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UNIFORM N-DICHOTOMY FOR EVOLUTIONARY PROCESS IN BANACH SPACES

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ABSTRACT. We study some properties of N-dichotomy for evolutionary process and generalize the theory of the uniform N-equistability using these properties.

1. Introduction

Throughout this paper, X is a real or complex Banach space and $L(X)$ is the set of all bounded linear operators from X into itself. Let T be the set defined by $T = \{(t,s) : 0 \leq s \leq t < \infty\}$. A mapping $P : T \to L(X)$ is called an evolutionary process ([1], [6]) if the following are satisfied :

(i) $P(t,s)P(s,t_0) = P(t,t_0)$ for all $0 \leq t_0 \leq s \leq t$,
(ii) $P(t,t)x = x$ for all $x \in X$,
(iii) $P(t,s)$ is strongly continuous in s on $[0,t]$ and in t on $[s,\infty)$,
(iv) there is a nondecreasing function $p : [0,\infty) \to (0,\infty)$ such that

$$\|P(t,s)\| \leq p(t-s) \qquad \text{for all } \ (t,s) \in T.$$

Let $L^\infty_{t_0}(X)$ be the space of X-valued functions f defined almost everywhere on $[t_0,\infty)$ such that f is strongly measurable and essentially bounded, and let $X_1(t_0)$ be the set $X_1(t_0) = \{x \in X \ : \ P(\cdot,t_0)x \in$

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$L^\infty_{t_0}(X)\}$. If $X_2(t_0)$ is a complementary subspace of $X_1(t_0)$ then we denote by $P_1(t_0)$ the projection along $X_2(t_0)$ and $P_2(t_0) = I - P_1(t_0)$ the projection along $X_1(t_0)$.

We also denote :

$P_1(t, t_0) = P(t, t_0)P_1(t_0)$ and $P_2(t, t_0) = P(t, t_0)P_2(t_0)$.

In what follows we denote by $\mathcal{N}$ the set of all functions $N : R_+ \to R_+$ satisfying the following conditions :

(i) N is nondecreasing on $[0, \infty)$,

(ii) N is continuous on $[0, \infty)$ and $N(0) = 0$,

(iii) $N(uv) \leq N(u) \cdot N(v)$ for all $u \geq 0$ and $v \geq 0$.

EXAMPLE. *Let* $N : R_+ \to R_+$, $N(u) = u$ *for* $u \in [0, 1]$ *and* $N(u) = u^2$ *for* $u > 1$. *Then we know that* $N \in \mathcal{N}$.

DEFINITION 1.1. (cf. [2], [3], [4]) An evolutionary process P is said to be :

(i) uniformly exponentially dichotomic (and we write P is u.e.d) if there are $M_1, M_2, \nu_1, \nu_2 > 0$ such that

$$\|P_1(t, t_0)x\| \leq M_1 \cdot \exp[-\nu_1(t - s)] \cdot \|P_1(s, t_0)x\| \tag{1.1}$$

and

$$\|P_2(t, t_0)x\| \geq M_2 \cdot \exp[\nu_2(t - s)] \cdot \|P_2(s, t_0)x\|, \tag{1.2}$$

for all $t \geq s \geq t_0 \geq 0$ and $x \in X$,

(ii) N-uniformly exponentially dichotomic (and we write P is N-u.e.d) if there are M_1, M_2, ν_1, $\nu_2 > 0$ such that

$$N(\|P_1(t, t_0)x\|) \leq M_1 \cdot \exp[-\nu_1(t - s)] \cdot N(\|P_1(s, t_0)x\|), \tag{1.3}$$

and

$$N(\|P_2(t, t_0)x\|) \geq M_2 \cdot \exp[\nu_2(t - s)] \cdot N(\|P_2(s, t_0)x\|), \tag{1.4}$$

for all $t \geq s \geq t_0 \geq 0$ and $x \in X$.

