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SOME CHARACTERIZATIONS OF BEST APPROXIMATION ELEMENT FROM SUBSPACES IN LINEAR 2-NORMED SPACES

S. S. Kim and S. S. Dragomir

Abstract. In this paper, we shall give new characterizations of best approximation element in linear 2-normed spaces in terms of bounded linear 2-functionals and 2-hyperplanes.

1. Introduction

Let X be a linear space of dimension greater than 1, and let $\|\cdot,\cdot\| : X \times X \to R$ be a function with the following conditions:

(N_1) $\|x, y\| = 0$ if and only if x and y are linearly dependent,
(N_2) $\|x, y\| = \|y, x\|$,
(N_3) $\|\alpha x, y\| = |\alpha| \|x, y\|$, where α is real,
(N_4) $\|x + y, z\| \le \|x, z\| + \|y, z\|$.

$\|\cdot,\cdot\|$ is called a *2-norm* on X and $(X, \|\cdot,\cdot\|)$ a *linear 2-normed space*([6]).

Let A, C be a subspaces of X. A bilinear functional $f : A \times C \to R$ is called a *bounded linear 2-functional* if there is a real constant $K > 0$ such that $|f(x, y)| \le K\|x, y\|$ for $x, y \in X$([12]).

For a bounded linear 2-functional we have

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$$\|f\| = \inf\{K : |f(x,y)| \le K\|x,y\| \quad \text{for all} \quad x,y \in X\}.$$

Additional properties of bounded linear 2-functionals may be found in [4], [5], [9] and [12].

Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space and $V(x_1, x_2, ..., x_n)$ be a subspace of X generated by $x_1,\ x_2, ...,\ x_n$ in X. For all $x,\ y \in X$, define

$$\rho_\pm(x,z)(y) = \lim_{t\to 0^\pm} \frac{\|x+ty,z\|^2 - \|x,z\|^2}{2t}$$

for any real t and $z \in X \setminus V(x,y)$.

Theorem 1.1([1], [2]). *We have some properties of* $\rho_\pm$:

(1) $\rho_\pm(\alpha x, z)(\beta y) = \alpha\beta\rho_\pm(x,z)(y)$ *for* $\alpha\beta \ge 0$.
(2) $\rho_\pm(x,z)(\alpha x + y) = \alpha\rho_\pm(x,z)(x) + \rho_\pm(x,z)(y)$ *for all* $\alpha \in R$.
(3) $\rho_\pm(x,z)(y+y') \le (\rho_\pm(x,z)(x))^{1/2}(\rho_\pm(y,z)(y))^{1/2} = \rho_\pm(x,z)(y')$.
(4) $\rho_+(x,z)(-y) = \rho_+(-x,z)(y) = -\rho_-(x,z)(y)$.
(5) $\rho_+(x,z)(x) = \rho_-(x,z)(x) = \|x,z\|^2$.
(6) $(X, \|\cdot,\cdot\|)$ *is smooth at* $x_o \in X\setminus\{0\}$ *if and only if* $\rho_+(x,z)(y) = \rho_-(x,z)(y)$.
(7) $x\perp_z(\alpha x+y)$ *if and only if* $\rho_-(x,z)(y) \le -\alpha\|x,z\|^2 \le \rho_+(x,z)(y)$ *where* $\perp_z$ *is orthogonality([7]), that is,* $x\perp_z y$ means $\|x+ty,z\| \ge \|x,z\|$ *for all* $t \in R$.

Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space. For a subspace G of X, let $[x,G]$ be the subspace of X generated by x and G, where $x \in X\setminus\overline{G}$. Then for $z \in X\setminus[x,G]$, an element $g_o \in G$ is called the *best approximation element* of x by G (with respect to z) if

$$\|x - g_o, z\| \le \|x - g, z\|$$

for all $g \in G$([10]). The set of all elements of best approximation of x by G with respect to z is denoted by $P_{G,z}(x)$, that is,

$$P_{G,z}(x) = \{g_o \in G : \|x - g_o, z\| \leq \|x - g, z\|\}.$$

The following theorem gives a relationship between orthogonality and best approximation in linear 2-normed spaces.

