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PRIMARY DECOMPOSITIONS IN NON-COMMUTATIVE LATTICES

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1. Introduction

A partially ordered set P is said to be inductive if every totally ordered subset has an upper bound. Then Zorn's Lemma asserts that every nonempty inductive set contains at least one maximal element.

A lattice is a partially ordered set L in which every subset with two elements has both a greatest lower bound and a least upper bound in L. The least upper bound (greatest lower bound) of the subset $\{a, b\}$ of L is called the join of a and b (meet of a and b) and is denoted by $a \vee b$ $(a \wedge b)$.

A multiplicative lattice is a lattice L on which a binary operation $(a, b) \mapsto ab$ from $L \times L$ into L, called multiplication, is defined for each pair of elements in L, and satisfies the following conditions;

(1) $ab \leq a \wedge b$ for all a, b in L.
(2) $a(b \vee c) = ab \vee ac$ and $(b \vee c)a = ba \vee ca$ for all a, b, c in L.

It is easy to show that a multiplicative lattice L also satisfies;

(3) $a(b \wedge c) \leq ab \wedge ac$ and $(b \wedge c)a \leq ba \wedge ca$ for all a, b, c in L.
(4) if $a \leq b$ in L, then $ac \leq bc$ and $ca \leq cb$ for all a, b and c in L.

A multiplicative lattice L is associative if $a(bc) = (ab)c$ for all elements a, b and c in L, and commutative if $ab = ba$ for all a and b in L.

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A lattice L satisfies the ascending chain condition, abbreviated the ACC, if for every ascending sequence $a_1 \leq a_2 \leq a_3 \leq \ldots$ of elements of L there exists a positive integer n such that $a_k = a_n$ for all $k \geq n$. Then it is easy to show that the ACC, even in a partially ordered set, is equivalent to the maximal condition : That is, every nonempty subset S of L has a maximal element in S; more precisely, there exists an element m in S such that if $m \leq s$ for some s in S, then $m = s$. Note that the maximal element need be unique.

An element a of a lattice L is meet-irreducible, if $a = b \wedge c$ for b and c in L implies that either $b = a$ or $c = a$, that is, if a is not the meet of two elements of L, each strictly greater than a.

The following result is easy to verify and its proof is an analogue of that in non-commutative rings.

PROPERTY 1. *If a lattice L satisfies the ACC, then every element of L is a meet of a finite number of meet-irreducible elements of L.*

A lattice L is semi-modular if, for all element a, b and c in L the relations $b \wedge c < a < c < b \vee c$ imply that there exists an element t in L satisfying $b \wedge c < t \leq b$ and $a = (a \vee t) \wedge c$.

A lattice L is modular if, for all elements a, b and c in L, $a \leq c$ implies that $a \vee (b \wedge c) = (a \vee b) \wedge c$.

For every ring R, the lattice $L(R)$ of all ideals of R partially ordered by set inclusion is modular. More generally, a power set of any set with set union and set intersection as binary operations and set inclusion as a partial order forms a modular lattice. It is easy to show that a modular lattice is also semi-modular (for a proof, see [4]), but the converse is not true in general.

In this chapter, all of the lattices under consideration are assumed to be associative and semi-modular unless otherwise stated and they also have a unique greatest element, denoted by e, and a least element, denoted by 0. If a lattice L is multiplicative, then these elements also satisfy $ea = ae = a$ and $0a = a0 = 0$ for all elements a in L.

An element q of a multiplicative lattice L is right-primary if, for all a and b in L, $ab \leq q$ and $b \not\leq q$ imply that $a^n \leq q$ for some positive integer n . In the commutative case, there is no need to distinguish between right- and left-primary elements, and an element with the given property will be called primary.

An element p in multiplicative lattice L is prime if, for all a and b in L, $ab \leq p$ implies that either $a \leq p$ or $b \leq p$.

Let a be an arbitrary element of a multiplicative lattice L and consider the following set N of elements of L

$$N = \{b \in L \mid b^k \leq a \quad \text{for some positive integer } k\}.$$

Then, since the element a itself is in N, N is nonempty subset of L. Hence, if L satisfies the ACC, then N contains a maximal element. Moreover, N is closed under the join operation. For, if b and c are in N so that $b^k \leq a$ and $c^m \leq a$ for some positive integers k and m, then in each term appearing in the expansion of $(b \vee c)^{k+m}$ either b appears at least k times or c appears at least m times so that each term is either less than or equal to b^k or is less than or equal to c^m. For instance, a term like $bcbc\cdots$ in which b and c appear alternately has either b at least k times or c at least m times. So, in any case, it is less than or equal to $b^k \vee c^m$, which is less than or equal to a. Thus $(b \vee c)^{k+m} \leq a$, so $b \vee c$ is also contained in N and thus N has exactly one maximal element, called the radical of a and denoted by $Rad(a)$. Since $Rad(a)$ itself is an element of the set N, there exists a positive integer k such that

(1) $(Rad(a))^k \leq a$.

Other useful properties of radicals which can be found in Lesieur and Croisot [4] are

(2) If $a \leq b$, then $Rad(a) \leq Rad(b)$.

(3) $Rad(a_1 a_2 \cdots a_k) = Rad(a_1 \wedge a_2 \wedge \cdots \wedge a_k)$
$= Rad(a_1) \wedge Rad(a_2) \wedge \cdots \wedge Rad(a_k)$.

Hence, in particular, $Rad(a^n) = Rad(a)$ for all positive integer n.

(4) $Rad(Rad(a)) = Rad(a)$.

(5) If p is a prime element, then $Rad(p) = p$.

If a lattice L is commutative, then a routine application of definitions will show that if q is primary element, then $Rad(q)$ is a prime element. In this case, q is said to be p-primary where $p = Rad(q)$. Without commutativity, a similar result can be obtained for left- and right-primary elements (See [4]).

Note that q is a primary element if and only if $ab \leq q$ and $a \not\leq Rad(q)$ imply that $b \leq q$.

Let a and b be two elements of a multiplicative lattice L. If there exists an element c in L satisfying $cb \leq a$ and such that $xb \leq a$ for x in L implies $x \leq c$, then c is unique and is called the left-residual of a by b and is denoted by $a :_l b$. Similarly, an element d in L satisfying $bd \leq a$ and such that $bx \leq a$ for x in L implies $x \leq d$ is called the right residual of a by b and is denoted by $a :_r b$. Needless to say, in commutative case, $a :_l b = a :_r b$ for all a and b in L. A lattice L is left-residuated (right-residuated) if $a :_l b$ ($a :_r b$) exists in L for every pair of elements a and b in L, and is residuated if it is both left- and right-residuated.

The following result is easy to verify.

THEOREM 1. *If a multiplicative lattice L satisfies the ACC, then it is residuated.*

PROOF. Let a and b be two arbitrary elements of L. Consider the set N;

$$N = \{c \in L \mid bc \leq a\}.$$

Then, N is clearly nonempty since a is in N. Hence by the ACC, N contains a maximal element c. If $bx \leq a$ for some x in L, then $b(x \vee c) = bx \vee bc$ is less than or equal to a so that $x \vee c$ is contained in N and $c \leq x \vee c$, and thus by the maximality of c in N, $c = x \vee c$, proving $x \leq c$. Hence, $c = a :_r b$ exists in L. A similar argument shows that $a :_l b$ also exists in L and hence L is residuated.

The following properties of the residuals are easy to verify (for more details, refer to [4]).

(1.1) $b(a :_r b) \leq a$ and $a \leq (a :_r b) \wedge (a :_r a)$.

(1.2) $b \leq c$ implies that $a :_r c \leq a :_r b$ and $b :_r a \leq c :_r a$.