It is clear that the uniform exponential dichotomy is a particular case (when $N(u) = u$) of the N-uniformly exponential dichotomy.

LEMMA 1.1. *Let $\varphi : T \to R_+$ be a function. If there exist positive numbers H, δ, η with $\eta > 1$ such that*

$$\eta\varphi(s+\delta, t_0) \le \varphi(s, t_0) \quad \text{and} \quad \varphi(t, t_0) \le H\varphi(s, t_0), \tag{1.5}$$

for all $t \ge s \ge t_0 \ge 0$, then there are $K, \nu > 0$ such that

$$\varphi(t, t_0) \le K \cdot \exp[-v(t-s)]\varphi(s, t_0), \tag{1.6}$$

for all $t \ge s \ge t_0 \ge 0$.

PROOF. Let $t \ge s \ge t_0 \ge 0$ and $n = [(t-s) \cdot \delta^{-1}]$. Then we have

$$\varphi(t, t_0) \le H\varphi(s+n\delta, t_0) \le \eta^{-n}H\varphi(s, t_0) = K \cdot \exp[-\nu(t-s)] \cdot \varphi(s, t_0),$$

where $K = \eta H$ and $\nu = \delta^{-1} \cdot \ln \eta$. This completes the proof.

By the same method, we can also prove the following lemma.

LEMMA 1.2. *Let $\Psi : T \to [0, \infty)$ be a function. If there exist positive numbers H, δ, $\eta > 0$, with $\eta \in (0, 1)$ such that*

$$\eta\Psi(s+\delta, t_0) \ge \Psi(s, t_0) \quad \text{and} \quad \Psi(t, t_0) \ge H\Psi(s, t_0), \tag{1.7}$$

for all $t \ge s \ge t_0 \ge 0$, then there are $K, \nu > 0$ such that

$$\Psi(t, t_0) \ge K \ \ \exp[\nu(t-s)]\Psi(s, t_0). \tag{1.8}$$

LEMMA 1.3. *Let $g : R_+ \to R_+^*$ be a continuous function on R_+ with* $\inf\{g(u) : u \ge 0\} < 1$ *and $x \in X$ such that*

$$N(\|P_1(t, t_0)x\|) \le g(t-s) \cdot N(\|P_1(s, t_0)x\|), \tag{1.9}$$

for all $t \ge s \ge t_0 \ge 0$. Then there exist $M, \nu > 0$ such that

$$N(\|P_1(t, t_0)x\|) \le M \cdot \exp[-\nu(t-s)] \cdot N(\|P_1(s, t_0)x\|). \tag{1.10}$$

PROOF. Since $\inf\{g(u) : u \geq 0\} < 1$, there is $\delta > 0$ such that $g(\delta) < 1$. Let $n = [(t-s)\cdot\delta^{-1}] \in N$. It is clear that there is $r \in [0,\delta)$ such that $t = s + n\delta + r$. Hence we have

$$\begin{aligned}
N(\|P_1(t,t_0)x\|) &= N(\|P(t,s+n\delta)P_1(s+n\delta,t_0)x\|)\\
&\leq N(P(t-s-n\delta))\cdot N(\|P_1(s+n\delta,t_0)\|)N(P(\delta))\\
&\quad \cdot g(\delta)\cdot N(\|P_1(s+(n-1)\delta,t_0)\|)\\
&\leq \cdots\\
&\leq N(P(\delta))\cdot (g(\delta))^n \cdot N(\|P_1(s,t_0)x\|)\\
&= N(P(\delta))\cdot \exp(-\nu n\delta)\cdot N(\|P_1(s,t_0)x\|)\\
&= N(P(\delta))\cdot \exp[-\nu(t-s)]\cdot \exp(\nu r)\cdot N(\|P_1(s,t_0)x\|)\\
&= M\cdot \exp[-\nu(t-s)]\cdot N(\|P_1(s,t_0)x\|),
\end{aligned}$$

for all $t \leq s \leq t_0 \leq 0$, where $\nu = \delta^{-1}\cdot \ln(g(\delta)) > 0$ and $M = N(P(\delta)) \times \exp(vr) > 0$. This completes the proof.

