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Equivariant Real Vector Bundles over a Circle

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원위에서의 Equivariant Real Vector Bundles

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Let G be a compact Lie group and let $\rho : G \rightarrow O(2)$ be a homomorphism. Denote by V the G-module associated with ρ and by $S(V)$ the unit circle of V. In this paper, we show that if G is abelian, then a real G-vector bundle over $S(V)$ is isomorphic to Whitney sum of real G-line or G-plane bundles.

군 G가 compact Lie군이며 $\rho : G \rightarrow O(2)$ 가 homomorphism일 때 군 G가 가환군이면 원위에서 실 G-vector bundle은 실 G-line bundle들의 Whitney 합이거나 G-plane bundle들의 Whitney 합과 isomorphic 하다는 것을 보였다.

Key words : real G-vector bundle, equivariant vector bundle, G-plane bundle, Whitney sum.

Ⅰ. Decomposition

Let G be a compact Lie group and let $\rho : G \rightarrow O(2)$ be a homomorphism. If G is abelian, $\rho(G)$ is an abelian subgroup of $O(2)$; so it is contained in $SO(2)$ or isomorphic to D_1 or D_2, where D_n denotes the dihedral subgroup of $O(2)$ generated by the reflection matrix with respect to the x-axis and the rotation matrix of angle $2\pi/n$. When $\rho(G)$ is not contained in $SO(2)$, we may assume that $\rho(G) = D_n$ (or $O(2)$). Denote by V the G-module associated with ρ and by $S(V)$ the unit circle of V. Note that effectiveness of the G-action is equivalent to the injectivity of ρ.

Proposition A. (Kim, 1993). *A real G-vector bundle over $S(V)$ is isomorphic to Whitney sum of real G-line bundles if the G-action on $S(V)$ is effective.*

The effectiveness assumption cannot be dropped in the proposition above but we obtain the following result.

Proposition B. If G is abelian, then a real G-vector bundle over $S(V)$ is isomorphic to Whitney sum of real G-line or G-plane bundles.

II. Proof of Proposition B

Since the real G-line bundles are classified by (Kim and Masuda, 1994), we classify real G-plane bundles over $S(V)$ when G is abelian.

Let $E \to S(V)$ be a real G-plane bundle. Since the action of $H = \ker \rho$ on $S(V)$ is trivial, the fibers of E define a real 2-dimensional H-module F. If F is irreducible, then the action of H induces a complex structure so that E becomes a complex G-line bundle, which is analyzed in (Cho et al.,). So we may assume that F is not irreducible. If F is the direct sum of non-isomorphic real 1-dimensional H-modules, then E decomposes into Whitney sum of real G-line bundles accordingly. Therefore we may assume that F is the direct sum of a same 1-dimensional H-module χ. Moreover we may assume that the G-action on E is effective. If χ is the trivial H-module, then H must be the trivial group by the effectiveness of the G-action on E; so the G-action on $S(V)$ is effective. It follows from Proposition A that E decomposes into Whitney sum of real G-line bundles. Thus we may assume that χ is the nontrivial H-module and H is of order 2. We have a short exact sequence

$$(*) \quad 1 \to H \to G \xrightarrow{\rho} \rho(G) \to 1.$$

If this exact sequence splits, then χ extended to a G-module $\tilde{\chi}$. A real G-line bundle $E \otimes_{\mathbb{R}} \tilde{\chi}$ has the trivial H-action. Making the G-action on the bundle effective, we may reduce to the case where the G-action on the base $S(V)$ is effective. It follows again from Proposition A that E decomposes into Whitney sum of real G-line bundles. Thus we may assume that the exact sequence $(*)$ does not split in the sequel.

We consider three cases.

Case 1. The case where $\rho(G) = D_1$. Since the exact sequence $(*)$ dose not split, G must be isomorphic to $\mathbb{Z}_4$. This case is studied in (Cho and Suh, 1997) in detail but we shall give a different argument for later's convenience. Since the G-action on E is effective, G acts on the fibers E_p and E_q as rotation; so their exterior products $\bigwedge^2 E_p$ and $\bigwedge^2 E_q$ are both trivial G-modules. This implies that a real G-line bundle $\bigwedge^2 E$ is trivial with fiber $\mathbb{R}$. Therefore one can choose a G-invariant nondegenerate 2-form. This together with a G-invariant metric on E defines a G-invariant complex structure on E. Therefore E is realification of a complex G-line bundle. So there are two types of G-plane bundles by (Cho et al.,).

case 2. The case where $\rho(G) = D_2$. In this case G must be isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$. Let s and t denote element of G which are respectively of order 4 and 2 and generate G. There are four elements of order 4, those are s, s^3, st and s^3t.

If all of them map to $-1 \in D_2$ by ρ, then 1, s^2, t and $s^2 t$ are in $H = \ker \rho$ which contradicts that H is of order 2. Therefore we may assume that $\rho(s)$ is the reflection with respect to the x-axis. Then $\rho(t)$ is -1 or the reflection with the y-axis, but we may assume that $\rho(t)$ is the reflection with respect to the y-axis.