(1.3) $(a_1 \wedge \cdots \wedge a_n) :_r c = (a_1 :_r c) \wedge \cdots \wedge (a_n :_r c)$.

(1.4) $a :_r (b_1 \vee \cdots \vee b_n) = (a :_r b_1) \wedge \cdots \wedge (a_n :_r b_n)$.

(1.5) $(a :_r b) :_r c = a :_r bc$.

(1.6) $a :_r b = a :_r (a \vee b)$, and $a :_r b = e$ implies that $b \leq a$.

Similar properties hold for left-residuals.
Since $((a :_r b) :_l c) \leq a :_r b$ implies that $(b((a :_r b) :_l c)c \leq a$ and thus $b((a :_r b) :_l c$, it follows that $(a :_r b) :_l c \leq (a :_l c) :_r b$. Changing the roles of the right- and left-residuals shows that

(1.7) $(a :_r b) :_l c = (a :_l c) :_r b$.

Further, the following proposition holds.

(1.8) $a :_l (bc) = (a :_l c) :_l b$.

In a multiplicative lattice satisfying the ACC, the radical of a primary element takes a special form.

PROPOSITION 1. *Let q be a primary element, not equal to e, in a multiplicative lattice L satisfying the ACC, and let $p = Rad(q)$. Then there exists an element t in L such that $t \not\leq q$ and $p = q :_l t$.*

PROOF. Let q be a primary element, $p = Rad(q)$ and consider the set N of elements in L;

$$N = \{q :_l t \mid t \not\leq q\}.$$

Since e is in N, N is nonempty and contains a maximal element $p^* = q :_l t$ for some fixed element t not less than or equal to q. Since $p^* = (q :_l t)t \leq q$ and $t \not\leq q$ for q a primary element, it follows that $p^* \leq Rad(q) = p$. Next, assume that $be \leq p^*$ and $c \not\leq p^*$ for b and c in L. Then $ct \not\leq q$ and so $p^* = q :_l t \leq q :_l ct$. Thus, from the maximality of p^* in N, $p^* = q :_l ct$ and since $bc \leq q :_l t = p^*$ we obtain $bct \leq q$. So, it follows that $b \leq q :_l ct = p^*$, proving that p^* is a prime element. Since $p^n \leq q$ for some positive integer n, $q \leq p^*$ and p^* is a prime element, $p \leq p^*$. Hence, $p = p^* = q :_l t$ where $t \not\leq q$.

The following proposition gives relations between (right-) primary elements and their residuals and radicals in a multiplicative lattices satisfying the ACC.

PROPOSITION 2. *Let q be an element in a multiplicative lattice L satisfying the ACC. Then*

(1) *q is a primary element if and only if $q :_l r \leq Rad(q)$ for all r in L such that $r \not\leq q$. If q is p-primary, then $q :_l r$ is also p-primary where $p = Rad(q)$.*

(2) *q is a primary element if and only if $q = q :_r t$ for all t in L such that $t \not\leq Rad(q)$.*

PROOF. (1) If q is a primary element and if $r \not\leq q$, then $(q :_l r)r \leq q$ implies that $q :_l r \leq Rad(q)$. Clearly, $Rad(q :_l r) = Rad(q)$. To show that $q :_l r$ is a p-primary element, assume that $bc \leq q :_l r$ and $c \not\leq q :_l r$ for b and c in L. Then $b(cr) \leq q$ and $cr \not\leq q$ imply that $b^n \leq q$ for some positive integer n and so $b^n \leq q :_l r$, proving $q :_l r$ is in fact a p-primary element.

(2) If $t \not\leq Rad(q)$ for a primary element q, then $t(q :_r t) \leq q$ implies that $q :_r t \leq q$ and so $q = q :_r t$. Conversely, if $bc \leq q$ and $b \not\leq Rad(q)$ for b and c in L, then $c \leq q :_r b = q :_r b = q$, showing q is primary.

There is an interesting relationship between the radical of an element and an arbitrary element which is not less than or equal to the radical. This relation can be applied to determine the structure of radicals.

PROPOSITION 3. *Let L be a multiplicative lattice satisfying the ACC, and a and b be two elements in L such that $a \neq e$ and $b \not\leq Rad(a)$. Then there exists a prime element p in L such that $a \leq p$, $b \not\leq p$ and $a :_l b^k \leq p$ for all positive integers k.*

PROOF. For a positive integer n, let $b_n = a :_l b^n$. Then $a \leq b_1 \leq \cdots \leq b_n \leq \cdots$ is an ascending sequence of elements in L. By the ACC, there exists a positive integer k such that $b_k = b_n$ for all $n \geq k$. Consider the set N of elements in L;

$$N = \{d \in L \mid b_k \leq d \quad \text{and} \quad b \not\leq Rad(d)\}.$$

Then b_k is in N. For, if not, then $b \leq Rad(b_k)$, so $b^n \leq b_k = a :_l b^k$ for some positive integer n and thus $b^{n+k} = b^n b^k \leq (a :_l b^k)b^k \leq a$, which means that $b \leq Rad(a)$, contradicting the assumption on b. Thus N is nonempty and again by the ACC, there exists a maximal element p in L. Then $b^k = a :_l b^k \leq p$ and $b \not\leq p$. To show that p is a prime element, assume that $rs \leq p$, but $r \not\leq p$ and $s \not\leq p$ for some r and s in L. Then, $p < r \vee p$ and $p < s \vee p$, i.e., p is strictly less than both $r \vee p$ and $s \vee p$. So from the maximality of p in N, we get that $b \leq Rad(r \vee p)$ and

$b \leq Rad(s \vee p)$. Thus $b^n \leq r \vee p$ and $b^m \leq s \vee p$ for some positive integers n and m so that $b^{n+m} = b^n b^m \leq (r \vee p)(s \vee p) = rs \vee rp \vee ps \vee p^2 \leq p$ and thus $b \leq Rad(p)$, contradicting the fact that p is in N. Hence p is a desired prime element, and this completes the proof.

A prime element p is a minimal prime of a if $p \geq a$ and for every prime element p', $p \geq p' \geq a$ implies that $p = p'$.

In commutative rings, if a prime ideal P contains an ideal A, then P contains a minimal prime ideal of A [6]. A similar result holds in multiplicative lattices satisfying the ACC.

To show this, note first that in such lattices, the assumption of the ACC can be used to show the existence of a greatest lower bound for every nonempty subset in the following way :

Let L be a lattice with a least element 0 satisfying the ACC, and S be an arbitrary nonempty subset of L. Since the element 0 itself is a lower bound for S, it follows that the set N of all lower bounds for S is nonempty. Hence, by the ACC, N contains a maximal element m. Note that N is closed under the join operation. For, if m_1 and m_2 are in N, then $m_1 \vee m_2 \leq s \vee s = s$ for all s in S so that $m_1 \vee m_2$ is also a lower bound for S. This implies that N has a unique maximal element m, which is obviously a greatest lower bound for S, and denoted by $\wedge_{s \in S} s$. In this way, a meet for the elements in an arbitrary subset of L can be defined by taking its greatest lower bound.

In particular, if a subset S consists of countably infinite number of elements s_n, then its greatest lower bound is denoted by $\wedge_n s_n$.

Consider a chain of prime elements $p_1 \geq p_2 \geq \cdots \geq p_n \geq \cdots$ in a multiplicative lattice L satisfying the ACC. Then, by the argument above, its greatest lower bound $\wedge_n p_n$ exists in L. Let $p = \wedge_n p_n$. To show that p is also a prime element, assume that $bc \leq p$ for b and c in L and $b \not\leq p$. Then $b \not\leq p_n$ for some positive integer n and thus $b \not\leq p_k$ for all $k \geq n$ since the given chain is descending. Therefore, for all k such that $k \geq n$, $bc \leq p_k$ and $b \not\leq p_k$ imply that $c \leq p_k$. It is clear that, for $m \leq n$, $p_m \geq p_n$ and $c \leq p_n$ imply that $c \leq p_m$. Hence $c \leq p_k$ for all positive integers k and so $c \leq p$, proving p is indeed a prime element in L. Now let a be an arbitrary element in L such that $a \neq e$. Then, clearly $Rad(a) \neq e$, and thus from Proposition 3, there exists a prime element p in L such that $a \leq p$ and $p \neq c$. As mentioned earlier, there is a minimal prime element p^* of a such that $p^* \leq p$.