And we have the following corresponding lemma.

LEMMA 1.4. *Let* $h : [0,\infty) \to (0,\infty)$ *be a continuous function on* $[0,\infty)$ *with* $\sup\{h(u) : u \geq 0\} > 1$ *and* $x \in X$ *such that*

$$N(\|P_2(t,t_0)x\|) \geq h(t-s)N(\|P_2(s,t_0)x\|), \tag{1.11}$$

for all $t \geq s \geq t_0 \geq 0$. *Then there exist* M', $\nu' > 0$ *such that*

$$N(\|P_2(t,t_0)x\|) \geq M'\cdot \exp[\nu'(t-s)]\cdot N(\|P_2(s,t_0)x\|). \tag{1.12}$$

2. N-dichotomy for evolutionary process

THEOREM 2.1. *The following statements are equivalent :*

(a) *There exists* $N \in \mathcal{N}$ *such that* P *is* N*-u.e.d. ;*
(b) *The evolutionary process* P *is u.e.d. ;*
(c) *For every* $N \in \mathcal{N}$ *the evolutionary process* P *is* N*-u.e.d..*

PROOF. (a) ⇒ (b): Let $s \geq t_0 \geq 0$, $u \geq 0$ and $x \in X$. From (1.3) we obtain

$$M_1^{-1} \exp(\nu_1 u) \cdot N(\|P_1(s+u, t_0)x\|) \leq N(\|P_1(s, t_0)x\|). \tag{2.1}$$

Since

$$\lim_{u \to \infty} M_1^{-1} \exp(\nu_1 u) = \infty,$$

there exists $\delta > 0$ such that

$$N(2) \cdot N(\|P_1(s+u, t_0)x\|) \leq N(\|P_1(s, t_0)x\|), \tag{2.2}$$

for all $s \geq t_0 \geq 0$, $u \geq \delta$, and consequently

$$2\|P_1(s+u, t_0)x\| \leq \|P_1(s, t_0)x\| \tag{2.3}$$

for all $s \geq t_0 \geq 0$, $u \geq \delta$ and $x \in X$. On the other hand, if $s \leq t \leq s+\delta$, then

$$\begin{aligned} \|P_1(t, t_0)x\| &= \|P(t, s)P_1(s, t_0)x\| \\ &\leq M \exp[\omega(t-s)] \cdot \|P_1(s, t_0)x\| \\ &\leq M \exp(\omega\delta) \cdot \|P_1(s, t_0)x\|. \end{aligned} \tag{2.4}$$

From (2.3), (2.4) and Lemma 1.1, we obtain that there exist $M_1', \nu_1' > 0$ such that

$$\|P_1(t, t_0)x\| \leq M_1' \exp[-\nu'(t-s)] \cdot \|P_1(s, t_0)x\|$$

for all $t \geq s \geq t_0 \geq 0$ and $x \in X$.

Let $t \geq s \geq t_0 \geq 0$ and $\eta > 0$ such that $N(\eta) \leq M_2$. Then

$$\begin{aligned} N(\|P_2(t, t_0)x\|) &\geq N(\eta) \cdot N(\|P_2(s, t_0)x\|) \\ &\geq N(\eta \cdot \|P_2(s, t_0)x\|), \end{aligned}$$

and hence

$$\|P_2(t, t_0)x\| \geq \eta\|P_2(s, t_0)x\| \tag{2.5}$$

for all $t \geq s \geq t_0 \geq 0$. Since $\lim_{u\to\infty} M_2 \cdot \exp(\nu_2 u) = \infty$, there exists $\tau > 0$ such that

$$N(\|P_2(s+\delta, t_0)x) \geq N(2) \cdot N(\|P_2(s, t_0)x\|),$$

and hence

$$\|P_2(s+\delta, t_0)x\| \geq 2\|P_2(s, t_0)x\| \tag{2.6}$$

for all $s \geq t_0 \geq 0$ and $x \in X$. From (2.5), (2.6) and Lemma 1.2, we obtain that there exist M_2', $\nu_2' > 0$ such that

$$\|P_2(t, t_0)x\| \geq M_2' \exp[\nu_2'(t-s)] \cdot \|P_2(s, t_0)x\|$$

for all $t \geq s \geq t_0 \geq 0$ and $x \in X$.

