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DUALITY IN THE OPTIMAL CONTROL PROBLEMS OF NONLINEAR PARABOLIC SYSTEMS

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ABSTRACT. In this paper, we study the duality theory of nonlinear parabolic systems . The main objective is to prove the duality theorem under general conditions within an infinite-dimensional framework. As an application, an example is given.

1. Introduction

Recently, many authors have studied optimal control problems of various practical systems [1, 3, 4, 6]. The crucial to these developments is to establish the optimality systems which characterize the optimal control. The optimal control problems is to find existence condition of infimum cost functional of primal systems. The purpose of duality theory show that the infimum cost functional of given systems is equal to suprimum cost functional of dual systems. Thus we can solve optimal control problem of given systems by finding a solution of the optimal control problem of dual systems. Up to now, most of authors dealt with duality theory of linear systems. Chan[2] and Tanimoto[10] have studied the duality theory for linear parabolic optimal control problem. Park and Lee [7, 8, 9] dealt with duality theory for the corresponding linear, nonlinear hyperbolic optimal control systems. In this paper, we study duality theory for strongly nonlinear control system governed by

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parabolic equations. The main purpose of this paper is to apply results to nonlinear parabolic optimal control systems and to prove the duality theorem under general conditions within an infinite-dimensional framework. Our approach is based on an infinite dimensional version of the duality theorem for the optimality systems. To this end, we invoke necessary and sufficient conditions of optimality for distributed control systems given in [5].

2. Preliminaries

Let $I = [0,T](T > 0)$ and X, H be separable Hilbert spaces. We assume that $X \subset H$, the injection of X into H is continuous and X is dense in H. Identifying H its dual (pivot space), we have $X \hookrightarrow H \hookrightarrow X^*$. By $< \cdot, \cdot >$, we will denote the duality brackets for the pair (X, X^*) and by $(\cdot,\cdot)$ the inner product in H. The two are compatible in the sense that $< \cdot, \cdot > |_{X \times H} = (\cdot,\cdot)$. Also by $\|\cdot\|$(resp. $|\cdot|, \|\cdot\|_*$), we will denote the norm of X(resp. H, X^*). We define a function space $W(0,T)$ by

$$W(0,T) = \{x | x \in L^2(0,T;X), \dot{x} \in L^2(0,T;X^*)\}$$

with an inner product

$$(x_1, x_2)_{W(0,T)} = \int_0^T \{(x_1(t), x_2(t))_X + (\dot{x}_1(t), \dot{x}_2(t))_{X^*}\}dt,$$

where $\dot{x} = \dfrac{dx}{dt}$. This space furnished with the

$$\|x\|_{W(0,T)} = \left(\|x\|^2_{L^2(0,T,X)} + \|\dot{x}\|^2_{L^2(0,T;X^*)} \right)^{\frac{1}{2}},$$

Furthermore, it is well known that $W(0,T) \hookrightarrow C(I,H) = \{y : I \to H \text{ continuous}\}$ continuously. Finally, since $X \hookrightarrow H$ compactly, we have that $W(0,T) \hookrightarrow L^2(H)$ compactly. The control space will be modelled by a separable, reflexive Banach space Y. By $P_{fc}(Y)$ we will denote the family of nonempty, closed convex subset of Y. We consider the following optimal control problem :

$$(P)\begin{cases} J(x,u) = l(x(T)) + \int_0^T L(t,x(t),u(t))dt \to \inf \\ \text{subject to} \\ \begin{cases} \dot{x}(t) + A(t)x(t) = B(t)u(t) + f(t,x(t)) \text{ a.e.} \\ x(0) = x_0, \;\; u(t) \in U(t) \text{ a.e }, u(\cdot) \text{ measurable.} \end{cases} \end{cases} \tag{2.1}$$

We will need the following hypotheses on the data of the optimal control problem (P).

