C}_j{}^i{}_m \overline{P}^m{}_{k0} + \overline{M}\,\overline{C}_j{}^i{}_k - \overline{C}_j{}^i{}_m \overline{M}^m \overline{y}_k = C_j{}^i{}_m P^m{}_{k0} + M C_j{}^i{}_k - C_j{}^i{}_m M^m y_k.$$

Hence, by putting

$$Q^{*i}_j{}_k = C_j{}^i{}_m P^m{}_{k0} + M C_j{}^i{}_k - C_j{}^i{}_m M^m y_k, \tag{5.4}$$

we obtain

$$\overline{Q}^{*i}_j{}_k = Q^{*i}_j{}_k.$$

That is, the tensor field $Q^{*i}_j{}_k(x, y)$ is invariant under the conformal changes of the given (G, N)-structure.

Next, by means of (5.1(d)) and (5.3), we have

$$\dot{\partial}_h \overline{F}_j{}^i{}_k = \dot{\partial}_h F_j{}^i{}_k + 2\mathring{C}^{im}{}_h(\overline{M}_m - M_m) G_{jk} - 2G^{im}(\overline{M}_m - M_m)\mathring{C}_{jkh}.$$

On the other hand, it is easily seen that $\overline{C}^{im}{}_h \overline{G}_{jk} = C^{im}{}_h G_{jk}$.

The equation just above is rewritten as follows:

$$\begin{aligned} \dot{\partial}_h \overline{F}_j{}^i{}_k &- 2\mathring{\overline{C}}^{im}{}_h \overline{M}_m \overline{G}_{jk} + 2\overline{G}^{im} \overline{M}_m \mathring{\overline{C}}_{jkh} \\ &= \dot{\partial}_h F_j{}^i{}_k - 2\mathring{C}^{im}{}_h M_m G_{jk} + 2G^{im} M_m \mathring{C}_{jkh}. \end{aligned}$$

So, if we put

$$F^{*i}_h{}_{jk} = \dot{\partial}_h F_j{}^i{}_k - 2\mathring{C}_h{}^i{}_m M^m G_{jk} + 2M^i \mathring{C}_{jkh}, \tag{5.5}$$

then we obtain $\overline{F}^{*}_{h}{}^{i}{}_{jk} = F^{*i}_{h}{}_{jk}$. Thus the tensor field $F^{*i}_{h}{}_{jk}(x,y)$ is also invariant under the conformal changes of the given (G,N)-structure.

Next, as for the tensor $Q_{h}{}^{i}{}_{jk}$, from (5.1(d)) and (5.1(e)), it follows that

$$\begin{aligned}\overline{Q}_{h}{}^{i}{}_{jk} =& X_{j}C_{h}{}^{i}{}_{k} + \overline{F}_{m}{}^{i}{}_{j}C_{h}{}^{m}{}_{k} - \overline{F}_{k}{}^{m}{}_{j}C_{m}{}^{i}{}_{k} - \overline{F}_{k}{}^{m}{}_{j}C_{h}{}^{i}{}_{m} - C_{h}{}^{i}{}_{m}\overline{P}^{m}{}_{jk}\\ =& Q_{h}{}^{i}{}_{jk} + \sigma_{m}(\delta^{i}_{j}C_{k}{}^{m}{}_{h} - G^{im}C_{hik} - \delta^{m}_{h}C_{j}{}^{i}{}_{k} + G_{hj}C^{im}{}_{k}).\end{aligned}$$

Using (5.3), we obtain that the tensor field $Q^{*i}_{h}{}_{jk}(x,y)$ is defined by

$$(5.6)\quad Q^{*i}_{h}{}_{jk} = Q_{h}{}^{i}{}_{jk} - M_{m}(\delta^{i}_{j}C_{k}{}^{m}{}_{h} - G^{im}C_{hik} - \delta^{m}_{h}C_{j}{}^{i}{}_{k} + G_{hj}C^{im}{}_{k}),$$

which give $\overline{Q}^{*}_{h}{}^{i}{}_{jk} = Q^{*i}_{h}{}_{jk}$. Thus $Q^{*i}_{h}{}_{jk}(x,y)$ is invariant under the conformal changes of the given (G,N)-structure.

Next, as for the tensor $P_{h}{}^{i}{}_{jk}$, from (5.1(d)) and (5.1(f)), it follows that

$$\begin{aligned}\overline{P}_{h}{}^{i}{}_{jk} =& P_{h}{}^{i}{}_{jk} + (\overline{M}_{r} - M_{r})C_{m}{}^{r}{}_{k}G_{hj}G^{im} - G^{ir}(\overline{M}_{r} - M_{r})G_{hm}C_{j}{}^{m}{}_{k}\\ &+ (\overline{M}_{h} - M_{h})C_{j}{}^{i}{}_{k} - (\overline{M}_{m} - M_{m})C_{h}{}^{m}{}_{k}\delta^{i}_{j}.\end{aligned}$$

Thus the tensor field $P_{h}{}^{i}{}_{jk}$, defined by

$$(5.7)\qquad \begin{aligned}P^{*i}_{h}{}_{jk} =& P_{h}{}^{i}{}_{jk} - M_{r}C_{m}{}^{r}{}_{k}G_{hj}G^{im} + G^{ir}M_{r}G_{hm}C_{j}{}^{m}{}_{k}\\ &- M_{h}C_{j}{}^{i}{}_{k} + M_{m}C_{h}{}^{m}{}_{k}\delta^{i}_{j},\end{aligned}$$

is also invariant under the conformal changes of the given (G,N)-structure.

Moreover, because of $\sigma_{i} = \sigma_{i}(x)$, (5.3) leads us to

$$(5.8)\qquad \dot{\partial}_{j}\overline{M}_{k} = \dot{\partial}_{j}M_{k},$$

that is, the tensor field $\dot{\partial}_{j}F_{k}$ itself is invariant under the conformal changes of the given (G,N)-structure.

