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SLLN for Pairwise Independent Random Variables

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쌍별독립인 확률변수에 대한 대수의 강법칙

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Let $\{f(n)\}$ be an increasing sequence such that $f(n)>0$ for each n and $f(n)\to\infty$. Let $\{X_n, n\geq 1\}$ be a sequence of pairwise independent random variables. In this paper we give sufficient conditions on $\{X_n, n\geq 1\}$ such that $\sum_{i=1}^{n}(X_i - EX_i)/f(n)$ converges to zero almost surely.

$\{f(n)\}$ 은 양의 수열로 $f(n)\to\infty$ 이며 $\{X_n, n\geq 1\}$ 은 쌍별독립인 확률변수 열일 때 정규화된 부분합 $\sum_{i=1}^{n}(X_i - EX_i)/f(n)$ 이 0에 수렴할 확률이 1이 되는 $\{X_n, n\geq 1\}$ 의 조건을 찾고자 한다.

Key words : Strong law of large numbers, pairwise independent random variables, almost sure convergence

Ⅰ. Introduction

Let $\{f(n)\}$ be an increasing sequence such that $f(n)>0$ for each n and $f(n)\to\infty$. Let $\{X_n, n\geq 1\}$ be a sequence of pairwise independent random variables. Recently, Sung[3] proved a SLLN(Strong Law of Large Numbers) for pairwise independent and identically distributed(pairwise i.i.d.) random variables.

Theorem 1.1. (Sung, 1997). Let $\{X_n, n\geq 1\}$ be pairwise i.i.d. random variables and let $b_n = f(n)$ for all $n\geq 1$. Assume that

(a) $x/f(x)\uparrow$;

(b) $f(x)/\log^2 x\uparrow\infty$;

(c) $b_n^2 \sum_{i=n}^{\infty} 1/b_i^2 = O(n)$;

(d) $b_n^2 (\sum_{i=n}^{\infty} \log^2 i/b_i^2)/\log^2 n = O(n)$;

(e) $(\sum_{i=1}^{n} b_i/i)/b_n = O(1)$;

(f) $C_1\leq b_n^2/\{nf(b_n)\}\leq C_2$ for some constants $C_1>0$ and $C_2>0$;

(g) $b_n^2/\{f(b_n/\log^2 n)n\log^2 n\}\geq C_3$

for some constant $C_3>0$.

Then $E[X_1^2/f(|X_1|)]<\infty$ implies $\sum_{i=1}^{n}(X_i - EX_i)/b_n\to 0$ almost surely.

In this paper, we will obtain a SLLN for pairwise independent, but not necessarily identically distributed, random variables. Furthermore, our result implies Theorem 1.1.

II. Main Result

To prove our main result, we need the following lemma which is well known(Loeve, 1997, p. 124).

Lemma 2.1. Let $\{X_n, n\geq 1\}$ be a sequence of orthogonal random variables. If $\sum_{n=1}^{\infty}\log^2 n\ EX_n^2<\infty$, then $\sum_{n=1}^{\infty}X_n$ converges almost surely.

Chandra and Goswami (Chandra and Goswami, 1992) proved a SLLN for pairwise independent random variables.

Lemma 2.2. (Chandra and Goswami, 1992). Let $\{X_n, n\geq 1\}$ be a sequence of pairwise independent random variables and let $b_n = f(n)$ for all $n\geq 1$. Assume that

(i) $\sum_{i=1}^{n}E|X_i - EX_i|/b_n\leq C$ for some constant $C>0$;

(ii) $\sum_{n=1}^{\infty}Var(X_n)/b_n^2<\infty$.

Then $\sum_{i=1}^{n}(X_i - EX_i)/b_n\to 0$ almost surely.

Now, we state and prove our main result.

