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THE EXTENSION OF SOLUTIONS OF COMPLEX PARTIAL DIFFERENTIAL EQUATIONS

JUNG HO PARK

1. Introduction

In the theory of functions of several complex variables, the following phenomenon occurs: There exists a region in $\mathbb{C}^n$(with $n > 1$), where $\mathbb{C}^n$ is the space of n-complex variables, such that all functions holomorphic in this region can be holomorphically extended to a larger region. A typical example for this extension phenomenon is the Hartogs extension theorem for holomorphic functions of several complex variables. Consider holomorphic functions of several complex variables as solutions of the Cauchy-Rieman system

$$\frac{\partial w}{\partial \bar{z}} = 0$$

or

$$\begin{cases} \frac{\partial u}{\partial x_j} - \frac{\partial v}{\partial y_j} = 0 \\ \frac{\partial v}{\partial x_j} + \frac{\partial u}{\partial y_j} = 0, \end{cases}$$

where $w = u + iv$, $z_j = x_j + \sqrt{-1}y_j$, $j = 1, 2, \cdots, n$.

Then the Hartogs extension theorem is in fact about the extension of the solution of a first order system of partial differential equations

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with constant coefficients. In [5], such extension phenomena for the solution of the system

$$\frac{\partial w}{\partial \bar{z}_j} = f_j(z_1, z_2, \cdots, z_n, w) \quad (j = 1, 2, \cdots, n)$$

were investigated and the system

$$\sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij}^{(l)}(x) \frac{\partial u_i}{\partial x_j} = 0 \qquad (l = 1, 2, \cdots, L),$$

where $A_{ij}^{(l)}$ are real analytic functions, was investigated by Son [6].

In this paper, we confine ourselves to the extension problem of the solution of the generalized p-independent linear partial differential equations with one unknown function in a simply connected open set $\Omega \subset \mathbb{C}^n$:

$$\sum_{j=1}^{n} a_{ij}(z) \frac{\partial u}{\partial z_j} = 0, \ \ 1 \leq i \leq p \ , \ \ 1 \leq p \leq n-1,$$

where $a_{ij}(z) = a_{ij}(z_1, z_2, \cdots, z_n)$ are given holomorphic functions in Ω.

2. Extension theorems

We consider the following generalized p-independent linear partial differential equations with one unknown function in a simply connected open set $\Omega \subset \mathbb{C}^n$:

$$\sum_{j=1}^{n} a_{ij}(z) \frac{\partial u}{\partial z_j} = 0, \quad 1 \leq i \leq p \ , \ \ 1 \leq p \leq n-1,$$

where $a_{ij}(z) = a_{ij}(z_1, z_2, \cdots, z_n)$ are given holomorphic functions in Ω.

The solvability is guaranteed by Begehr and Dzhuraev [1] with the following property.

PROPOSITION 2.1. *Let Ω be an open set in $\mathbb{C}^n$. Consider the generalized p-independent linear partial differential equations*

$$\sum_{j=1}^{n} a_{ij}(z)\frac{\partial u}{\partial z_j} = 0, \quad 1 \le i \le p \ , \ 1 \le p \le n-1.$$

If these equations are complete, that is, the commutators,

$$[L_k, L_l] = L_k L_l - L_l L_k = \sum_{h=1}^{p} \lambda_{kl}^h L_h \ (1 \le k < l \le p),$$

where $L_i u = \dfrac{\partial u}{\partial z_i} + \displaystyle\sum_{j=p+1}^{n} \lambda_{ij}(z)\frac{\partial u}{\partial z_j} = 0$ *and* λ_{kl}^h, *are constants, then these partial differential equations have a holomorphic solution in* Ω.

THEOREM 2.2. *Let Ω be a simply connected open set in $\mathbb{C}^n$ and K be a compact subset of Ω such that $\Omega \backslash K$ is connected. If for every $f \in C_0^\infty(\Omega)$ there exists a solution $v \in C_0^\infty(\mathbb{C}^n)$ of the inhomogeneous equations*

$$\sum_{j=1}^{n} a_{ij}(z)\frac{\partial u}{\partial z_j} = f$$

and the corresponding homogeneous equations

$$\sum_{j=1}^{n} a_{ij}(z)\frac{\partial u}{\partial z_j} = 0$$

are complete then every solution of the homogeneous system in $\Omega \backslash K$ can be extended to Ω.