To show this, consider the set P of all prime elements t in L such that $a \leq t$ and $t \leq p$. Then p is in P and therefore P is nonempty. Define a new relation " $\leq'$ " in P as follows : for p_1 and p_2 in P, $p_1 \leq' p_2$ if and only if $p_2 \leq p_1$ in L. Then, since $\leq$ is a partial order on L, it follows that $\leq'$ is also a partial order on P. Moreover, P is inductive. To see this , let Q be an arbitrary totally ordered subset of P. Then $q = \wedge_{t \in Q} t$ exists in L and it is also a prime element such that $a \leq q$ and $q \leq p$. Therefore q is contained in P. Note also that $a \leq t$ in L implies that $t \leq' q$ in P for all t in Q, showing q is an upper bound for Q. Thus P is inductive and so P contains a maximal element p^* with respect to the relation $\leq'$ by the Zorn's lemma. Then p^* is a prime element such that $a \leq p^* \leq p$. Suppose that p_1 is a prime element satisfying $a \leq p_1 \leq p^*$. Then p_1 is in P and $p^* \leq' p_1$. Then, from the the maximality of p^* in P, $p^* = p_1$. This show that p^* is in fact a minimal prime element of a. Consequently, the following proposition has been proved.

PROPOSITION 4. *Every element in a multiplicative lattice satisfying the ACC has at least one minimal prime element.*

COROLLARY 1. *Let $a \neq e$ be an arbitrary element in a multiplicative lattice L satisfying the ACC. Then $Rad(a)$ is the meet of all minimal prime elements of L.*

PROOF. Let $r = \wedge\{p \in L \mid p \text{ is a minimal prime element of } a\}$. If p is a minimal prime element of a, then $a \leq p$ and hence $Rad(a) \leq Rad(p) = p$, $Rad(a) \leq r$. If $Rad(a) \neq r$, then there exists a prime element p_1 in L such that $a \leq p_1$ by Proposition 3. Thus there exists a minimal prime element p of a such that $p \leq p_1$ and $r \not\leq p$, which contradicts the fact that r is a meet of all minimal prime elements of a. Hence $Rad(a) = r$.

As a matter of fact, the presence of the ACC in lattices provides more information about the structure of radicals in terms of the number of minimal prime elements.

PROPOSITION 5. *Let a be an arbitrary element in a multiplicative lattice L satisfying the ACC. Then a has only a finite number of minimal prime elements.*

PROOF. If a is a prime element, then a is the only minimal prime element of a, so there is nothing to prove. Therefore, assume that a is not a prime element and suppose that a has an infinite number of minimal prime elements p_i. Since a is not a prime element, there exists elements b and c in L such that $bc \leq a$, $b \nleq a$ and $c \nleq a$. Then both $a \vee b$ and $a \vee c$ are strictly greater than a, and $(a \vee b)(a \vee c) = (a \vee b)a \vee (a \vee b)c = a^2 \vee ba \vee ac \vee bc \leq a \vee bc = a \leq p_i$ for all i. Hence each p_i is greater than or equal to $a \vee b$ or to $a \vee c$. Note that either an infinite number of prime elements p_i must be greater than or equal to $a \vee b$. Assume that this is true for $a \vee b$ and let $b_1 = a \vee b$. Note that each minimal prime element p_i of a is also a minimal prime element of b_1. Hence b_1 also has an infinite number of minimal prime elements, and b_1 cannot be a prime element. Therefore, if a has an infinite number of minimal prime elements, then there exists an element with the same property, which is strictly greater then a and continuation of this argument leads to a contradiction of the ACC in L. This completes the proof for Proposition 5.

Note that the unique representation of a radical of an element a as a meet of a finite number of its minimal prime elements does not depend on any decomposition of a into meet-irreducible elements in a lattice.

A primary element q is a minimal primary of a if $a \leq q$ and for every primary element q', $a \leq q' \leq q$ implies that $q = q'$.

The following result proves the existence of a minimal primary element of an element whose radical is a prime element.

PROPOSITION 6. *Let a be an element in a multiplicative lattice L satisfying the ACC such that $a \neq e$. Let $p = Rad(a)$. If p is a prime element in L, then there exists a unique p-primary element q of a such that $a :_l q \nleq p$*

PROOF. Consider the following set N of elements of L;

$$N = \{t \in L \mid a \leq t \ \text{ and } \ a :_l t \nleq p\}.$$

Since $a \neq e$ implies that $p \neq e$ and $e = a :_l a \nleq p$, it follows that a is in N and therefore N is nonempty. Hence, by the ACC, N contains a maximal element q. Then $q \geq a$ and $a :_l q \nleq p$. Note that $(a :_l q)q \leq a \leq p$ and $a :_l q \nleq p$ imply that $a \leq q \leq p$ so that $Rad(q) = p$. To show that q is a primary element, assume that $bc \leq q$ and $c \nleq q$ for b and c in L. Then since q is strictly less than $q \vee c$ and q is maximal element in N, it follows from the properties (1.2) and (1.8) that

(a) $a :_l (q \vee c) \leq p$,
(b) $a :_l q \leq (a :_l (q \vee c)) :_l b$.

Suppose that $b \nleq p = Rad(q)$, i.e., $b^n \nleq q$ for all positive integers n. Then $p :_l b = p$ and from (a), $(a :_l (q \vee c)) :_l b \leq p :_l b = p$. However, $a :_l q \nleq p$ and thus $(a :_l (q \vee c)) :_l b \nleq p$ by (b), which is a contradiction. Hence $b \leq p = Rad(q)$, showing q is indeed a p-primary element. If there is a p-primary element t such that $a < t \leq q$, then $(a :_l q)q \leq a < t$ and $a :_l q \nleq p = Rad(q)$ imply that $q \leq t$ and thus $q = t$, which shows that q is a minimal primary element of a. The uniqueness follows form the fact that $q_1 \wedge q_2$ is again a p-primary element if q_1 and q_2 are both p-primary elements. This completes the proof for Proposition 6.

REMARK.

(a) *Taking right residuals in Proposition 6 shows that the same q also satisfies $a :_r q \nleq p$.*
(b) *In commutative rings, an ideal whose radical is a prime ideal is called primary.*

2. Conditions for a primary decomposition

A lattice L is said to satisfy the primary decomposition property if every element a of L can be expressed as a meet of a finite number of primary elements q_i of L. Such an expression is called a primary decomposition of a.

A primary decomposition $a = q_1 \wedge \cdots \wedge q_k$ is a normal decomposition of a if:

(1) No q_i is greater than or equal to the meet of the remaining q_j,

(2) $Rad(q_i) \neq Rad(q_j)$ if $i \neq j$.

It is well known that every primary decomposition can be reduced to a normal decomposition.

In general there can be more than one normal decomposition of an element a. However, the following are well-known(see [4]):

(1) The number of primary components in any two normal decompositions of a is the same,
(2) The set of radicals of the primary components occurring in any normal decomposition of a is unique.

The prime radical of a primary component which appears in any normal decomposition of a is called an associated prime element of a or a prime element belonging to a.

The following proposition gives a necessary and sufficient condition for a prime element to be an associated prime element for some element.