(b) $\Rightarrow$ (c): Let $t \geq s \geq t_0 \geq 0$ and $N \in \mathcal{N}$. From (1.1) it follows that

$$N(\|P_1(t, t_0)x\|) \leq N(M_1 \exp[-\nu_1(t-s)]) \cdot N(\|P_1(s, t_0)x\|) \tag{2.7}$$

and from $\lim_{u\to 0} N(u) = 0$, there exists $\delta_1 > 0$ such that

$$N(\|P_1(s+\delta_1, t_0)x\|) \leq \frac{1}{2} N(\|P_1(s, t_0)x\|) \tag{2.8}$$

for all $s \geq t_0 \geq 0$ and $x \in X$. On the other hand, it follows from (1.1) that

$$\|P_1(t, t_0)x\| \leq M_1 \|P_1(s, t_0)x\|$$

for all $t \geq s \geq t_0 \geq 0$ and $x \in X$. Using Lemma 1.1, we obtain that there exist M_1'', $\nu_1'' > 0$ such that

$$N(\|P_1(t, t_0)x\|) \leq M_1'' \exp[-\nu_1(t-s)] \cdot N(\|P_1(s, t_0)x\|)$$

for all $t \geq s \geq t_0 \geq 0$ and $x \in X$. From (1.2) we obtain

$$M_2^{-1} \exp[-\nu_2(t-s)] \cdot \|P_2(t, t_0)x\| \geq \|P_2(s, t_0)x\|$$

for all $t \geq s \geq t_0 \geq 0$, and hence

$$N(M_2^{-1}\exp[-\nu_2(t-s)]) \cdot N(\|P_2(t,t_0)x\|) \geq N(\|P_2(s,t_0)x\|).$$

Hence $\lim_{u\to 0} N(u) = 0$. Therefore, there exists $\tau_1 > 0$ such that

$$\frac{1}{2}N(\|P_2(s+\tau_1,t_0)x\|) \geq N(\|P_2(s,t_0)x\|)$$

for all $t \geq s \geq t_0 \geq 0$ and $x \in X$.

By (1.2) it follows

$$M_2^{-1}\|P_2(t,t_0)x\| \geq \|P_2(s,t_0)x\|$$

for all $t \geq s \geq t_0 \geq 0$, and consequently

$$N(\|P_2(t,t_0)x\|) \geq [N(M_2^{-1})]^{-1} \cdot N(\|P_2(s,t_0)x\|)$$

for all $t \geq s \geq t_0 \geq 0$ and $x \in X$.

(c) $\Rightarrow$ (a) is obvious.

Seo-Nam [5] proved the following theorem.

THEOREM 2.2. *The evolutionary process P is u.e.d. if and only if there exist $M, m > 0$ such that*

$$\int_t^\infty \|P_1(u,t_0)x\|du \leq M \cdot \|P_1(t,t_0)x\| \ , \tag{2.9}$$

$$\int_{t_0}^t \|P_2(u,t_0)x\|du \leq M \cdot \|P_2(t,t_0)x\| \ , \tag{2.10}$$

and

$$m \cdot \|P_2(t,t_0)x\| \leq \|P_2(t+1,t_0)x\| \tag{2.11}$$

for all $(t,t_0) \in T$ and $x \in X$.

Now, we are in a position to prove the main theorem in this paper.