THEOREM 1.2. ([4]) *Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space, G a linear subspace of X, $x \in X \setminus \overline{G}$ and $z \in X \setminus [x, G]$. Then $g_o \in P_{G,z}(x)$ if and only if $(x - g_o) \perp_z G$.*

In 1994 and 1990, I. Franić([4]) and S. Mabizela([9]) gave some characterizations of the best approximation in terms of bounded linear 2-functions, respectively. Also, some results on approximation theory in linear 2-normed spaces have been obtained by S.S. Kim, Y.J. Cho and T.D. Narang([8]), S. Elumalai, Y.J. Cho and S.S. Kim([3]) and R. Ravi([11]).

In this paper, new characterizations of best approximation in linear 2-normed spaces is given in terms of bounded linear 2-functionals and 2-hyperplanes.

2. Characterizations of best approximation

Let f be a non-zero linear 2-functional on $X \times V(z)$. Then we define the *2-hyperplane* H through the origin by

$$H = \{x \in X | f(x, z) = 0\}.$$

THEOREM 2.1. *Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space, f a non-zero bounded linear 2-functional on $X \times V(z)$ and H a 2-hyperplane through the origin, $x_o \in X \setminus H, z \in X \setminus [x, H]$ and $g_o \in H$. Then the following statements are equivalent:*

(1) *$g_o \in P_{H,z}(x_o)$;*

(2) *(a) For all $x \in X$*

$$\begin{aligned} &\rho_-\left(\frac{f(x_o,z)(x_o-g_o)}{\|x_o-g_o,z\|^2},z\right)(x) \\ &\le f(x,z) \le \rho_+\left(\frac{f(x_o,z)(x_o-g_o)}{\|x_o-g_o,z\|^2},z\right)(x), \text{ and} \end{aligned} \tag{2.1}$$

(b) $\quad \|f\| = \dfrac{|f(x_o,z)|}{\|x_o-g_o,z\|}.$

PROOF. (1) implies (2): Suppose that $g_o \in P_{H,z}(x_o)$. By Theorem 1.2, $(x_o-g_o)\perp_z H$. Let $w = x_o - g_o$ and $x \in X$. Then we have $f(x,z)w - f(w,z)x$ belong to H and so $w\perp_z(f(x,z)w - f(w,z)x)$. By Theorem 1.1,

$$\rho_-(w,z)(f(x,z)w - f(w,z)x) \le 0 \le \rho_+(w,z)(f(x,z)w - f(w,z)x)$$

for all $x \in X$ and $z \in X \setminus [x,H]$. Since

$$\begin{aligned} &\rho_\pm(w,z)(f(x,z)w - f(w,z)x) \\ &\quad = f(x,z)\|w,z\|^2 + \rho_\pm(w,z)(-f(w,z)x) \end{aligned}$$

and $w\perp_z H$, if w is any non-zero element of X, then $f(w,z) \ne 0$. Now we will consider two cases: $f(w,z) > 0$ and $f(w,z) < 0$.

Case 1. Suppose that $f(w,z) > 0$. Then we have

$$\begin{aligned} 0 &\le f(x,z)\|w,z\|^2 + \rho_+(w,z)(-f(w,z)x) \\ &= f(x,z)\|w,z\|^2 - \rho_-(f(w,z)w,z)(x) \end{aligned}$$

and so

$$f(x,z) \ge \rho_-\left(\frac{f(w,z)w}{\|w,z\|^2},z\right)(x).$$

On the other hand, we have

$$\begin{aligned} 0 &\ge f(x,z)\|w,z\|^2 + \rho_-(w,z)(-f(w,z)x) \\ &= f(x,z)\|w,z\|^2 - \rho_+(f(w,z)w,z)(x) \end{aligned}$$

and so

$$f(x,z) \le \rho_+\left(\frac{f(w,z)w}{\|w,z\|^2}, z\right)(x).$$

Therefore, it follows that

$$\rho_-\left(\frac{f(w,z)w}{\|w,z\|^2}, z\right)(x) \le f(x,z) \le \rho_+\left(\frac{f(w,z)w}{\|w,z\|^2}, z\right)(x).$$