PROPOSITION 7. *Let $a = q_1 \wedge \cdots \wedge q_m$ be a normal decomposition of an element a, not equal to e, in a multiplicative lattice L satisfying the ACC. Let p be a prime element in L. Then p is an associated prime element of a if and only if there exist an element t in L and an integer i such that $t \leq q_i$, $p = a :_l t$ and $t \leq q_j$ for all $i \neq j$.*

PROOF. Assume that p is an associated prime element of a, say $p = p_1$ where $p_1 = Rad(q_1)$. If $m = 1$, then $a = q_1$ is a primary element and $p = a :_l t$ for some element t in L such that $t \not\leq a$ by Proposition 1. Hence, assume that $m > 1$. Since $a = q_1 \wedge \cdots \wedge q_m$ is a normal decomposition, there exists an element b in L such that $b \not\leq q_1$ and b is less than or equal to the meet of the remaining q_j, i.e., $b \leq q_2 \wedge \cdots \wedge q_m$. Then $a :_l b = q_1 :_l b$ and $q_1 :_l b$ is a p_1-primary element by Proposotion2, so $p_1^{k_0} \leq q_1 :_l b$ for some positive integer k_0. Clearly $q_1 :_l b$ is less than or equal to $(q_1 :_l b) :_r p_1$. If $q_1 :_l b = (q_1 :_l b) :_r p_1$, then $q_1 :_l b = (q_1 :_l b) :_r p_1 = ((p_1 :_r p_1) :_r p_1 = (q_1 :_l b) :_r p_1^2$ by (1.5) and thus $q_1 :_l b$ is equal to $(q_1 :_l b) :_r p_1^k$ for all positive integer k. Then, in particular, $q_1 :_l b = (q_1 :_l b) :_r p_1^{k_o} = e$ since $p_1^{k_o} \leq q_1 :_l b$ and thus $b = eb = (q_1 :_l b)b \leq q_1$, contradicting the choice of b. Hence $q_1 :_l b$ is strictly less than $(q_1 :_l b) :_r p_1$, so there exists an element c in

L such that

$$c \leq (q_1 :_l b) :_r p_1 \quad \text{and} \quad c \not\leq q_1 :_l b.$$

Then $p_1(cb) \leq q_1$, but $cb \not\leq q_1$ so that $q_1 :_l cb \leq p_1$ by Proposition 2. But $p_1(cb) \leq q_1$ also implies that $p_1 \leq q_1 :_l cb$. Thus $p_1 = q_1 :_l cb = (q_1 :_l b) :_l c = (a :_l b) :_l c = a :_l cb$ by (1.8). Hence $p_1 = a :_l cb$ and $cb \not\leq q_1$ but $cb \leq q_i$ if $i > 1$.

Conversely, let $p = a :_l t$ for some t in L such that $t \not\leq q_i$, $t \leq q_j$ if $j \neq i$. Then $p = (q_1 :_l t) \wedge \cdots \wedge (q_m :_l t) = q_i :_l t$ is a p_i-primary element and so $p = Rad(p) = Rad(q_i :_l t) = p_i$. Therefore, p is an associated prime element of a. This completes the proof for Proposition 7.

COROLLARY 2. *Let a and L be as in Proposotion7. Then $Rad(a) = a :_l t$ for some element t in L such that $t \not\leq q_i$ for all i.*

PROOF. For each i, $p_i = Rad(q_i) = a :_l t_i$ for some element t_i in L such that $t_i \not\leq q_i$. Hence, $Rad(a) = Rad(q_1 \wedge \cdots \wedge q_m) = Rad(q_1) \wedge \cdots \wedge Rad(q_m) = (a :_l t_1) \wedge \cdots \wedge (a :_l t_m) = a :_l t$, where $t = t_i \vee \cdots \vee t_m$. Thus $t \not\leq q_i$, because $t_i \not\leq q_i$ and $t_i \leq t$ for all i.

Before presenting conditions under which the primary decomposition property holds in lattices, it is useful to provide an example to demonstrate that the ACC assumption even on a commutative lattice is not sufficient for the existence of primary decomposition of all elements.

It is easy to construct a lattice L which is multiplicative, associative and commutative satisfying the ACC. However, note that the meet-irreducible element 0 is not a primary element, since $ab \leq 0$, $b \not\leq 0$, but $a^n = a \not\leq 0$ for all positive integer n. Thus, unlike the situation for commutative rings, a condition besides the ACC is necessary to guarantee the existence of primary decompositions. The following are three major conditions which appear on the literature to guarantee the existence of primary decompositions under some assumptions.

(1) Ward and Dilworth [7].

(WD) : Given two elements a and b in a lattice, there exists a positive integer n such that

$$a^n \wedge b \leq ab.$$

(2) Barnes and Cunnea [1],[5].
(BC′) : Given two elements a and b in a lattice, there exists a positive integer n such that

$$b^n \wedge (a :_r b^n) \leq a.$$

(3) Barnes and Cunnea [1],[5].
(BC) : Given two elements a and b in a lattice, there exists a positive integer n such that

$$(a \vee b^n) \wedge (a :_r b^n) = a.$$

For convenience, the statement that every element of a lattice has a primary decomposition will be denoted by N. Ward and Dilworth have proved that the condition (WD) is equivalent to the condition (N) in a commutative, associative and modular lattice satisfying the ACC, and Lesieur has proved the same result for an associative and semi-modular lattice [3], and Kurata [2] and McCarthy [5] for a non-associative lattice. The condition (BC′) and (BC) were discovered by Barnes and Cunnea [1] in a commutative Noetherian ring, and the equivalence of (N) to the condition (BC′), in a residuated lattice satisfying the ACC, and (N) to the Condition (BC) in a modular and residuated lattice with the ACC, have been proved by McCarthy [5].

The main result in the this paper is to present another conditions, in addition to those listed before, on a (non-commutative) semi-modular lattice satisfying the ACC under which every element has a primary decomposition. Those conditions are applied to a direct construction of a normal decomposition without use of a meet-irreducible decomposition (See Property 1) .

To avoid repetitions, the following condition is denoted (*):

(*) If a and b are two arbitrary elements in a lattice, then there exists a positive integer n such that

$$b \wedge (a :_l b)^n \leq a.$$

To show that the condition (N) is equivalent to the condition (*), the following impotent result due to Lesieur [3] is needed.

LEAIEURE'S LEMMA. *Let a be a meet-irreducible element of a semi-modular lattice L. If $a \wedge b = c \wedge b$ for b and c in L with $a < c$, then $b \leq a$.*

PROOF. First, observer that $c \wedge b \leq a \leq c \leq c \vee b$. If $c \wedge b = a$, and thus $a \leq b$. Then $a \vee b = b$ and $c \wedge (a \vee b) = c \wedge b = a \wedge b = a$, but $a < c$. Hence, from the meet-irreducibility of a, $a \vee b$ is equal to a so that $b \leq a$. Next, if $c = c \vee b$, then $b \leq c$ and so $a \wedge b = c \wedge b = b$, hence $b \leq a$. Finally, assume that $c \wedge b < a$ and $c < c \vee b$. Then, from the semi-modularity of L, there exists an element t in L such that $c \wedge b < t \leq b$ and $a = (a \vee t) \wedge c$. But, then, since a is meet-irreducible and $a < c$, it follows that $a \vee t = a$, so $t \leq a$ and thus $t \leq a \wedge b \leq c \wedge b \leq t$, which is a contradiction. This completes the proof of the lemma.

The following theorem is one of the main result of this paper.