PROOF. Decompose Ω with open sets Ω_1 and Ω_2 satisfying the following conditions,

(1) $\Omega_1 \cup \Omega_2 = \Omega$,

(2) $\Omega_1 \cap \Omega_2 \cap K \neq \phi$, and

(3) $\{K \backslash (\Omega_1 \cap \Omega_2)\} \cap \Omega_k \neq \phi$ $(k = 1, 2)$,

then for $k = 1, 2$, $\Omega_k \cap K$ is compact in Ω_k and $\Omega_k \backslash (\Omega_k \cap K)$ is simply connected.

Let u be a solution of $\sum_{j=1}^{n} a_{ij}(z)\frac{\partial u}{\partial z_j} = 0$ in $\Omega\backslash K$ and φ_k be a function of the class $C_0^\infty(\Omega_k)$ such that the restriction of φ_k to a neighborhood of $\Omega \cap K$ equals to 1. Define

$$u_0^k = \begin{cases} (1-\varphi_k)u & \text{in } \Omega_k\backslash(\Omega_k \cap K) \\ 0 & \text{in } \Omega_k \cap K. \end{cases}$$

Then we have $u_0^k \in C^\infty(\Omega_k)$. Let v_k be a solution of $\sum_{j=1}^{n} a_{ij}(z)\frac{\partial u}{\partial z_j} = f$, where

$$f = \begin{cases} -\sum_{j=1}^{n} a_{ij}(z)u\frac{\partial \varphi_k}{\partial z_j} & \text{in } \Omega_k\backslash(\Omega_k \cap K) \\ 0 & \text{in } \Omega_k \cap K. \end{cases}$$

Put $U_k = u_0^k - v_k$. Then U_k is a solution of $\sum_{j=1}^{n} a_{ij}(z)\frac{\partial u}{\partial z_j} = 0$ in Ω_k.

Because, in $\Omega_k \cap K$, $u_0^k = 0$ and $f = 0$ imply

$$\sum_{j=1}^{n} a_{ij}(z)\frac{\partial U_k}{\partial z_j} = -\sum_{j=1}^{n} a_{ij}(z)\frac{\partial v_k}{\partial z_j} = -f = 0.$$

Also in $\Omega_k\backslash(\Omega_k \cap K)$, we have

$$\begin{aligned}
\sum_{j=1}^{n} a_{ij}(z)\frac{\partial U_k}{\partial z_j} &= \sum_{j=1}^{n} a_{ij}(z)\frac{\partial(u - \varphi_k u - v_k)}{\partial z_j} \\
&= \sum_{j=1}^{n} a_{ij}(z)\frac{\partial u}{\partial z_j} - \sum_{j=1}^{n} a_{ij}(z)\frac{\partial(\varphi_k u)}{\partial z_j} - \sum_{j=1}^{n} a_{ij}(z)\frac{\partial v_k}{\partial z_j} \\
&= \sum_{j=1}^{n} a_{ij}(z)\frac{\partial u}{\partial z_j} - \sum_{j=1}^{n} a_{ij}(z)\varphi_k\frac{\partial u}{\partial z_j} \\
&\quad - \sum_{j=1}^{n} a_{ij}(z)u\frac{\partial \varphi_k}{\partial z_j} - \sum_{j=1}^{n} a_{ij}(z)\frac{\partial v_k}{\partial z_j} \\
&= 0 - 0 + f - f = 0.
\end{aligned}$$

From the assumption of the theorem, there exists a solution $v_k \in C_0^\infty(\mathbb{C}^n)$ such that $\sum_{j=1}^n a_{ij}(z)\frac{\partial v_k}{\partial z_j} = f$ and so, v_k is a solution of $\sum_{j=1}^n a_{ij}(z)\frac{\partial v_k}{\partial z_j} = 0$ in $\mathbb{C}^n \backslash supp\ \varphi_k$ because $f|_{\mathbb{C}^n\backslash supp\ \varphi_k} = 0$.

Since $v_k \in C_0^\infty(\mathbb{C}^n)$, we have $v_k = 0$ for large enough $|z|$.