Case 2. Suppose that $f(w,z) < 0$. For any $x, y \in X$ and $z \in X \setminus V(x,y)$,

$$\rho_-(x,z)(y) = -\rho_+(x,z)(-y) = -\rho_+(-x,z)(y)$$

and

$$\rho_-(-x,z)(y) = -\rho_+(-x,z)(-y) = -\rho_+(x,z)(y)$$

hold. Since $f(w,z) < 0$, we have

$$\begin{aligned} 0 &\le f(x,z)\|w,z\|^2 + \rho_+(w,z)(-f(w,z)x) \\ &= f(x,z)\|w,z\|^2 - \rho_-(f(w,z)w,z)(x) \end{aligned}$$

and so

$$f(x,z) \ge \rho_-\left(\frac{f(w,z)w}{\|w,z\|^2}, z\right)(x).$$

Also, by the similar method we have

$$f(x,z) \le \rho_+\left(\frac{f(w,z)w}{\|w,z\|^2}, z\right)(x).$$

Therefore,

$$\rho_-\left(\frac{f(w,z)w}{\|w,z\|^2}, z\right)(x) \le f(x,z) \le \rho_+\left(\frac{f(w,z)w}{\|w,z\|^2}, z\right)(x).$$

Since $g_o \in H$, $f(w,z) = f(x_o,z)$ and so we obtain (a).

Next, let $u = f(x_o, z)(x_o - g_o)/\|x_o - g_o, z\|^2$. Then, by (a)

$$f(x, z) \leq \rho_+(u, z)(x) \leq \|x, z\|\|u, z\|$$

and

$$f(x, z) \geq \rho_-(u, z)(x) = -\rho_+(u, z)(-x) \geq \|x, z\|\|u, z\|.$$

Therefore, $-\|u, z\| \leq f(x, z)/\|x, z\| \leq \|u, z\|$ and hence $\|f\| \leq \|u, z\|$. On the other hand, we have

$$\|f\| \geq \frac{f(u, z)}{\|u, z\|} \geq \frac{\rho_-(u, z)(u)}{\|u, z\|} = \|u, z\|$$

and so we conclude that (b) holds.

(2) implies (1): From (a), for $x \in H$

$$\rho_-\left(\frac{f(x_o, z)(x_o - g_o)}{\|x_o - g_o, z\|^2}, z\right)(x) \leq 0 \leq \rho_+\left(\frac{f(x_o, z)(x_o - g_o)}{\|x_o - g_o, z\|^2}, z\right)(x).$$

Therefore, it follows that

$$\frac{f(x_o, z)(x_o - g_o)}{\|x_o - g_o, z\|^2} \perp_z H$$

and so since $f(x_o, z) \neq 0$, $(x_o - g_o) \perp_z H$. Therefore, by Theorem 1.2 we have $g_o \in P_{H,z}(x_o)$.

By Theorem 2.1, we obtain easily the following corollaries:

COROLLARY 2.2. *Let $(X, (\cdot, \cdot|\cdot))$ be a 2-inner product space, f a non-zero bounded linear 2-functional on $X \times V(z)$, H a 2-hyperplane through the origin, $x_o \in X \setminus H$, and $z \in X \setminus [x, H]$. Then there exists $g_o \in H$ such that*

$$f(x, z) = \left(x, \frac{f(x_o, z)(x_o - g_o)}{\|x_o - g_o, z\|^2} | z\right) \quad \textit{and} \quad \|f\| = \frac{|f(x_o, z)|}{\|x_o - g_o, z\|}.$$

COROLLARY 2.3. *Let $(X, \|\cdot,\cdot\|)$ be a smooth linear 2-normed space, f a non-zero bounded linear 2-functional on $X \times V(z)$, H a 2-hyperplane through the origin, $x_o \in X \setminus H, z \in X \setminus [x, H]$ and $g_o \in H$. Then the following statements are equivalent:*

(1) $g_o \in P_{H,z}(x_o)$;

$$(2)\ f(x,z) = \rho_+\left(\frac{f(x_o,z)(x_o - g_o)}{\|x_o - g_o, z\|^2}, z\right)(x) \quad \text{and} \quad \|f\| = \frac{|f(x_o,z)|}{\|x_o - g_o, z\|}.$$

Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space, G a linear subspace of X, $x \in X \setminus \overline{G}$ and $z \in X \setminus [x, G]$. If $P_{G,z}(x)$ has at least one element for every $x \in X$, then G is said to be *proximinal* ([10]).

