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BOUNDED ANALYTIC FUNCTIONS IN THE COMPLEX BALL AND THE HYPERBOLIC DISTANCE

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ABSTRACT. We give an easy proof that $\phi(z) = \frac{z_1^2}{1-z_2^2}$ induces the bounded composition operator $C_\phi : \mathcal{B} \to \bigcap H^p(B_2)$ defined by $C_\phi f = f \circ \phi$.

1. Introduction

For a bounded holomorphic map ϕ from the open unit ball B of $\mathbb{C}^n$ into the open unit disc U of $\mathbb{C}$, we in this paper consider the composition operator C_ϕ defined by $C_\phi f = f \circ \phi$. Historically, the study of composition operators on Bloch space $\mathcal{B}$ into a nice function space was initiated in the view point of the boundary behavior. P. Ahern observed that $C_\phi g \in BMOA(B)$ for all $g \in \mathcal{B}$ and for all monomials ϕ ([1]). Then there found out several examples of homogeneous polynomials and conditions for ϕ to have the property ([1], [2], [9]). If we restrict to $n = 1$, then the boundedness of $C_\phi : \mathcal{B} \to BMOA$ can be characterized by the membership $\phi \in \rho BMOA$, where $\rho BMOA$ is the hyperbolic $BMOA$ class of S. Yamashita ([8], [9], [12]). In view of

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a known parallelism between the Hardy space H^p and the Yamashita hyperbolic Hardy class ρH^p, the boundedness of $\mathcal{C}_\phi : \mathcal{B} \to H^{2p}$ was characterized by the membership $\phi \in \rho H^{2p}$ when $n = 1$ ([7]).

We, in this paper, consider $\mathcal{C}_\phi$ from $\mathcal{B}$ into $\bigcap H^p(B)$, where $H^p(B)$ is the Hardy space on the ball. Concerning the characterization of the boundedness of $\mathcal{C}_\phi$ in terms of the growth of ϕ, our objective is to give an application of the fact that

$$(1.1) \qquad \mathcal{C}_\phi : \mathcal{B} \to \bigcap H^p(B) \text{ bounded } \iff \phi \in \bigcap \rho H^p(B).$$

2. Preliminaries

Let B be the open unit ball of $\mathbb{C}^n$ and U denote B when $n = 1$. Let S be the boundary of B. The surface area measure on S normalized to have total mass one will be denoted by σ.

The Hardy space $H^p(B)$, $0 < p < \infty$, is defined to consist of those f holomorphic in B for which $\|f\|_{H^p} = \lim_{r \to 1} M_p(r, f) < \infty$, where

$$M_p(r, f) = \left(\int_S |f(r\zeta)|^p d\sigma(\zeta) \right)^{\frac{1}{p}}.$$

See [10], [4] or [5] for H^p spaces.

Let ρ denote the non-euclidean hyperbolic distance in U :

$$\rho(z, w) = \frac{1}{2} \log \frac{|1 - \bar{z}w| + |z - w|}{|1 - \bar{z}w| - |z - w|}, \quad z, w \in U.$$

For $0 < p < \infty$, the hyperbolic Hardy class $\rho H^p(B)$ consists of those holomorphic maps $\phi : B \to U$ for which

$$\sup_{0<r<1} M_p(r, \rho(\phi)) < \infty,$$

where $\rho(\phi)(z) = \rho(\phi(z), 0)$. See [13] for $n = 1$. Similarly, the hyperbolic $BMOA$ class $\rho BMOA(B)$ consists of those holomorphic maps of B into U for which

$$\sup_\tau \sup_{0<r<1} M_1(r, \rho(\phi \circ \tau)) < \infty,$$

where τ runs through all automorphisms of B. See [8] and [12].

Concerning the problem of characterizing the boundedness of the composition operators, there occurred general phenomenon saying that

$$C_\phi : \mathcal{B} \to Y \quad \text{bounded} \quad \Longleftrightarrow \phi \in \rho(Y), \tag{2.1}$$

where $\rho(Y)$ is the hyperbolic counterpart of Y in the sense that it consists of those bounded functions whose membership is characterized via hyperbolic distance $\rho(f(z), 0)$ in place of euclidean distance $|f(z)|$ that is used in the definition of the membership '$f \in Y$'. Examples of classes $\rho(Y)$ are $\rho H^p(\Delta)$ and $\rho BMOA(B)$. By [7, Theorem 1] and [8, Theorem], (2.1) is known to be true for these classes. Noting that

$$M_p(r, \rho(\phi)) \ < \ \infty \ \Longleftrightarrow \ M_p\left(r, log\frac{1}{1-|\phi|}\right) \ < \ \infty,$$

we obtain (1.1) by Theorem 1 in Section 3, and this says that (2.1) is true with $Y = \bigcap H^p(B)$.