In addition to the above equation, we have

$$\overline{\nabla}_{j}\overline{M}_{k} = \nabla_{j}M_{k} - \nabla_{j}\sigma_{k} - \sigma_{k}M_{j} - \sigma_{j}M_{k} + \sigma_{m}M^{m}G_{jk} - 2\sigma_{j}\sigma_{k} + \sigma_{m}\sigma^{m}G_{jk},$$

from which we have

$$\nabla_j \sigma_k = \overline{\nabla}_j \overline{M}_k - \nabla_j M_k + \sigma_k M_j + \sigma_j M_k - \sigma_m M^m G_{jk} + 2\sigma_j \sigma_k - \sigma_m \sigma^m G_{jk}. \tag{5.9}$$

Since $\nabla_j \sigma_k = \nabla_k \sigma_j$, we have

$$\overline{\nabla}_j \overline{M}_k - \overline{\nabla}_k \overline{M}_j = \nabla_j M_k - \nabla_k M_j.$$

Namely, the tensor field, defined by

$$\nabla_j M_k - \nabla_k M_j, \tag{5.10}$$

is also invariant under the conformal changes of the given (G, N)-structure.

Finally, on account of (5.9) and (5.3), we have

$$\begin{aligned}\sigma_{kj} =& \overline{\nabla}_j \overline{M}_k - \nabla_j M_k + \sigma_k M_j + \sigma_j M_k - \sigma_m M^m G_{jk} + \sigma_j \sigma_k - \frac{1}{2}\sigma_m \sigma^m G_{jk} \\ =& \overline{\nabla}_j \overline{M}_k - \nabla_j M_k + \overline{M}_j \overline{M}_k - M_j M_k - \frac{1}{2}\overline{M}_m \overline{M}^m \overline{G}_{jk} + \frac{1}{2} M_m M^m G_{jk}.\end{aligned}$$

The above is rewritten as

$$\begin{aligned}\sigma_{kj} =& \left(\overline{\nabla}_j \overline{M}_k + \overline{M}_j \overline{M}_k - \frac{1}{2}\overline{M}_m \overline{M}^m \overline{G}_{jk}\right) \\ &- \left(\nabla_j M_k + M_j M_k - \frac{1}{2} M_m M^m G_{jk}\right).\end{aligned}$$

Hence if we put

$$M_{ij} = \nabla_j M_k + M_j M_k - \frac{1}{2} M_m M^m G_{ik}, \tag{5.11}$$

then $\sigma_{kj} = \overline{M}_{kj} - M_{kj}$, from which we have

$$\begin{aligned}\overline{K}_h{}^i{}_{jk} =& K_h{}^i{}_{jk} + \delta^i_j(\overline{M}_{kh} - M_{kh}) - \delta^i_k(\overline{M}_{jh} - M_{jh}) \\ &- G_{hj} G^{im}(\overline{M}_{mk} - M_{mk}) + G_{hk} G^{im}(\overline{M}_{mj} - M_{mj}).\end{aligned}$$

Thus the tensor field $K^{*}_{h}{}^{i}{}_{jk}$, defined by

$$(5.12)\quad K^{*}_{h}{}^{i}{}_{jk} = K_{h}{}^{i}{}_{jk} - \delta^{i}_{j}M_{kh} + \delta^{i}_{k}M_{jh} + G_{hj}G^{im}M_{mk} - G_{hk}G^{im}M_{mj},$$

is also invariant under the conformal changes of the given (G,N)-structure.

Moreover, the tensor field $\overline{R}_{h}{}^{i}{}_{jk}$ is given by

$$\begin{aligned}\overline{R}_{h}{}^{i}{}_{jk} =& R_{h}{}^{i}{}_{jk} + \delta^{i}_{j}(\overline{M}_{kh} - M_{kh}) - \delta^{i}_{k}(\overline{M}_{jh} - M_{jh})\\ &- G_{hj}(\overline{M}^{i}{}_{k} - M^{i}{}_{k}) + G_{hk}(\overline{M}^{i}{}_{j} - M^{i}{}_{j}).\end{aligned}$$

Hence the tensor field $R^{*}_{h}{}^{i}{}_{jk}$, defined by

$$(5.13)\qquad R^{*}_{h}{}^{i}{}_{jk} = R_{h}{}^{i}{}_{jk} - \delta^{i}_{j}M_{kh} + \delta^{i}_{k}M_{jh} + G_{hj}M^{i}{}_{k} - G_{hk}M^{i}{}_{j},$$

is also invariant under the conformal changes of the given (G,N)-structure.

Summarizing, we have the following:

THEOREM 5.1. *Let $G_{ij}(x,y)$ be a generalized Finsler metric satisfying $C = \sqrt{C_m C^m} \neq 0$ and N be a non-linear connection. With respect to the (G,N)-connection, let $M = C_m F^{m}{}_{r0}C^{r}/C^{2}$ and $M_k = \dot{\partial}_k M$. Then the tensor fields $Q^{*}_{h}{}^{i}{}_{k}$, $F^{*}_{h}{}^{i}{}_{jk}$, $Q^{*}_{h}{}^{i}{}_{jk}$, $P^{*}_{h}{}^{i}{}_{jk}$, $K^{*}_{h}{}^{i}{}_{jk}$, $R^{*}_{h}{}^{i}{}_{jk}$ which are given respectively by (5.4), (5.5), (5.6), (5.7), (5.12), (5.13) and $\dot{\partial}_j M_k$, $\nabla_j M_k - \nabla_k M_j$ are all invariant under the conformal changes of the given (G,N)-structure.*

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