By the uniqueness of analytic continuation, $v_k = 0$ in the unbounded component A_k of $\mathbb{C}^n\backslash supp\ \varphi_k$. Put $A_k \cap (\Omega_k \backslash supp\ \varphi_k) = B_k$. Then B_k is nonempty open set such that $v_k = 0$ and $u = u_0^k$. Thus we have $U_k = u$ in B_k. Since $\Omega_k\backslash(\Omega_k \cap K)$ is simply connected, we have $U_k = u$ in $\Omega_k\backslash(\Omega_k \cap K)$.

Define

$$U = \begin{cases} U_1 = U_2 & \text{in } \Omega_1 \cap \Omega_2 \\ U_k & \text{in } \Omega_k\backslash(\Omega_1 \cap \Omega_2). \end{cases}$$

Then U is a solution of $\sum_{j=1}^n a_{ij}(z)\frac{\partial u}{\partial z_j} = 0$ in Ω and $U = u$ in $\Omega\backslash K$. We complete the proof.

We, now, apply this result to another differential equation $\frac{\partial u}{\partial z} = 0$.

Generally the equation $\frac{\partial u}{\partial z} = 0$, however, has no holomorphic solution in Ω so we cannot apply the uniqueness of analytic continuation. But Krantz and Park [4] proved the following efficient lemma.

LEMMA 2.3. *Let Ω be open in $\mathbb{C}^n$ and $f \in C^\infty(\Omega)$. Suppose that for all $w \in \Omega$ there exist an open neighborhood U of w with $U \subset \Omega$ and constants $C > 0,\ R > 0$ such that $|f^{(\alpha)}(z)| \leq C\frac{\alpha!}{R^\alpha}$ $\forall z \in U$, then f is a holomorphic function in Ω.*

PROOF. On U, since the inequality,

$$\sum_\alpha \frac{f^{(\alpha)}(w)}{\alpha!}(z-w)^\alpha \leq C \sum_\alpha \frac{1}{R^\alpha}(z-w)^\alpha,$$

is satisfied, $\sum_\alpha \frac{f^{(\alpha)}(w)}{\alpha!}(z-w)^\alpha$ converges on the open ball $V = \{z \in \mathbb{C}^n : |z-w| < R\}$, since $U \cap V \subset V$, $\sum_\alpha \frac{f^{(\alpha)}(w)}{\alpha!}(z-w)^\alpha$ converges to f on $U \cap V$. Moreover $U \cap V \subset \Omega$ being open and $w \in U \cap V$ imply that f is holomorphic in Ω.

Applying this to above equation, we can get the following result.

THEOREM 2.4. *If a C^∞ solution u of $\frac{\partial u}{\partial z} = 0$ satisfies the same conditions with Theorem 2.2 and Lemma 2.3, then u can be extended to Ω.*

PROOF. The condition of Lemma 2.3 makes u satisfy the uniqueness of analytic continuation and so it can be proved similarly with the proof of Theorem 2.2.

Above two theorems dealt with homogeneous partial differential equations then how about the inhomogeneous cases? We can get an answer about this question as following.

PROPOSITION 2.5. *If a C^∞ solution v of $\frac{\partial u}{\partial z} = 0$ satisfies the same conditions with Theorem 2.4, then the solution of the inhomogeneous equation $\frac{\partial u}{\partial z} = f$ can be extended to Ω.*

PROOF. Let v_1 be a solution of $\frac{\partial u}{\partial z} = f$ in $\Omega \backslash K$ and v_2 be a solution in Ω. Putting $v_1 - v_2 = v$, we get

$$\frac{\partial v}{\partial z} = \frac{\partial(v_1 - v_2)}{\partial z} = \frac{\partial v_1}{\partial z} - \frac{\partial v_2}{\partial z} = 0 \quad \text{in} \quad \Omega \backslash K.$$

That is, v is a solution of $\frac{\partial v}{\partial z} = 0$ in $\Omega \backslash K$.

By the result of Theorem 2.4, v can be extended to the whole Ω.

Since v_2 is a solution in Ω and $v_1 = v + v_2$, v_1 can be extended to a solution of $\frac{\partial u}{\partial z} = f$ in Ω.

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Department of Mathematics
Pusan National University
Pusan 609-735, Korea