THEOREM 2.3. *The evolutionary process P is N-u.e.d. if and only if there exist $N \in \mathcal{N}$ and $n, m > 0$ such that*

$$\int_t^\infty N(\|P_1(u,t_0)x\|)du \le M \cdot N(\|P_1(t,t_0)x\|), \tag{2.12}$$

$$\int_{t_0}^t N(\|P_2(u,t_0)x\|)du \le M \cdot N(\|P_2(t,t_0)x\|), \tag{2.13}$$

and

$$m \cdot N(\|P_2(t,t_0)x\|) \le N(\|P_2(t+1,t_0)x\|) \tag{2.14}$$

for all $(t, t_0) \in T$ and $x \in X$.

PROOF. The necessity is obvious.

For the sufficiency, let $t \ge s+1 > s \ge t_0 \ge 0$. Then we have

$$\begin{aligned}
&N(\|P_1(t,t_0)x\|) \cdot \int_0^1 (N(P(u)))^{-1} du \\
&= \int_0^1 N(\|P(t,\tau)P_1(\tau,t_0)x\|) \cdot [N(P(u))]^{-1} du \\
&\le \int_{t-1}^t N(P(t-\tau)) \cdot N(\|P_1(\tau,t_0)x\|) \cdot (N(P(t-\tau)))^{-1} d\tau \\
&= \int_{t-1}^t N(\|P_1(\tau,t_0)x\|) d\tau \\
&\le \int_s^\infty N(\|P_1(\tau,t_0)x\|) d\tau \\
&\le M \cdot N(\|P_1(s,t_0)x\|).
\end{aligned}$$

Therefore

$$N(\|P_1(t,t_0)x\|) \le M(\int_0^1 (N(P(u)))^{-1} du)^{-1} \cdot N(\|P_1(s,t_0)x\|)$$

for all $t \ge s+1 > s \ge t_0 \ge 0$.

If $t_0 \le s \le t < s+1$, then

$$\begin{aligned} N(\|P_1(t,t_0)x\|) &\le N(P(t-s)) \cdot N(\|P_1(s,t_0)x\|) \\ &\le N(P(1)) \cdot N(\|P_1(s,t_0)x\|), \end{aligned}$$

and hence

$$N(\|P_1(t,t_0)x\|) \le H \cdot N(\|P_1(s,t_0)x\|) \tag{2.15}$$

for all $t \ge s \ge t_0 \ge 0$ and $x \in X$, where

$$H = \max\{M \cdot (\int_0^1 N(P(u))du)^{-1}, N(P(1))\}.$$

Integrating (2.15) from s to t we obtain

$$\begin{aligned} (t-s)N(\|P_1(t,t_0)x\|) &\le H \int_s^t N(\|P_1(u,t_0)x\|)du \\ &\le H \int_s^\infty N(\|P_1(u,t_0)x\|)du \\ &\le H \cdot M \cdot N(\|P_1(s,t_0)x\|). \end{aligned} \tag{2.16}$$

Combining this and (2.15), we obtain

$$N(\|P_1(t,t_0)x\|) \le M(H+1) \cdot (t-s+1)^{-1} \cdot N(\|P_1(s,t_0)x\|) \tag{2.17}$$

for all $t \ge s \ge t_0 \ge 0$.

It follows from Lemma 1.3, there are $M_1 > 0$ and $\nu_1 > 0$ such that

$$N(\|P_1(t,t_0)x\|) \le M_1 \exp[-\nu_1(t-s)] \cdot N(\|P_1(s,t_0)x\|) \tag{2.18}$$

for all $t \ge s \ge t_0 \ge 0$.