Other undefined notations and terminologies of this paper will follow the book of W. Rudin [10] and of M. Stoll [11].

3. Bloch to $\bigcap H^p$ pullbacks

We let ϕ be a holomorphic map from B into U. We abbreviate $H^p(B)$ as H^p. The following results follows directly when $n = 1$ from [7].

THEOREM 1 [6]. *The composition operator $C_\phi : \mathcal{B} \to \bigcap H^p$ is bounded if and only if*

$$\sup_{0<r<1} \int_S \left(log\frac{1}{1-|\phi(r\zeta)|^2}\right)^p d\sigma(\zeta) < \infty \tag{3.1}$$

for all $p : 0 < p < \infty$.

Now, we give a nice application of Theorem 1 in the next Section.

4. An Example

Consider the function

$$F(z) = \frac{z_1^2}{1 - z_2^2}, \quad z = (z_1, z_2) \in B_2.$$

It was first considered in [3], and the authors there proved quite complicatedly that F takes Bloch functions to H^p for all $p : 0 < p < \infty$. We give a simple proof here. In view of Theorem 1, the fact can be verified by showing that $F \in \bigcap \rho H^p(B_2)$. Since F is holomorphic in B_2, $|F| < 1$, and $F(\zeta) = \lim_{r \to 1} F(r\zeta)$ exists almost every $\zeta \in S$, it is sufficient to prove that

$$\int_S \left(\log \frac{1}{1 - |F(\zeta)|^2}\right)^p \, d\sigma(\zeta) < \infty \tag{4.1}$$

for all $p : p > 1$. The following is easy to check :

$$\begin{aligned}
&\int_S \left(\log \frac{1}{1 - |F(\zeta)|^2}\right)^p d\sigma(\zeta) \\
&= \int_S \left(\log \frac{1}{1 - |\frac{\zeta_1^2}{1-\zeta_2^2}|^2}\right)^p d\sigma(\zeta) \\
&= \int_S \left(\log \frac{|1-\zeta_2|^2}{|1-\zeta_2^2|^2 - (1-|\zeta_2|^2)^2}\right)^p d\sigma(\zeta) \\
&= \int_U \left(\log \frac{|1-\omega^2|^2}{|1-\omega^2|^2 - (1-|\omega|^2)^2}\right)^p dv_1(\omega) \\
&= \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \left(\log \frac{1 - 2r^2 cos2\theta + r^4}{2r^2(1 - cos2\theta)}\right)^p r dr d\theta \\
&= \frac{4}{\pi} \int_0^1 \int_0^{\pi/2} \left(\log \frac{(1-r^2)^2 + 4r^2 sin^2\theta}{4r^2 sin^2\theta}\right)^p r dr d\theta \\
&= \frac{4}{\pi} \int_0^1 \int_0^{\pi/2} \left\{\log \left(1 + \left(\frac{1-r^2}{2r\, sin\theta}\right)^2\right)\right\}^p r dr d\theta \\
&\le \frac{4}{\pi} \int_0^1 \int_0^{\pi/2} \left\{\log \left(1 + \left(\frac{\pi\theta(1-r^2)}{8r}\right)^2\right)\right\}^p r dr d\theta.
\end{aligned}$$

Here ν_1 is the normalized volume measure on U. By changing variable $x = 1 + \left(\frac{\pi\theta(1-r^2)}{8r}\right)^2$, the last integral is

$$\begin{aligned}&\int_0^1 r dr \int_{1+c(r)}^{\infty} (\log x)^p \frac{1-r^2}{8r}\frac{\pi}{2}\frac{dx}{(x-1)^{3/2}} \\ =&\frac{\pi}{16}\int_0^1 (1-r^2) dr \int_{1+c(r)}^{\infty} \frac{(\log x)^p}{(x-1)^{3/2}} dx,\end{aligned}$$

where

$$c(r) = \left(\frac{1-r^2}{2r}\right)^2.$$

Noting that

$$\int_2^{\infty} \frac{(\log x)^p}{(x-1)^{3/2}} dx < \infty$$

and

$$\int_1^2 \frac{(\log x)^p}{(x-1)^{3/2}} dx \le \int_1^2 (x-1)^{p-3/2} dx < \infty$$

for $p > \frac{1}{2}$, we obtain (4.1).

It was not known whether F had the Bloch-BMO pullback property([3]). Concerning this problem, it was mentioned in [3] that the previously known methods (used by P. Ahern and W. Rudin) do not work for this F. See Remark (a) and (b) of [3]. In a coming paper of the author the problem will be settled.

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