THEOREM 2. *Let L be a semi-modular, multiplicative lattice satisfying the ACC. Then the conditions (N) and (*) are equivalent.*

PROOF. First assume that the condition (N) holds in L and let a and b be two arbitrary elements in L. Let $a = q_1 \wedge \cdots \wedge q_k$ be a primary decomposition of a where each q_i is a primary element. Assume that the q_i's are arranged so that $b \leq q_1, \cdots, b \leq q_m$ for $m \leq k$, and $b \not\leq q_i$ if $i > m$. Then $a :_l b = (q_1 :_l b) \wedge \cdots \wedge (q_m :_l b) \wedge (q_{m+1} :_l b) \wedge \cdots \wedge (q_k :_l b) = (q_{m+1} :_l b) \wedge \cdots \wedge (q_k :_l b)$, and each $q_i :_l b \leq Rad(q_i)$ for $i > m$ by Proposition 2. So there exists a positive integer n such that $(q_i :_l b)^n$ is less than or equal to q_i for all $i > m$. Then, clearly $(a :_l b)^n \leq q_{m+1} \wedge \cdots \wedge q_k$, and thus

$$b \wedge (a :_l b)^n \leq q_1 \wedge \cdots \wedge q_m \wedge q_{m+1} \wedge \cdots \wedge q_k = a,$$

which proves that the condition (*) holds. Thus (N) implies the condition(*). To show the condition (*) implies the condition (N), it suffices by property (1.1), to prove that if a is a meet-irreducible element of L, then a is a primary element. Assume that a is meet-irreducible, $bc \leq a$ and $c \not\leq a$ for b and c in L. By the condition (*), there is a positive integer n such that

$$(a \vee c) \wedge (a :_l (a \vee c))^n \leq a.$$

Then $bc \leq a$ implies that $b \leq a :_l c = a :_l (a \vee c)$, so $b^n \leq (a :_l (a \vee c))^n$ and thus $(a \vee c) \wedge b^n \leq a$, $(a \vee c) \wedge b^n \nleq a \wedge b^n$. However, $a \wedge b^n \leq (a \vee c) \wedge b^n$ for all positive integers n so that

$$(a \vee c) \wedge b^n = a \wedge b^n.$$

But $a \vee c > a$, i.e., $a \vee c$ is strictly greater than a. Then, by the Lesieur's Lemma, $b^n \leq a$, proving a is a primary element. Hence, the condition (*) implies the (N) in L.

Note that, in the proof of Theorem 2, the positive integer n was chosen so that $(a :_l b)^n \leq q_{m+1} \wedge \cdots \wedge q_k$. Therefore, $a \vee (a :_l b)^n \leq a \vee (q_{m+1} \wedge \cdots \wedge q_k) = q_{m+1} \wedge \cdots \wedge q_k$ since each $q_i \geq a$. Also note that $b \leq q_1 \wedge \cdots \wedge q_m$ implies $a \vee b \leq a \vee (q_1 \wedge \cdots \wedge q_m) = q_1 \wedge \cdots \wedge q_m$. Hence, it follows that $(a \vee b) \wedge (a \vee (a :_l b)^n)$ is less than or equal to $q_1 \wedge \cdots \wedge q_m \wedge \cdots \wedge q_k = a$ and so $(a \vee b) \wedge (a \vee (a :_l b)^n) = a$. Conversely, if $a = (a \vee b) \wedge (a \vee (a :_l b)^n) \leq a$, which is the condition (*) and thus the condition(N) holds in L. Hence, the following result has been proved.

COROLLARY 3. *Let L be a semi-modular, multiplicative lattice satisfying the ACC. Then the following conditions are equivalent:*

(1) *L satisfies the condition (N);*
(2) *L satisfies the following condition :*
(**) *Given two elements a and b in L, there exists a positive integer n such that*

$$a = (a \vee b) \wedge (a \vee (a :_l b)^n).$$

It is well known that if R is a commutative Noetherian ring, then the lattice L(R) of all ideals of R is modular (and hence semi-modular) and satisfies the condition (N) and thus the condition (*) by [6] and Theorem 2. Hence,

COROLLARY 4. *Let R be a commutative Noetherian ring and let A and B be ideals of R. Then, there exists a positive integer n such that $(A + B) \cap (A + (A : B)^n) = A$.*

Without using the primary decomposition in Noetherian rings, the following Krull Intersection Theorem can be obtain (see [1]).

COROLLARY 5. *Let R be a commutative Noetherian ring and M be an ideal of R. Set $A = \cap_n M^n$. Then $A = AM$.*

PROOF. By Corollary 4, there exists a positive integer n such that $A \cap (AM : A)^n$ is contained in AM. Clearly, M is contained in $AM : A$, so $A = A \cap M^n$ is contained in $A \cap (AM : A)^n$, which show that A is contained in AM. The reverse inclusion is obvious.

A similar result holds in a lattice satisfying the condition (*).

COROLLARY 6. *Let L be a residuated lattice satisfying the condition (*), and let a be an arbitrary element of L. If $\wedge_n a^n$ exists in L, and $x \in L$ is less than or equal to $\wedge_n a^n$, then $x = ax$.*

PROOF. By (*), there exists a positive integer n such that

$$x \wedge (ax :_l x)^n \leq ax.$$

Note that $ax \leq ax$ implies that $a \leq ax :_l x$ so $x \leq x \wedge a^n \leq x \wedge (ax :_l x)^n \leq ax$. But $ax \leq x$ is always true and so $x = ax$.

COROLLARY 7. *Let L be a multiplicative lattice satisfying the ACC and the condition (*). If a and x are elements of L such that $x \leq \wedge_n a^n$, then $x = ax$.*

PROOF. The conclusion follows from the fact that $\wedge_n a^n$ exists in L for every element a and from Corollary 6.

COROLLARY 8. *Let L be a semi-modular, multiplicative lattice satisfying the ACC and the condition (N). If $c = \wedge_k (a \vee b^k)$, then $c = a \vee bc$.*

PROOF. Note that $a \vee bc \leq c$ is always true since $a \leq c$ and $bc \leq c$. Since the condition (N) holds in L, it follows that the condition (**) holds in L by Corollary 3. Hence, there exists a positive integer n such that

$$a \vee bc = [(a \vee bc) \vee c] \wedge [(a \vee bc) \vee ((a \vee bc) :_l c)^n].$$

Since $bc \leq a \vee bc$, $b \leq (a \vee bc) :_l c$, we have $b^n \leq ((a \vee bc) :_l c)^n$ and hence $a \vee bc \geq c \wedge [(a \vee bc) \vee b^n] = c \wedge [(a \vee b^n) \vee bc] \geq c \wedge (c \vee bc) = c$. Therefore, $a \vee bc \geq c$ and thus $c = a \vee bc$.

The following result gives a direct relationship between the condition (WD) and condition (*) in all (left-) residuated lattices (not necessarily satisfying the ACC).

THEOREM 3. *Let L be a (left-) residuated lattice. Then the conditions (WD) and (*) are equivalent.*

PROOF. Assume that the condition (WD) holds in L and let a and b be two arbitrary elements in L. Then there exists a positive integer n such that $(a :_l b)^n \wedge b \leq (a :_l b)b$. But $(a :_l b)b \leq a$ and so $(a :_l b)^n \wedge b \leq a$, and the condition (*) holds. Thus the condition (WD) implies the condition (*) in L. Conversely, assume that the condition (*) is true in L and choose a positive integer n so that, for a and b in L , $b \wedge (ab :_l b)^n \leq ab$. Then $ab \leq ab$ implies that $a \leq ab :_l b$, hence a^n is less than or equal to $(ab :_l b)^n$ and thus $a^n \wedge b \leq ab$, which is the condition (WD). This completes the proof for Theorem 3.

In Theorem 1, it has been proved that the assumption of the ACC implies that every multiplicative lattice is residuated. The following theorem asserts that the conditions (BC′) and (*) are equivalent in such lattices.