$(H1)$ Let $a(t;\phi,\varphi)$, $t \in I$ be a bilinear form on $X \times X$ satisfying

(i) $a(t;\phi,\varphi) = a(t;\varphi,\phi), \forall \phi,\varphi \in X, \forall t \in I$,

(ii) $\exists c > 0$ such that

$$|a(t;\phi,\varphi)| \leq c\|\phi\|_X \|\varphi\|_X, \forall \phi,\varphi \in X, \forall t \in I$$

and $\exists \alpha > 0, \lambda_\imath \in R$ such that

$$a(t;\phi,\phi) \geq \alpha\|\phi\|_X^2, \forall \phi \in X, \forall t \in I.$$

Under these assumptions, we can define the operator $A(t) \in \mathcal{L}(X,X^*)$ for $t \in I$ defined by the relation

$$a(t;\phi,\varphi) = \langle A(t)\phi,\varphi\rangle_{X^*,X} \text{ for all } \phi,\varphi \in X.$$

$(H2)$ $f(t,x) : I \times X \to H$ is a map

(i) $t \to f(t,x)$ is (strongly) measurable in H for all x , $f(t,\cdot)$ is sequentially weakly continuous and $f(t,\cdot)$ is convex,

(ii) $f(t,\cdot)$ is continuously Gateaux differentiable,

(iii) $\|f(t,x)\| \leq k_1(t) + m_1\|x\|^{-1/2}, k_1(\cdot) \in L_+^2, m_1 \geq 0$,

$\|f_x(t,x)\| \leq k_2(t) + m_2\|x\|^{-1/2}$ a.e., $k_2(\cdot) \in L_+^2$, $m_2 \geq 0$

$(H3)$ $B \in L^\infty(I,\mathcal{L}(Y,H))$.

$(H4)$ $L : I \times H \times Y \to \mathbb{R}$ is an integrand such that

(i) $t \to L(t,x,u)$ is measurable,

(ii) $(x,u) \to L(t,x,u)$ is continuous and convex in u,

(iii) $x \to L(t,x,u)$ is convex, Gateaux differentiable and $(x,u) \to L_x(t,x,u)$ is continuous,

(iv)$|L(t,x,u)| \le k_1(t) + m_1(t)(|x|^2 + \|u\|^2)$ a.e., $\|L_x(t,x,u)\|_{\mathcal{L}(H)} \le k_2(t) + m_2(t)(|x|^2 + \|u\|^2)$ a.e. with $k_1(\cdot), k_2(\cdot) \in L^1_+, m_1(\cdot), m_2(\cdot) \in L^\infty_+$.

$(H5)$ $l : H \to \mathbb{R}$ is continuously Gateaux differentiable and convex.

$(H6)$ $U : I \to P_{wkc}(Y) := \{A \in P_{fc}(Y)$ and A is weakly compact $\}$ is measurable and $U(t) \subseteq W$ a.e., with $W \in P_{wkc}(Y)$.

THEOREM 2.1. ([6]) *If hypotheses $(H1)-(H6)$ hold, $x_0 \in H$ and (x,u) solves problem (P) then there exist $p(\cdot) \in W(0,T)$ and $v(\cdot) \in L^2(Y^*)$ such that*

(i) *State equation:*

$$\dot{x}(t) + A(t)x(t) = f(t,x(t)) + B(t)u(t), \quad x(0) = x_0.$$

(ii) *Adjoint equation:*

$$-\dot{p}(t) + A^*(t)p(t) - f_x(t,x(t))p(t) = L_x(t,x(t),u(t)) \quad a.e.,$$
$$p(T) = -l_x(x(T)), \text{ where } A^* \text{ is adjoint operator of } A.$$

(iii) *Minimum principle:*

$$\inf_{u \in U(t)} (v(t) + 2B^*(t)p(t), u - \hat{u}(t)) = 0 \ a.e.$$

We associate another optimal control problem to (P), which is called the dual problem of (P). In order to describe it, we need a function $K : I \times H \to R$ defined by

$$K(t,p(t)) = \inf_{u \in U(t)} \left\{ L(t,x(t),u(t)) + < p(t), B(t)u(t) > \right\}.$$