LEMMA 2.4. ([10]) *Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space and H be a 2-hyperplane through the origin. Then H is proximinal if and only if there exists a non-zero $x \in X$ such that $0 \in P_{H,z}(x)$.*

From Theorem 2.1 and Lemma 2.4, we obtain easily the following:

THEOREM 2.5. *Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space, f a non-zero bounded linear 2-functional on $X \times V(z)$ and H a 2-hyperplane through the origin. Then the following statements are equivalent:*

(1) H is proximinal;

(2) For non-zero $u \in X$ and $z \in X \setminus V(x, u)$,

(a) $\rho_-(u,z)(x) \le f(x,z) \le \rho_+(u,z)(x)$

(b) $\|f\| = \|u, z\|$.

COROLLARY 2.6. *Let $(X, \|\cdot,\cdot\|)$ be a smooth linear 2-normed space and H a 2-hyperplane through the origin. Then H is proximinal if and only if there exists a non-zero $u \in X$ such that $f(x,z) = \rho_+(u,z)(x)$ for all $x \in X$ and $\|f\| = \|u, z\|$.*

3. A variational characterization of best approximation

In this section, we will give a variational characterizations of best approximation element.

THEOREM 3.1. *Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space, f be a non-zero bounded linear 2-functional on $X \times V(z)$ and a non-zero element $w \in X$. Then the following statements are equivalent:*

(1) *The following inequality holds,*

$$\rho_-(w,z)(x) \leq f(x,z) \leq \rho_+(w,z)(x) \quad \text{for all} \quad x \in X, \tag{3.1}$$

(2) *The element w minimize the quadratic functional $F_{f_z} : X \to R$ defined by*

$$F_{f_z}(u) = \|u, z\|^2 - 2f(u,z).$$

PROOF. (i) $\Rightarrow$ (ii): If w satisfies the relation (3.1), then we have $f(w,z) = \|w,z\|^2$ for $x = w$. Now, let $u \in X$. Then we have

$$\begin{aligned} F_{f_z}(u) - F_{f_z}(w) &= \|u,z\|^2 - 2f(u,z) + \|w,z\|^2 \\ &\geq \|u,z\|^2 - 2\rho_+(w,z)(u) + \|w,z\|^2 \\ &\geq \|u,z\|^2 - 2\|u,z\|\|w,z\| + \|w,z\|^2 \\ &= (\|u,z\| - \|w,z\|)^2 \geq 0, \end{aligned}$$

and so w minimize the functional F_{f_z}.

(ii) $\Rightarrow$ (i): Suppose that w minimize the functional F_{f_z}. Then we have

$$F_{f_z}(w + \lambda u) - F_{f_z}(w) \geq 0$$

for all $u \in X$ and $\lambda \in R$. On the other hand, since $F_{f_z}(w + \lambda u) - F_{f_z}(w) = \|w + \lambda u, z\|^2 - \|w,z\|^2 - 2\lambda f(u,z)$ we have

$$2\lambda f(u,z) \leq \|w + \lambda u, z\|^2 - \|w,z\|^2 \tag{3.2}$$

for all $u \in X$ and $\lambda \in R$. Now, we assume that $\lambda > 0$. Then by (3.2) we have

$$f(u,z) \leq \frac{\|w + \lambda u, z\|^2 - \|w,z\|^2}{2\lambda} \quad \text{for all} \quad u \in X,$$

which gives $f(u,z) \leq \rho_+(w,z)(u)$ for $\lambda \to 0^+$ and all $u \in X$. Putting $(-u)$ instead of u, we have $f(u,z) \geq -\rho_+(w,z)(-u)) = \rho_-(w,z)(u)$ for all $u \in X$. Therefore, we have the relation (3.1).

By Theorem 3.1, we obtain the following:

COROLLARY 3.2. *Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space and f a non-zero bounded linear 2-functional on $X \times V(z)$ and a non-zero element $w \in X$. Then w is a element of smoothness of X and it minimizes the functional F_{f_z} if and only if*

$$f(x,z) = \rho_+(w,z)(x) \quad \text{for all} \quad x \in X.$$

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S. S. Kim
Department of Mathematics
Dongeui University
Pusan 614-714, Korea
E-mail: sskim@hyomin.dongeui.ac.kr

S. S. Dragomir
School of Communications and Informatics
Victoria University of Techology
Melbourne City, MC 8001 Australia
E-mail: sever@matilda.vu.edu.au