THEOREM 4. *Let L be a multiplicative lattice satisfying the ACC. Then, the conditions (BC') and (*) are equivalent.*

PROOF. Assume that the condition (BC′) holds in L and let a and b be arbitrary element in L. Then there exists a positive integer n such that

$$(a :_l b)^n \wedge [a :_r (a :_l b)^n] \leq a.$$

Observe that $(a :_l b)b \leq a$ implies that $b \leq a :_r (a :_l b) \leq a :_r (a :_l b)^n$. Thus, $b \wedge (a :_l b)^n \leq a$, which is the condition (*). Conversely, assume that the condition (*) holds in L: if a and b are arbitrary elements in L, then $b \wedge (a :_l b)^n \leq a$ for some positive integer n. First, by the ACC, we can choose a positive integer k so that $a :_r b^k = a :_r b^t$ for all positive integers t such that $t \geq k$. Next, by the condition (*), choose a positive integer m such that

$$(a :_r b^k) \wedge [a :_l (a :_r b^k)]^m \leq a.$$

Now $b^k(a :_r b^k) \leq a$ implies that $b^k \leq a :_l (a :_r b^k)$. Set $n = km$. Then $a :_r b^k = a :_r b^n$ and $b^n = (b^k)^m \leq [(a :_l (a :_r b^k)]^m$ so that $b^n \wedge (a :_r b^n) \leq [a :_l (a :_r b^k)]^m \wedge (a :_r b^k) \leq a$, which is the condition (BC′). Thus the condition (*) implies the condition (BC′) in L. This completes the proof for Theorem 4.

It is well known that the condition (BC) is equivalent to the conditions (N) and (WD) in a modular lattice satisfying the ACC [5]. However, it has not been known whether or not the condition (N) implies the condition (BC) even in semi-modular lattices. The following theorem and its corollary settle this matter.

THEOREM 5. *Let L be a multiplicative lattice satisfying the ACC. Then, the conditions (BC) and (**) are equivalent.*

PROOF. Assume that the condition (BC) holds in L and let a and b be two arbitrary elements in L. Then, there exists a positive integer n such that

$$a = [a \vee (a :_l b)^n \wedge [a :_r (a :_l b)^n].$$

Since $(a :_l b)b \leq a$, $b \leq a :_r (a :_l b)$ we have $b \leq a :_r (a :_l b)^n$ so that $[a \vee (a :_l b)^n \wedge (a \vee b) \leq a$. Since the reverse inequality is always true, it follows that $a = [a \vee (a :_l b)^n] \wedge (a \vee b)$, which is the condition (**). Thus the condition (BC) implies the condition (**) in L . Conversely, assume that the condition (**) holds in L and let a and b be arbitrary elements in L. By the ACC, there exists a positive integer k such that $a :_r b^k = a :_r b^n$ for all positive integer n such that $n \geq k$. Next, using the condition (**), choose a positive integer m so that

$$a = [a \vee (a :_r b^k)] \wedge [a \vee (a :_l (a :_r b^k))^m].$$

Then $b^k(a :_r b^k) \leq a$ implies that $b^k \leq a :_l (a :_r b^k)$ and thus $b^{km} = (b^k)^m \leq [a :_l (a :_r b^k)]^m$. Let $n = km$. Then $a :_r b^n = a :_r b^k$ and thus $(a :_r b^n) \wedge (a \vee b^n) \leq (a :_r b^k) \wedge [a \vee (a :_l (a :_r b^k))^m] \leq a$. But $(a :_r b^n) \wedge (a \vee b^n) \geq a$ is always true, so $a = (a :_r b^n) \wedge (a \vee b^n)$, which is the condition (BC). Thus the condition (**) implies that the condition (BC) in L. This completes the proof for Theorem 5.

COROLLARY 9. *Let L be a semi-modular, multiplicative lattice satisfying the ACC. Then the conditions (N), (WD), (BC′), (*) and (**) are all equivalent in L.*

PROOF. Note that the following equivalences already have been proved : (WD) if and only if (*) in Theorem 3, and (N) if and only if (*) in Theorem 2, and (N) if and only if (**) in corollary 3, and (*) if and only if (BC′) in Theorem 4, and finally (**) if and only if (BC) in Theorem 5. Hence, all the conditions are equivalent in L.

3. Construction of a primary decomposition

In this chapter, let L be a (not necessarily commutative) multiplicative, associative and modular lattice satisfying the ACC and the condition (*).

It has been proved that L under these assumptions satisfies the condition (N). In the proof of this, the fact that every element of L can be expressed as a meet of a finite number of meet-irreducible elements of L was used. In this chapter, a direct construction of a primary decomposition will be given without use of the irreducibility of elements in L. For this purpose, a condition under which a prime element is a minimal prime element of some element is given in the following proposition.

PROPOSITION 8. *Let a be an arbitrary element of L such that $a \neq e$, and let p be a prime element of L. Then p is a minimal prime element of a if and only if there exists an element q in L such that $a \leq q$, $Rad(q) = p$ and $a :_l q \not\leq p$.*

PROOF. Assume that there exists an element q in L satisfying $a \leq q$, $Rad(q) = p$ and $a :_l q \not\leq p$. Let n be a positive integer such that $p^n \leq q$. If a itself is a prime element and $a \neq p$, then $a :_l p^n = a$, since $(a :_l p^n)p^n \leq a$ and $p^n \not\leq a$. But then $a :_l q \leq a :_l p^n = a \leq p$, which is a contradiction. Hence, assume that a is not a prime element, and suppose that there exists a prime element p^* such that $a < p^* < p$, that is $a \neq p^*$ and $p^* \neq p$. Then $a :_l p^n \leq p^* :_l p^n = p^* \leq p$, but

$a :_l q \nleq p$ and $a :_l q \leq a :_l p^n \nleq p$, which is a contradiction. Hence, p is a minimal prime element of a.

Conversely, assume that p is a minimal prime element of a. Note that $p \neq e$. For, if not, then $Rad(a) = p = e$ implies that $a = e$, which is not the case. Consider the following set N of elements in L;

$$N = \{t \in L \mid a \leq t \text{ and } a :_l t \nleq p\}.$$

Since $a \leq a$ and $a :_l a = e \nleq p$, it follows that a is contained in N, so N is nonempty and it contains a maximal element p by the ACC. Then $a \leq q$ and $a :_l q \nleq p$. Note that since $(a :_l q)q \leq a \leq p$ and $a :_l q \nleq p$ for p a prime element, it follows that $q \leq p$ so that $Rad(q) \leq Rad(p) = p$. Also note that if $bc \leq q$ and $c \nleq q$ for b and c in L, then $b \leq p$. Now $c \nleq q$ implies that $q \vee c$ is strictly greater than q and thus $a :_l (q \vee c) \leq p$ by the maximality of q in N. Since $b(q \vee c) = bq \vee bc \leq bq \vee q = q$, $a :_l q \leq a :_l b(q \vee c) = (a :_l (q \vee c)) :_l b$ by Property (1.8) and thus $(a :_l (q \vee c)) :_l b \nleq p$. Therefore, if $b \nleq p$, then $p :_l b = p$, which is a contradiction. Thus $b \leq p$. Now, assume that $Rad(q) \neq p$. Since $Rad(q) \leq p$ and p is also a minimal prime element of q, it follows that $Rad(q) \leq p \wedge t$ where t is a meet of the remaining minimal prime element(s) of q. Let k be the smallest positive integer such that $(Rad(q))^k \leq q$, i.e., $(Rad(q))^n \nleq q$ if $n < k$. If $k = 1$,then $Rad(q) = q$ so that $tp \leq t \wedge p = q$ and $t \nleq p$ imply that $p \leq q$, and thus $p = q$. So $a :_l p = a :_l q \nleq p$. Assume that $k > 1$. Then $tp(Rad(q))^{k-1} \leq (t \wedge p)(Rad(q))^{k-1} = Rad(q))^k \leq q$ and $t \nleq p$ imply that $p(Rad(q))^{k-1} \leq q$. By (*) and (WD) from Theorem 3, there exists a positive integer n_1 such that $p^{n_1} \vee (Rad(q))^{k-1} \leq p(Rad(q))^{k-1}$. Then $(Rad(q))^{k-1}p^{n_1} \leq q$ and thus $(t_p)(Rad(q))^{k-2}p^{n_1} \leq (Rad(q))^{k-1}p^{n_1} \leq q$ and $t \nleq p$ imply that $p(Rad(q))^{k-2}p^{n_1} \leq q$. Again by the condition (*), there exists a positive integer n_2 such that

$$p^{n_2} \wedge (Red(q))^{k-2} \leq p(Rad(q))^{k-2},$$

and hence

$$(Rad(q))^{k-2}p^{n_2}p^{n_1} \leq [p^{n_2} \wedge (Rad(q))^{k-2}]p^{n_1} \leq p(Rad(q))^{k-2}p^{n_1} \leq q.$$