As observed in Case 1 $\bigwedge^2 E$ is trivial with fiber $\mathbb{R}$ as $<s>$-line bundle where $<s>$ denotes the order 4 subgroup of G generate by s. If t preserves an orientation of E, then t acts trivially on fibers of $\bigwedge^2 E$; hence $\bigwedge^2 E$ is trivial with fiber $\mathbb{R}$ as G-line bundle. Then as in case 1 one can choose a nondegenerate G-invariant 2-form on E so that E becomes a complex G-line bundle which is trivial by Theorem 2.2.

In the sequel we may assume that t reverses an orientation of E.

Claim. We identify $\mathbb{C}$ with $\mathbb{R}^2$. If t reverses an orientation of E, then E is isomorphic to a real G-plane bundle $S^1 \times \mathbb{C} \to S^1$ with G-action given by

$$s(z, v) = (\bar{z}, i\bar{z}v), \quad t(z, v) = (-\bar{z}, \bar{v}).$$

Proof. As remarked above E admits a complex structure preserved by the action of s. Suppose that the $<s>$-complex line bundle structure on E is trivial. Then the orientations on E induced by the action of s at the $<s>$-fixed points agree. Since t commutes with s, this means that t preserves the orientation, which is a contradiction. Therefore the $<s>$-complex line bundle structure on E is nontrivial. It follows from theorem 2.2 that we may assume that the action of s on $E \,(= S^1 \times \mathbb{C})$ is given by

$$s(z, v) = (\bar{z}, i\bar{z}v).$$

The action of t on E is described as

$$t(z, v) = (-\bar{z}, t(z)v)$$

with $t(z) \in GL(2, \mathbb{R})$ where v is viewed as an element of $\mathbb{R}^2$ through the identification of $\mathbb{C}$ with $\mathbb{R}^2$. Since t is of order 2 and commutes with s, we have

$$t(-\bar{z})t(z) = 1 \quad \text{and} \quad t(\bar{z})i\bar{z} = -izt(z)$$

where $i, z \in \mathbb{C}^* = GL(1, \mathbb{C})$ are viewed as elements of $GL(2, \mathbb{R})$ through the natural inclusion $GL(1, \mathbb{C}) \subset GL(2, \mathbb{R})$. These identities say that $t(z)$ is determined for all $z \in S^1$ once it is determined for $z = \exp i\theta \in S^1$ with $0 \le \theta \le \pi/2$, and that $t(i)$ is of order 2. Moreover det $t(1)$ is negative since $t(1)i = -it(1)$; so det $t(i)$ is also negative because $t(z) \in GL(2, \mathbb{R})$ is a continuous function of z.

A continuous map $C: S^1 \to GL(2, \mathbb{R})$ defines a coordinate change of the bundle $E = S^1 \times \mathbb{C}$ by $(z, v) \to (z, C(z)v)$. The commutativity with the action of s on E is equivalent to this identity :

$$C(\bar{z})i\bar{z} = i\bar{z}C(z) \tag{1}$$

Thus, to prove the claim is equivalent to finding the map C which satisfies (1) and this identity

$$(2) \qquad C(-\bar{z})\,t(z) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} C(z)$$

The identities (1) and (2) say that $C(z)$ is determined for all $z \in S^1$ if it is determined for $z \in S^1$ in the quarter circle as is so for $t(z)$. Moreover (1) says that $C(1)$ must be an element of $GL(1, \mathbb{C}) \subset GL(2, \mathbb{R})$ and (2) says that

$$c(i)\,t(i) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} c(i)\,.$$

As remarked above $t(i)$ is of order 2 and has a negative determinant. It is elementary to check that there exists $C(i) \in GL(2, \mathbb{R})$ which satisfies the above identity and has a positive determinant. We take $C(1)$ to be the identity matrix so that we can connect $C(1)$ and $c(i)$ in $GL(2, \mathbb{R})$ along the quarter circle. Then $C(z)$ is defined for all $z \in S^1$ by (1) and (2). This gives an isomorphism between E and the plane bundle in the claim.

Case 3. The case where $\rho(G) \subset SO(2)$. Since the exact sequence (＊) does not split, G is finite cyclic or $SO(2)$ and $\rho: G \to SO(2)$ lifts to the double covering $\chi: SO(2) \to SO(2)$, i.e. there is a homomorphism $\hat{\rho}: G \to SO(2)$ such that $\chi\hat{\rho} = \rho$. The $\hat{\rho}$ defines a real 2-dimensional G-module $\hat{V}$. Consider the real G-line bundle

$$\gamma: S(\hat{V}) \times_{\mathbb{Z}_2} \mathbb{R} \to S(\hat{V})/\mathbb{Z}_2 = S(V)\,,$$

where $\mathbb{Z}_2 = \{\pm 1\}$ acts on $S(\hat{V})$ and $\mathbb{R}$ as scalar multiplication. The subgroup H acts trivially on the base $S(V)$ and nontrivially on fibers, so $E \otimes_{\mathbb{R}} \gamma$ has the trivial H-action. Making the G-action on the tensor bundle effective, we may assume that the G-action on the base is effective. Therefore the tensor bundle decomposes into Whitney sum of real G-line bundles, hence so is E because $\gamma \otimes_{\mathbb{R}} \gamma$ is trivial.

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IV. References

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