Let $x \in X$ and $t \ge s \ge t_0 + 1 > t_0 \ge 0$. Then we have

$$\begin{aligned} &N(\|P_2(s,t_0)x\|) \cdot \int_0^1 (N(P(u)))^{-1}du \\ &= N(\|P_2(s,t_0)x\|) \cdot \int_{s-1}^s (N(P(s-\tau)))^{-1}d\tau \\ &\le \int_{t_0}^t N(\|P_2(\tau,t_0)x\|)d\tau \\ &\le M \cdot N(\|P_2(t,t_0)x\|). \end{aligned}$$

Therefore, for $t \geq v+1 > v \geq t_0$

$$N(\|P_2(t,t_0)x\|) \geq M^{-1}\int_0^1 (N(P(u)))^{-1}du \cdot N(\|P_2(v+1,t_0)x\|)K \cdot N(\|P_2(v,t_0)x\|), \tag{2.19}$$

where

$$K = M^{-1}\cdot m \cdot \int_0^1 (N(P(u)))^{-1}du.$$

Integrating (2.19) from $v+1$ to t, we obtain

$$\int_{v+1}^t N(\|P_2(\tau,t_0)x\|)d\tau \geq K(t-v-1)\cdot N(\|P_2(v,t_0)x\|) \tag{2.20}$$

for all $t \geq v+1 > v \geq t_0 \geq 0$.

Therefore,

$$\begin{aligned} K(t-v-1)\cdot N(\|P_2(v,t_0)x\|) &\leq \int_{t_0}^t N(\|P_2(\tau,t_0)x\|)d\tau \\ &\leq M\cdot N(\|P_2(t,t_0)x\|) \end{aligned} \tag{2.21}$$

for all $t \geq v+1 > v \geq t_0 \geq 0$. Hence from (2.19) we obtain

$$\begin{aligned} N(\|P_2(t,t_0)x\|) &\geq K(t-v)\cdot (M+1)^{-1}\cdot N(\|P_2(v,t_0)x\|) \\ &\geq K_1(t-v+1)\cdot N(\|P_2(t,t_0)x\|) \end{aligned} \tag{2.22}$$

for all $t \geq v+1 > v \geq t_0 \geq 0$ and $x \in X$, where $K_1 = K\cdot[2(M+1)]^{-1}$.

If $v \leq t < v+1$, then

$$\begin{aligned} N(\|P_2(v+1,t_0)x\|) &= N(\|P(v+1,t)P_2(t,t_0)x\|) \\ &\leq N(P(v+1-t))\cdot N(\|P_2(t,t_0)x\|). \end{aligned}$$

Therefore,

$$\begin{aligned} N(\|P_2(t,t_0)x\|) &\geq (N(P(1)))^{-1}\cdot N(\|P_2(v+1,t_0)x\|) \\ &\geq m\cdot N(P(1))^{-1}\cdot N(\|P_2(v,t_0)x\|), \end{aligned} \tag{2.23}$$

and

$$N(\|P_2(t,t_0)x\|) \geq m \cdot N(P(1))^{-1} \cdot N(\|P_2(v,t_0)x\|) \cdot (t-v) \tag{2.24}$$

for all $0 \leq t_0 \leq v \leq t < v+1$.

Combining (2.23) and (2.24), we have

$$N(\|P_2(t,t_0)x\|) \geq K_2(t-v+1) \cdot N(\|P_2(v,t_0)x\|) \tag{2.25}$$

for all $0 \leq t_0 \leq v \leq t < v+1$, where

$$K_2 = (m+1) \cdot [2N(P(1))]^{-1} > 0.$$

From (2.24) and (2.25), we have

$$N(\|P_2(t,t_0)x\|) \geq K'(t-v+1) \cdot N(\|P_2(v,t_0)x\|)$$

for all $t \geq s \geq t_0 \geq 0$ and $x \in X$, where $K' = \min\{K_1, K_2\}$.

From Lemma 1.4, we know that there are $M_2 > 0$ and $\nu_2 > 0$ such that

$$N(\|P_2(t,t_0)x\|) \geq M_2 \exp[\nu_2(t-s)] \cdot N(\|P_2(s,t_0)x\|)$$

for all $t \geq s \geq t_0 \geq 0$ and $x \in X$. This completes the proof.

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