We call the following optimal control problem of the dual problem :

$$(D)\begin{cases} \displaystyle\int_0^T \Big\{K(t,p(t)) - \langle p(t), \dot{x} + A(t)x(t) - f(t,x(t))\rangle\Big\}dt \\ + l(x(T)) \to \sup \quad \text{subject to} \\ \begin{cases} -\dot{p}(t) + A^*(t)p(t) - f_x(t,x(t))p(t) = L_x(t,x(t),u(t)) \ \text{a.e.}, \\ p(T) = -l_x(x(T)). \end{cases} \end{cases} \tag{2.2}$$

As for the differential equation in (2.2), we restrict ourselves to only solutions which belong to $W(0,T)$. Hence, if there exists no solution $(p,x) \in W(0,T) \times W(0,T)$ in (2.2), then we define the supremum of problem (D) to be $-\infty$.

3. Duality

We prove a duality theorem saying that the infimum of (P) is equal to the supremum of (D).

THEOREM 3.1. *Under the assumptions (H1)–(H6), the infimum of (P) is equal to or greater than the supremum of (D).*

PROOF. We first prove a weak duality theorem i.e., the infimum of (P) is equal to or greater than the supremum of (D). Let $u(\cdot) \in U(t)$ be any admissible control and fix it for a moment. Due to Theorem 2.1, there exists a solution $x \in W(0,T)$ of (2.1) with any initial condition $x(0) = x_0 \in H$. Let $\bar{x}$ be any one of such solutions corresponding to u. That is,

$$\begin{aligned} \dot{\bar{x}}(t) + A(t)\bar{x}(t) &= f(t,\bar{x}(t)) + B(t)u(t) \\ \bar{x}(0) &= x_0 \in H. \end{aligned} \tag{3.1}$$

To this u we associate the following problem (D_u):

$$\int_0^T \Big\{L(t,x(t),u(t)) - \langle p(t), \dot{x}(t) + A(t)x(t) - f(t,x(t)) - B(t)u(t)\rangle\Big\}dt$$
$$+ l(x(T)) \quad \to \sup \quad \text{subject to (2.2).}$$

If there is no solution $(p(\cdot), x(\cdot)) \in W(0,T) \times W(0,T)$ satisfying (2.2), then we define the supremum of (D_u) to be $-\infty$. If $(p^\circ(\cdot), x^\circ(\cdot)) \in W(0,T) \times W(0,T)$ is an arbitrary solution of (2.2), then we can show that the value

(3.2)
$$
\begin{aligned}
&= \int_0^T L(t, \bar{x}(t), u(t))dt + l(\bar{x}(T)) - \int_0^T \Big\{ L(t, x^\circ(t), u(t)) \\
&\quad - \langle p^\circ(t), \dot{x}^\circ(t) + A(t)x^\circ(t) - f(t, x^\circ(t)) - B(t)u(t)\rangle \Big\} dt - l(x^\circ(T))
\end{aligned}
$$

is nonnegative. To do this, note that

$$B(t)u(t) = \dot{\bar{x}}(t) + A(t)\bar{x}(t) - f(t, \bar{x}(t)) \text{ a.e.}$$

By using convexity of f, L, l with respect to x and integration by part with $p^\circ(T) = l_x(x^\circ(T))$ and $x^\circ(0) = \bar{x}(0)$, we see that

