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ON CLASSES OF CERTAIN ANALYTIC FUNCTIONS DEFINED BY MULTIPLIER TRANSFORMATIONS

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Abstract. The purpose of the present paper is to introduce a new class $\mathcal{P}_{n,p}(\alpha)$ of analytic functions defined by a multiplier transformation and to investigate some properties for the class $\mathcal{P}_{n,p}(\alpha)$. Furthermore, we consider an integral of functions belonging to the class $\mathcal{P}_{n,p}(\alpha)$.

1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathbb{N} = \{1, 2, ...\})$$

which are analytic in the unit disk $\mathcal{U} = \{z : |z| < 1\}$. For any integer n, we define the multiplier transformation $I^n f$ of functions $f \in A_p$ by

$$I^n f(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{k+p+1}{p+1} \right)^{-n} a_{k+p} z^{k+p}.$$

Obviously, we have

$$I^n(I^m f(z)) = I^{n+m} f(z)$$

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for all integers m and n. For $p = 1$, the operators I^n are the multiplier transformations studied by Uralegaddi and Somanatha [7] and are closely related to the multiplier transformations introduced by Flett [2].

For any integer n, let $\mathcal{P}_{n,p}(\alpha)$ denote the class of functions $f \in \mathcal{A}_p$ satisfying the condition

$$\mathrm{Re}\left\{\frac{(I^n f(z))'}{z^{p-1}}\right\} > \alpha \quad (0 \le \alpha < p,\ z \in \mathcal{U}).$$

In the present paper, we prove that for the classes $\mathcal{P}_{n,p}(\alpha)$ of functions in $\mathcal{A}_p$, $\mathcal{P}_{n,p}(\alpha) \subset \mathcal{P}_{n+1,p}(\alpha)$ holds. Since $\mathcal{P}_{0,p}(\alpha)$ is the class of functions which satisfy the condition

$$\mathrm{Re}\left\{\frac{f(z)'}{z^{p-1}}\right\} > \alpha \quad (0 \le \alpha < p,\ z \in U),$$

all functions in $\mathcal{P}_{n,p}(\alpha)$ are p-valent for nonpositive integers n [6]. We also obtain a sufficient condition for p-valence. Furthermore, we investigate some properties in connection with certain integral transform.

2. Properties of the class $\mathcal{P}_{n,p}(\alpha)$

In order to derive our results, we need the following lemma due to Jack [3].

LEMMA 2.1. *Let w be non-constant analytic in $\mathcal{U} = \{z : |z| < 1\}$, $w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, we have $z_0 w'(z_0) = kw(z_0)$ where k is a real number, $k \ge 1$.*

With the help of Lemma 2.1, we now derive :

THEOREM 2.1. *For any integer n, $\mathcal{P}_{n,p}(\alpha) \subset \mathcal{P}_{n+1,p}(\beta)$, where*

$$\beta = \frac{2(p+1)\alpha + p}{2(p+1)}. \tag{2.1}$$

PROOF. Let $f \in \mathcal{P}_{n,p}(\alpha)$. Define an analytic function w in $\mathcal{U}$ by

$$\frac{(I^{n+1}f(z))'}{z^{p-1}} = \frac{p+(2\beta-p)w(z)}{1+w(z)}, \tag{2.2}$$

where β is given by (2.1). Clearly, $w(0) = 0$ and $w(z) \neq -1$. Using the identity

$$z(I^n f(z))' = (p+1)I^{n-1}f(z) - I^n f(z)$$

and differentiating (2.2), we obtain

$$\frac{(I^n f(z))'}{z^{p-1}} = \frac{p+(2\beta-p)w(z)}{1+w(z)} - \frac{2(p-\beta)zw'(z)}{(p+1)(1+w(z))^2}. \tag{2.3}$$

We claim that $|w(z)| < 1$ for $z \in U$. Otherwise, by Lemma 2.1, there exists a point z_0 in $\mathcal{U}$ such that

$$z_0 w'(z_0) = kw(z_0) \tag{2.4}$$

where $|w(z_0)| = 1$ and $k \geq 1$. The equation (2.3) in conjunction with (2.4) gives

$$\frac{(I^n f(z_0))'}{z_0^{p-1}} = \frac{p+(2\beta-p)w(z_0)}{1+w(z_0)} - \frac{2(p-\beta)kw(z_0)}{(p+1)(1+w(z_0))^2}. \tag{2.5}$$

Writing $w(z_0) = u + iv$ and taking the real part of (2.5), we have

$$\begin{aligned} \mathrm{Re}\left\{\frac{(I^n f(z_0))'}{z_0^{p-1}} - \alpha\right\} &= \beta - \alpha - 2(p-\beta)k\mathrm{Re}\left\{\frac{u+iv}{(p+1)(1+u+iv)^2}\right\} \\ &\leq \beta - \alpha - \frac{p-\beta}{2(p+1)} = 0. \end{aligned} \tag{2.6}$$

This contradicts the hypothesis that $f \in \mathcal{P}_{n,p}(\alpha)$. Hence $|w(z)| < 1$ for $z \in U$ and it follows from(2.2) that $f \in \mathcal{P}_{n+1,p}(\alpha)$.