Continuing this way, there exist positive integers $n_1, n_2, \cdots, n_{k-1}$ such that

$$(Rad(q))^{k-(k-1)}p^{n_{k-1}} \cdots p^{n_2}p^{n_1} \leq q.$$

Then form

$$tp(p^{n_{k-1}} \cdots p^{n_2}p^{n_1}) \leq (Rad(q))(p^{n_{k-1}} \cdots p^{n_2}p^{n_1}) \leq q$$

and $t \not\leq p$, we get $p^{n_1+n_2+\cdots+n_{k-1}+1} \leq q$, which means that $p \leq Rad(q)$. Since $Rad(q) \leq p$ is true, it follows that $Rad(q) = p$, which contradicts the assumption $Rad(q) \neq p$. Therefore, q is an element in L such that $a \leq q$, $Rad(q) = p$ and $a :_l q \not\leq p$. This completes the proof for Proposition 8.

Lemma 1. *Let L be a residuated lattice. Then*

(1) *For all elements a, b and c in L, $(a :_l b)(b :_l c) \leq a :_l c$.*
(2) *If p is a prime element in L, $a :_l (b \wedge c) \not\leq p$ and $c :_l d \not\leq p$, then $a :_l (b \wedge d) \not\leq p$.*

Proof. (1) Note that $(a :_l b)(b :_l c)c \leq (a :_l b)b \leq a$ implies that

$$(a :_l b)(b :_l c) \leq a :_l c$$

by the definition of lift-residuals.
(2) Observe that

$$\begin{aligned}(a :_l (b \wedge c))(c :_l d)(b \wedge d) &\leq (a :_l (b \wedge c))[(c :_l d)b \wedge (c :_l d)d] \\ &\leq (a :_l (b \wedge c))(b \wedge c) \leq a\end{aligned}$$

implies $(a :_l (b \wedge c))(c :_l d) \leq a :_l (b \wedge d)$. Now the conclusion follows from the definition of a prime element.

Corollary 10. *If p is a minimal prime element of an element $a \neq e$, then there exists a p-primary element q in L such that $a \leq q$ and $a :_l q \not\leq p$.*

PROOF. By Proposition 8, there exists an element q' in L such that $a \leq q'$, $Rad(q') = p$ and $a :_l q' \not\leq p$. Then by Proposition 6 in Chapter 1, there exists a p-primary element q of q' such that $q' :_l q \not\leq p$. Since $(a :_l q')(q' :_l q) \leq a :_l q$ by Lemma 1.(1),it follows that $a :_l q \not\leq p$. Now $a \leq q$ follows from the facts that $a \leq q'$ and $q' \leq q$.

Now, assume that, in addition to the ACC and the condition (*), L satisfies the modularity condition. Let a be an arbitrary element of L, not equal to e. To show that a has a primary decomposition in L, first let $p_{11}, \ldots, p_{1k_1}$ be the collection of all minimal prime elements of a. Recall that by Proposition 5,the number of such prime elements is finite. For each i, there exists a p_{1i} -primary element q_{1i} such that a $\leq q_{1i}$ and $a :_l q_{1i} \not\leq p_{1i}$ by Corollary 10. Set $q_1 = q_{11} \wedge \cdots \wedge q_{1k_1}$. If $a :_l q_1 = e$, then $q \leq a$ by Proposition 6 and thus $a = q_1 = q_{11} \wedge \cdots \wedge q_{1k_1}$ is a normal decomposition of a. If $a :_l q_1 \neq e$, then using the condition (*) choose a positive integer n such that

$$q_1 \wedge (a :_l q_1)^n \leq a. \tag{1}$$

Let $p_{11}, \cdots, p_{2k_2}$ be the set of all minimal prime elements of $a :_l q_1$. Observe that $p_{2i} \not\leq p_{1j}$ for all i and j since $a :_l q_{1j} \not\leq p_{1j}$, and $a :_l q_{1j} \leq a :_l q_1 \leq p_{2i}$. Note that each p_{2i} is also a minimal prime element of $(a :_l q_1)^n$. For, if there exists a prime element p such that $(a :_l q_1)^n \leq p \leq p_{2i}$, then $a :_l q_1 \leq p$. Since p_{2i} is a minimal prime element of $a :_l q_1$, $p = p_{2i}$, and hence p_{2i} is in fact a minimal prime element of $(a :_l q_1)^n$. Therefore, by Corollary 10 , for each $i = 1, \cdots, k_2$, there exists a p_{2i}-primary element s_{2i} in L such that $(a :_l q_1)^n \leq s_{2i}$ and $(a :_l q_1)^n :_l s_{2i} \not\leq p_{2i}$. Since $p_{2i} \not\leq p_{1j}$ for all i and j and $Rad(s_{2i}) = p_{2i}$, it follows that, $s_{2i} \not\leq q_1 = q_{11} \wedge \cdots \wedge q_{1k_1}$ for all i.
Note that, $a :_l (q_1 \wedge s_{2i}) \not\leq p_{2i}$ for all i, since

$$\begin{aligned}
(a :_l q_1)^n :_l s_{2i} \leq& (a :_l q_1)^n :_l (q_1 \wedge s_{2i}) \\
=& e \wedge [(a :_l q_1)^n :_l (q_1 \wedge s_{2i})] \\
=& [q_1 :_l (q_1 \wedge s_{2i})] \wedge [(a :_l q_1)^n :_l (q_1 \wedge s_{2i})] \\
=& (q_1 \wedge (a :_l q_1)^n) :_l (q_1 \wedge s_{2i}) \\
\leq& a :_l (q_1 \wedge s_{2i})
\end{aligned}$$

implies that $(a :_l q_1)^n :_l s_{2i} \leq a :_l (q_1 \wedge s_{2i})$ and $(a :_l q_1)^n :_l s_{2i} \not\leq p_{2i}$ by the choice of s_{2i}. Note also that,for all $i = 1, \ldots, k_2$,

$$(2) \qquad a :_l (q_1 \wedge (a \vee s_{2i})) \not\leq p_{2i}.$$

By the modularity of L, $a \leq q_1$ implies that $(q_1 \wedge (a \vee s_{2i})) = a \vee (q_1 \wedge s_{2i})$, so $a :_l (q_1 \vee (a \wedge s_{2i})) = a :_l (a \vee (q_1 \wedge s_{2i})) = a :_l (q_1 \wedge s_{2i}) \not\leq p_{2i}$, which proves the inequality (2). Since $s_{2i} \leq a \vee s_{2i} \leq p_{2i}$, it follows that $p_{2i} = Rad(s_{2i}) \leq Rad(a \vee s_{2i}) \leq Rad(p_{2i}) = p_{2i}$ and thus $Rad(a \vee s_{2i}) = p_{2i}$. So, again by Corollary 10, for each $i = 1, \cdots, k_2$, there exists a p_{2i} -primary element q_{2i} such that

$$(3) \qquad a \vee s_{2i} \leq q_{2i} \text{ and } (a \vee s_{2i}) :_l q_{2i} \not\leq p_{2i}.$$