(3.2)
$$
\begin{aligned}
&= \int_0^T \{L(t, \bar{x}(t), u(t)) - L(t, x^\circ(t), u(t))\}dt + l(\bar{x}(T)) - l(x^\circ(T)) \\
&\quad + \int_0^T \langle p^\circ(t), \dot{x}^\circ(t) - \dot{\bar{x}}(t)\rangle dt + \int_0^T \langle p^\circ(t), A(t)x^\circ(t) - A(t)\bar{x}(t))\rangle \\
&\quad + \int_0^T \langle p^\circ(t), f(t, \bar{x}(t)) - f(t, x^\circ(t))\rangle dt \\
&\geq \int_0^T \Big(L_x(t, x^\circ(t), u(t)), \bar{x}(t) - x^\circ(t) \Big) dt + \Big(l_x(x^\circ(T)), \bar{x}(T) - x^\circ(T) \Big) \\
&\quad + \Big(p^\circ(t), x^\circ(t) - \bar{x}(t) \Big)\Big|_0^T - \int_0^T \langle \dot{p}^\circ(t), x^\circ(t) - \bar{x}(t)\rangle dt \\
&\quad + \int_0^T \langle A^*(t)p^\circ(t), x^\circ(t) - \bar{x}(t)\rangle dt + \int_0^T \langle f_x(t, x^\circ(t))p^\circ(t), \bar{x}(t) - x^\circ(t)\rangle \\
&= \int_0^T \Big\langle - \dot{p}^\circ(t) + A^*(t)p^\circ(t) - f_x(t, x^\circ(t)) - L_x(t, x^\circ(t), u(t)), x^\circ(t) - \bar{x}(t) \\
&= 0.
\end{aligned}
$$

Let us denote by $p(\bar{x}, u)$ the value

$$\int_0^T L(t, \bar{x}(t), u(t))dt + l(\bar{x}(T)),$$

and by $d(u)$ the supremum of problem (D_u). Then the above argument show that

$$p(\bar{x}, u) \geq d(u) \tag{3.3}$$

holds for every $u \in U(t)$ and every $\bar{x}$ satisfying (3.1) together with u. If we denote by $J(p, x, u)$ the objective function of (D_u), that is,

$$\begin{aligned} J(p, x, u) = l(x(T)) + \int_0^T \Big\{ & L(t, x(t), u(t)) + \langle p(t), B(t)u(t) \rangle \\ & - \langle p(t), \dot{x}(t) + A(t)x(t) - f(t, x(t)) \rangle \Big\} dt, \end{aligned}$$

then it follows from (3.3) and a well-known inequality of game theory [11] that

$$\inf_{\bar{x}, u} p(\bar{x}, u) \geq \inf_u d(u) = \inf_u \sup_{p,x} J(p, x, u) \geq \sup_{p,x} \inf_u J(p, x, u). \tag{3.4}$$

For a given $u(\cdot) \in U(t)$, we have

$$\int_0^T \{L(t, x(t), u(t)) + \langle p(t), B(t)u(t) \rangle\} dt \geq \int_0^T K(t, p(t))dt,$$

since $L(t, x(t), u(t)) + \langle p(t), B(t)u(t) \rangle \geq K(t, p(t))$ a.e., by definition of the function K. Hence, it follows that

$$\inf_u J(p, x, u) \geq l(x(T)) + \int_0^T \{K(t, p(t)) - \langle p(t), \dot{x}(t) + A(t)x(t) - f(t, x(t)) \rangle\} dt$$

for every solution (p, x) of (2.2). If we denote by $\inf(P)$ the infimum of (P) and by $\sup(D)$ the supremum of (D), it is obvious that

$\inf_{\bar{x},u} p(\bar{x},u) = \inf(P)$ and $\sup_{p,x} \inf_u J(p,x,u) \geq \sup(D)$ by the above inequality. Therefore, we conclude from (3.4) that $\inf(P) \geq \sup(D)$.

We next prove the duality theorem that under certain conditions the infimum of (P) coincides with the supremum of (D). If $(x^\circ(0), x^\circ(t), u^\circ(t)) \in H \times W(0,T) \times U(t)$ attains the infimum of (P). By Theorem 2.1, there exists $p^\circ \in W(0,T)$ satisfying