Since β is greater than α in Theorem 2.1, we have :

COROLLARY 2.1. *For any integer n, $\mathcal{P}_{n,p}(\alpha) \subset \mathcal{P}_{n+1,p}(\alpha)$.*

REMARK 2.1. Since $\mathcal{P}_{0,p}(\alpha)$ is the class of p-valent functions [6], it follows from Theorem 2.1 that all functions in $\mathcal{P}_{n,p}(\alpha)$ are p-valent for any nonpositive integers.

Next, we prove :

THEOREM 2.2. *Let $f \in \mathcal{P}_{n,p}(\alpha)$ and let F_c be the integral operator defined by*

$$F_c(z) = \frac{p+c}{z^c} \int_0^z t^{c-1} f(t)dt \quad (c > -p). \tag{2.7}$$

Then $F_c \in \mathcal{P}_{n,p}(\alpha)$.

PROOF. From the definition of F_c, we obtain

$$z(I^n F_c(z))' = (c+p)I^n f(z) - cI^n F_c(z). \tag{2.8}$$

Define an analytic function w in $\mathcal{U}$ by

$$\frac{(I^n F_c(z))'}{z^{p-1}} = \frac{p + (2\beta - p)w(z)}{1 + w(z)}. \tag{2.9}$$

Then, $w(0) = 0$ and $w(z) \neq -1$. Using the identity (2.8) and differentiating (2.9), we have

$$\frac{(I^n f(z))'}{z^{p-1}} = \frac{p + (2\beta - p)w(z)}{1 + w(z)} - \frac{2(p-\beta)zw'(z)}{(c+p)(1+w(z))^2}.$$

Now proceeding as in the proof of Theorem 2.1, we can show that $F_c \in \mathcal{P}_{n,p}(\alpha)$.

THEOREM 2.3. *Let $f \in \mathcal{A}_p$ and satisfy the condition*

$$Re\left\{\frac{(I^n f(z))'}{z^{p-1}}\right\} > \alpha - \frac{p-\alpha}{2(p+c)} \quad (0 \leq \alpha < p; z \in \mathcal{U}).$$

Then $F_c \in \mathcal{P}_{n,p}(\alpha)$, where F_c is given by (2.7).

PROOF. The proof of this theorem is similar to that of Theorem 2.2 and so we omit it.

Putting $n = 0$ and $\alpha = 0$ in Theorem 2.3, we have the following

COROLLARY 2.2. *If $f \in \mathcal{A}_p$ and satisfies the condition*

$$Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > -\frac{p}{2(p+c)} \quad (z \in \mathcal{U}),$$

then the integral operator F_c defined by (2.7) belongs to $\mathcal{P}_{0,p}(0)$.

REMARK 2.2. For $p = 1$, Corollary 2.2 is stronger than the result of Bernardi [1] ; $\text{Re}\{f'(z)\} > 0$ implies $\text{Re}\{F_c'(z)\} > 0$. If we further put $c = 1$, we also extends the result obtained by Libera [4].

Finally, we obtain a converse of Theorem 2.2 in the following

THEOREM 2.4. *Let $F_c \in \mathcal{P}_{n,p}(\alpha)$ and let f be defined as in (2.7). Then $f \in \mathcal{P}_{n,p}(\alpha)$ in $|z| < r(p,c)$, where*

$$r(p,c) = \frac{p+c}{1+\sqrt{(p+c)^2+1}}. \tag{2.10}$$

Then the result is sharp.

PROOF. Since $F_c \in \mathcal{P}_{n,p}(\alpha)$, we can write

$$z(I^n F_c(z))' = z^p[\alpha + (p-\alpha)u(z)], \tag{2.11}$$

where u is analytic in $\mathcal{U}, u(0) = 1$ and $\text{Re}\{u(z)\} > 0$ in $\mathcal{U}$. Using (2.8) and differentiating (2.11), we get

$$\frac{\frac{(I^n f(z))'}{z^{p-1}} - \alpha}{p-\alpha} = u(z) + \frac{zu'(z)}{p+c}. \tag{2.12}$$

Using the well-known estimate $|zu'(z)| \leq 2r/(1-r^2)\mathrm{Re}\{u(z)\}$ ($|z| = r$), (2.12) yields

$$\mathrm{Re}\left\{\frac{(I^n f(z))' - \alpha}{p-\alpha}\right\} \geq \left(1 - \frac{2r}{(p+c)(1-r^2)}\right)\mathrm{Re}u(z). \qquad (2.13)$$

The right-hand side of (2.13) is positive provided $r < r(p,c)$ given by (2.10). Hence $f \in \mathcal{P}_{n,p}(\alpha)$ for $|z| < r(p,c)$. The result is sharp for the function f defined by

$$f(z) = \frac{z^{1-c}}{p+c}(z^c F_c(z))' \quad (c > -p; z \in \mathcal{U}),$$

where F_c is given by

$$(I^n F_c(z))' = z^{p-1}\left(\frac{p + (2\alpha - p)z}{1+z}\right) \quad (0 \leq \alpha < p; z \in \mathcal{U}).$$

REMARK 2.3. Taking $n = \alpha = 0$ and $p = 1$ in Theorem 2.4, we obtain the result by Bernardi [1]. If we further put $c = 1$, then we have the result obtained by Livingston [5].

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