Theorem 2.3. Let $\{A_n\}$, $\{B_n\}$, $\{C_n\}$ be sequences of Borel subsets in R^1 such that $A_n\cup B_n\cup C_n = R^1, A_n\cap B_n = B_n\cap C_n = A_n\cap C_n = \emptyset$ for each $n\geq 1$. Let $\{X_n, n\geq 1\}$ be a sequence of pairwise independent random variables and let $b_n = f(n)$ for all $n\geq 1$. Assume that

(i) $\sum_{n=1}^{\infty}P(X_n\in C_n)<\infty$;

(ii) $\sum_{i=1}^{n}EX_iI(X_i\in C_i)/b_n\to 0$;

(iii) $\sum_{i=1}^{n}E|X_i|I(X_i\in B_i)/b_n\leq C$ for some constant $C>0$;

(iv) $\sum_{n=1}^{\infty}Var(X_nI(X_n\in B_n))/b_n^2<\infty$;

(v) $\sum_{n=1}^{\infty}\frac{\log^2 n}{b_n^2}Var(X_nI(X_n\in A_n))<\infty$.

Then $\sum_{i=1}^{n}(X_i - EX_i)/b_n\to 0$ almost surely.

Proof. Note that for each $n\geq 1$

$$X_n = X_nI(X_n\in A_n) + X_nI(X_n\in B_n) + X_nI(X_n\in C_n).$$

It follows by condition (v) that

$$\sum_{n=1}^{\infty}\log^2 n\ E\left(\frac{X_nI(X_n\in A_n) - EX_nI(X_n\in A_n)}{b_n}\right)^2 = \sum_{n=1}^{\infty}\frac{\log^2 n}{b_n^2}Var(X_nI(X_n\in A_n))<\infty.$$

By Lemma 2.1, we have that $\sum_{n=1}^{\infty}\frac{X_nI(X_n\in A_n) - EX_nI(X_n\in A_n)}{b_n}$ converges almost surely, which implies

$$\frac{1}{b_n}\sum_{i=1}^{n}(X_iI(X_i\in A_i) - EX_iI(X_i\in A_i))\to 0 \text{ almost surely} \quad (1)$$

by the Kronecker lemma. Conditions (iii) and

(iv) imply

$$\frac{1}{b_n}\sum_{i=1}^{n}(X_i I(X_i \in B_i) - EX_i I(X_i \in B_i)) \to 0$$
almost surely (2)

by Lemma 2.2. Condition (i) implies

$$\frac{1}{b_n}\sum_{i=1}^{n} X_i I(X_i \in C_i) \to 0 \quad \text{almost surely} \quad (3)$$

by the Borel-Cantelli lemma. Combining (1), (2), (3), and (ii) implies the result.

The following corollary was proved by (Chandra and Goswami, 1992)

Corollary 2.4. Let $\{X_n, n \geq 1\}$ be a sequence of pairwise independent random variables such that there is a sequence $\{B_n\}$ of Borel subsets in R^1 satisfying the following conditions:

(i) $\sum_{n=1}^{\infty} P(X_n \in B_n^c) < \infty$;

(ii) $\sum_{i=1}^{n} E(X_i I(X_i \in B_i^c))/b_n \to 0$;

(iii) $\sum_{i=1}^{n} E|X_i| I(X_i \in B_i)/b_n \leq C$ for some constant $C > 0$;

(iv) $\sum_{n=1}^{\infty} Var(X_n I(X_n \in B_n))/b_n^2 < \infty$;

Then $\sum_{i=1}^{n}(X_i - EX_i)/b_n \to 0$ almost surely.

Proof. Let $A_n = \emptyset, C_n = B_n^c$ for all $n \geq 1$. Then the result follows easily by Theorem 2.3.

Remark. Under the conditions of Theorem 1.1, $E[X_1^2/f(|X_1|)] < \infty \Leftrightarrow \sum_{n=1}^{\infty} P(|X_1| > b_n) < \infty$.

Taking $A_n = [-\frac{b_n}{\log^2 n}, \frac{b_n}{\log^2 n}]$,

$B_n = [-b_n, -\frac{b_n}{\log^2 n}) \cup (\frac{b_n}{\log^2 n}, b_n]$,

$C_n = (-\infty, -b_n) \cup (b_n, \infty)$, Theorem 1.1 follows by Theorem 2.3. So Theorem 2.3 is an extension of Theorem 1.1.

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IV. References

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