(3.5)
$$\begin{aligned} &-\dot{p}^\circ(t) + A^*(t)p^\circ(t) - f_x(t,x^\circ(t))p^\circ(t) = L_x(t,x^\circ(t),u^\circ(t)) \text{ a.e.} \\ &p(T) = -l_x(x^\circ(T)) \end{aligned}$$

$$(\partial_u L(t,x^\circ(t),u^\circ(t)) + 2B^*(t)p^\circ(t), v - u^\circ(t))_{Y,Y^*} = 0, \tag{3.6}$$

for all $v \in U(t)$ a.e., where $B^*(t)$ denotes the adjoint operator of $B(t)$, $(\cdot,\cdot)_{Y,Y^*}$ the bilinear form on the dual pair (Y, Y^*) and $\partial_u L(t,x,u)$ the derivative of $L(t,x,u)$ with respect to u. By convexity of L, (3.6) implies $L(t,x^\circ(t),v(t)) + \langle p^\circ(t), B(t)v(t)\rangle \geq L(t,x^\circ(t),u^\circ(t)) + \langle p^\circ(t), B(t)u^\circ(t)\rangle$ for all $v \in U(t)$ a.e., from which we have

$$K(t,p^\circ(t)) = L(t,x^\circ(t),u^\circ(t)) + \langle p^\circ(t), B(t)u^\circ(t)\rangle \text{ a.e.} \tag{3.7}$$

From (3.5) we see that $(p^\circ(t), x^\circ(t))$ is a solution of (2.2). Thus from (3.7) that we obtain

$$\begin{aligned} &K(t,p^\circ(t)) - \langle p^\circ, \dot{x}^\circ(t) + A(t)x^\circ(t) - f(t,x^\circ(t))\rangle \\ &\quad = L(t,x^\circ(t),u^\circ(t)) + \langle p^\circ(t), B(t)u^\circ(t) - \dot{x}^\circ(t) - A(t)x^\circ(t) + f(t,x^\circ(t))\rangle \\ &\quad = L(t,x^\circ(t),u^\circ(t)) \text{ a.e.,} \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} &\int_0^T \{L(t,x^\circ(t),u^\circ(t))dt + l(x^\circ(T)) \\ &= \int_0^T \{K(t,p^\circ(t)) - \langle p^\circ, \dot{x}^\circ(t) + A(t)x^\circ(t) - f(t,x^\circ(t))\rangle + l(x^\circ(T))\}dt. \end{aligned}$$

By weak duality theorem, the infimum of (P) is equal to the supremum of (D) and that (p°, x°) attains the supremum of (D). This completes the proof.

References

[1] N. U. Ahmed and X. Xiang, *Nonlinear uncertain systems and necessary conditions of optimality*, SIAM J. Control Optim. **35(5)** (1997), 1755-1772.

[2] W. L. Chan, *Duality in the optimal control of non-well-posed parabolic equations with positive control*, J. Math. Anal. Appl. **107** (1985), 509-519.

[3] J. H. Ha and S. Nakagiri, *Optimal control problems for hyperbolic distributed parameter systems with damping terms*, to appear in Funkcialaj Ekvacioj.

[4] J. L. Lions, *Optimal control of systems governed by partial differential equations*, Springer-verlag Berlin Heidelberg New York, 1971.

[5] A. Papageorgiou and N. S Papageorgiou, *Necessary and sufficient conditions for optimality in nonlinear distributed parameter systems with variable initial state*, J. Math. Soc. Japan. **42** (1990), 387-396.

[6] N. S. Papageorgiou, *On the optimal control of strongly nonlinear evolution equations*, J. Math. Anal. Appl. **164** (1992), 83-103.

[7] J. Y. Park and M. J. Lee, *Duality in the optimal control of distributed parameter systems governed by hyperbolic equations*, preprint.

[8] J. Y. Park and M. J. Lee, *Duality in the optimal control problems for hyperbolic distributed parameter systems with damping terms*, J. Math. Anal. Appl. **227** (1998), 449-461.

[9] J. Y. Park and M. J. Lee, *Duality in the optimal control problems for nonlinear hyperbolic distributed parameter systems*, preprint.

[10] S. Tanimoto, *Duality in the optimal control of non-well-posed distributed systems*, J. Math. Anal. Appl. **171** (1992), 277-287.

[11] R. Teman and I. Ekeland,, *Convex analysis and variational problems*, Dunod, Paris, 1972.

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