Note that each p_{2i} is a prime element. The inequalities (2) and (3) imply that $a :_l (q_1 \wedge q_{2i}) \not\leq p_{2i}$ by Lemma 1.(2). Thus $q_1 \wedge q_{21} \wedge \cdots \wedge q_{2k_2} = q_{11} \wedge \cdots \wedge q_{1k_1} \wedge \cdots \wedge q_{2k_2}$ is a meet of primary elements satisfying

$$a :_l (q_1 \wedge q_{21} \wedge \cdots \wedge q_{2k_2}) \not\leq Rad(a :_l q_1) = p_{21} \wedge \cdots \wedge p_{2k_2},$$

because of the facts that $a :_l (q_1 \wedge q_{2i}) \not\leq q_{2i}$ and $a :_l (q_1 \wedge q_{2i}) \leq a :_l (q_1 \wedge q_{21} \wedge \cdots \wedge q_{2k_2})$. If $a :_l (q_1 \wedge q_{21} \wedge \cdots \wedge q_{2k_2}) = e$, then $a = q_1 \wedge q_{21} \wedge \cdots \wedge q_{2k_2} = q_{11} \wedge \cdots \wedge q_{1k_1} \wedge \cdots \wedge q_{2k_2}$ is a primary decomposition for a. If $a :_l (q_1 \wedge q_{21} \wedge \cdots \wedge q_{2k_2}) \neq e$, then $q_1 \wedge q_{21} \wedge \cdots \wedge q_{2k_2}$ is strictly greater than a. Then the same argument as before can be applied to obtain p_{3i}-primary elements q_{3i} such that $a :_l (q_1 \wedge q_{21} \wedge \cdots \wedge q_{2k_2} \wedge q_{3i}) \not\leq p_{3i}$, where p_{3i} is a minimal prime element of $a :_l (q_1 \wedge q_{21} \wedge \cdots \wedge q_{2k_2})$. Continuing this argument, a descending sequence of elements in L can be constructed in the following way :

$$a_1 = q_1 = q_{11} \wedge \cdots \wedge q_{1k_1},$$

and more generally, $a_{n+1} = a_n \wedge q_{n+1}$, where q_{n+1} is the meet of all of the primary elements of $a :_l a_n$ as found above, if $a :_l a_n \neq e$. Then, since the sequence $\{a_n\}$ is descending, it follows that the elements $a :_l a_n$ form an ascending sequence and hence, by the ACC in L, there exists a positive integer n such that $a :_l a_n = a :_l a_k$ for all positive integer $k \geq n$. If $a :_l a_n \neq e$, then $a :_l a_n$ has a minimal prime element p, but $a :_l a_{n+1} \not\leq p$ by the construction of the element a_{n+1}. Thus $a :_l a_n = e$, which means $a = a_n = q_1 \wedge \cdots \wedge q_n$, where $q_i = q_{i1} \wedge \cdots \wedge q_{ik_i}$ is a primary decomposition for a.

It remains to show that $a = q_1 \wedge \cdots \wedge q_n$, where $q_i = q_{i1} \wedge \cdots \wedge q_{ik_i}$ is a meet of p_{ij}-primary elements q_{ij}, p_{ij} is a minimal prime element of $a :_l (q_1 \wedge \cdots \wedge q_{i-1})$, is a normal decomposition for a. Recall that the elements a_i, q_i and p_{ij} satisfy the following properties :

$$(4) \qquad a :_l (a_{i-1} \wedge q_{ij}) \not\leq p_{ij} \quad \text{for all} \quad j = 1, \cdots k_i \quad \text{and} \quad i \geq 2,$$

where p_{ij} is a minimal prime element of $a :_l a_{i-1} = a :_l (q_{11} \wedge \cdots \wedge q_{i-1,1} \wedge \cdots \wedge q_{i-1,k_{i-1}})$. First, note that $p_{ij} \neq p_{st}$ for all j and t if $i \neq s$. For, assume that $s > i$ and recall that $a :_l (a_{i-1} \wedge q_{ij}) \not\leq p_{ij}$, but $a :_l (a_{i-1} \wedge q_{ij}) \leq a :_l a_{s-1} \leq p_{st}$ for all j and t. Also, by construction, $p_{ij} \neq p_{ik}$ if $j \neq k$. Thus $p_{ij} = p_{st}$ for $i \neq s$ or $j \neq t$, since the radicals of distinct primary components are different. Next, suppose that some q_{ij} is greater than or equal to the meet of the remaining primary components appearing in the decomposition of a. We claim that the product $q_{i+1} \cdots q_n \not\leq Rad(q_{ij}) = p_{ij}$. If not, then for some $s \geq i+1$, $q_s \leq p_{ij}$ since p_{ij} is a prime element. Then $q_s = q_{s1} \wedge \cdots \wedge q_{sk_s} \leq p_{ij}$ implies that, for some t, $q_{st} \leq p_{ij}$. Therefore, $p_{st} = Rad(q_{st}) \leq p_{ij}$. However, this contradicts the facts that $a :_l a_i = a :_l (q_1 \wedge \cdots \wedge q_i) \not\leq p_{ij}$ and $a :_l a_i \leq a :_l a_{s-1} \leq p_{st}$. This proves that $q_{i+1} \cdots q_n \not\leq Rad(q_{ij}) = p_{ij}$. Let q_i/q_{ij} be the meet of the remaining primary components in q_i after deleting q_{ij}. Then $(q_{i+1} \cdots q_n)(q_1 \wedge \cdots \wedge q_{i-1} \wedge (q_i/q_{ij})) \leq q_{ij}$. Since q_{ij} is a primary element and $q_{i+1} \cdots q_n \not\leq Rad(q_{ij})$, it follows that $q_1 \wedge \cdots \wedge q_{i-1} \wedge (q_i/q_{ij}) \leq q_{ij}$. Since the prime elements p_{it} are all distinct for $t = 1, \cdots, k_i$, it follows that, for $t \neq j$, $(q_{ij})^k \not\leq q_{ij}$ and $q_{it} \not\leq Rad(q_{ij})$. Thus $(q_1 \wedge \cdots \wedge q_{i-1}) \wedge (q_i/q_{ij}) \leq q_{ij}$, $(q_1/q_{ij})(q_1 \wedge \cdots \wedge q_{i-1}) \leq q_{ij}$, and $q_1/q_{ij} \not\leq Rad(q_{ij})$. Therefore, $a_{i-1} = q_1 \wedge \cdots \wedge q_{i-1} \leq q_{ij}$. But then $a_{i-1} = a_{i-1} \wedge q_{ij}$ and thus $a :_l a_{i-1} = a :_l (a_{i-1} \wedge q_{ij}) \not\leq p_{ij}$ by the inequality (4), contradicting the fact that $p_{ij} \geq a :_l a_{i-1}$. Therefore, for all i and j, q_{ij} cannot be greater than or equal to the meet of the remaining primary component in $a = q_{11} \wedge \cdots \wedge q_{1k_1} \wedge \cdots \wedge q_{n1} \wedge \cdots \wedge q_{nk_n}$. Thus the following theorem has been proved.

THEOREM 6. *Let $a \neq e$ be an arbitrary element in a modular, multiplicative lattice L satisfying the ACC and the condition (*). Then the given construction yields a (automatically normal) primary decomposition of a in L.*

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