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Study on Sequential Convergence Spaces, Convergence Spaces and Topological Spaces

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수열수렴공간, 수렴공간 그리고 위상공간에 대한 연구

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In this paper, we obtain the relations among the categories such as Seq, Conv and Top by using various functors between them.

본 논문에서는 Seq, Conv 그리고 Top 카테고리 사이의 여러 가지 functor를 이용하여 그들 사이의 관계를 조사한다.

Key words : Sequential convergence, Convergence, Adjoint, Functor

Ⅰ. Introduction

Convergence spaces, defined by axioms of Binz, have been studied from various points of view (Binz, 1975). In first countable topological spaces, one can restrict oneself to sequence in studying convergence and continuity. However, for more general spaces it seems to be assumed that sequences are not enough. But it appears that in some senses sequences are adequate for all spaces considered up to now in analysis. With this consideration, sequential convergence spaces have been studied from various point of view.

In this paper, we obtain the relations among the categories such as sequential convergence spaces, convergence spaces and topological spaces. For general categorical background, we refer to Adámek et al., (1990).

Ⅱ. Main Results

Let Top be the category of all topological spaces and continuous maps between them and Conv be the category of all convergence spaces and continuous maps between them (Binz, 1975).

Definition 2.1

If A is a subset of a convergence spaces (X, c), then the closure of A in (X, c) is the set $\overline{A} = \{x \in X \mid$ there is a filter containing A and converging to $x\}$

Proposition 2.2 (Kim, 1975)

For any convergence space (X, c), the followings hold :

1) $A \subseteq \overline{A}$
2) $\overline{\emptyset} = \emptyset$
3) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Definition 2.3

A subset A of a convergence space (X, c) is said to be closed in (X, c) if $\overline{A} = A$.

Corollary 2.4 (Kim, 1975)

For any convergence space (X, c), $\{A \mid A$ is closed in $(X, c)\}$ is the family of closed sets for some topology on X .

The topology defined by $\{A^c \subseteq X \mid A$ is closed in $X\}$ is called the topology induced by the convergence space X . And if $f: X \to Y$ is a continuous map in Conv, then $f: X \to Y$ is also continuous in the induced topologies. Thus, we can define a functor τ: Conv → Top by $\tau(X, c) = (X, \tau(c))$, where $\tau(c)$ is the topology induced by X and $\tau(f) = f$. Conversely, we can also define a functor ω: Top → Conv by $\omega(X, \Im) = (X, \omega(\Im))$, where $\omega(\Im)$ is the convergence structure defined by $\omega(\Im)(x)$ is the set of all filters converging to x and $\omega(f) = f$.

Proposition 2.5

1) $\tau \circ \omega = 1$
2) $(X, c) = \omega \circ \tau(X, c) \Leftrightarrow (X, c) = \omega(X, \Im)$ for some topology $\Im$ on X .

Proof. 1) Let $(X, \Im)$ be any topological space. We will show that $\tau \circ \omega(\Im) = \Im$. Note that $A \in \tau \circ \omega(\Im)$ iff any filter converging to some point of A contains A itself. Thus, if $A \in \Im$ and $\mathcal{F}$ is any filter converging to $x \in A$, then $\mathfrak{N}_x \subseteq \mathcal{F}$ and $A \in \mathfrak{N}_x$. Hence, $A \in \mathcal{F}$ i.e. $A \in \tau \circ \omega(\Im)$. Conversely, let $x \in A$. Then $\mathfrak{N}_x$ is a filter converging to x . Thus $A \in \mathfrak{N}_x$. Since this is true for all $x \in A$, we must $A \in \Im$.

2) ($\Leftarrow$) Since , $c = \omega(\Im)$
$\omega \circ \tau(c) = \omega \circ \tau \circ \omega(\Im) = \omega(\Im)$
$= c$ ($\Rightarrow$) $\tau(c)$ is a topology on X and
$\omega \circ \tau(c) = c$

Remark. For a convergence space (X, c) , $c(x) \subseteq \omega(\tau(c))(x)$.

Theorem 2.6

The functor τ is a left adjoint of ω .

Proof. We will show that for each $(X, c) \in$ Conv, $(1_X, (X, \tau(c)))$ is ω-universal map. First of all, it is easy to see that $1_X : (X, c) \to \omega \circ \tau(X, c)$ is continuous. Let $(X', \Im')$ be any topological space and $f: (X, c) \to (X', \omega(\Im'))$ be any continuous map.

Then $\tau(f) = f: \tau(X, c) = (X, \tau(c)) \to \tau(X', \omega(\Im')) = (X', \tau(\omega(\Im')))$ is continuous. Now, since $\Im' = \tau(\omega(\Im'))$ by definition, $f: (X, \tau(c)) \to (X', \Im')$ is continuous. Hence τ is a left adjoint of ω.

Proposition 2.7

The functor ω is full, faithful and embedding.

Proof. Let $f: \omega(X, \Im) \to \omega(X', \Im')$ be continuous.

Then $\tau(f) = f: (X, \Im) \to (X', \Im')$ is continuous by proposition 2.5. Thus, ω is full. Faithfulness is immediate from the definition of ω.

Let $\omega(X, \Im) = \omega(X', \Im')$.

Then $\tau(\omega(X, \Im)) = \tau(\omega(X', \Im'))$. That is $(X, \Im) = (X', \Im')$.

Hence ω is embedding.

Remarks

1) By Proposition 2.7, we may consider that Top is a full subcategory of Conv.

2) The functor ω is not dense.

Let $X = \{1, 2, 3\}$.

Let $c(1) = \{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_{1,2}\}$, $c(2) = \{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_{2,3}\}$ and $c(3) = \{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_{3,1}\}$ where $\mathcal{F}_i = \{A \subseteq X \mid i \in A\}$, $\mathcal{F}_{i,j} = \{A \subseteq X \mid i, j \in A\}$.

Then (X, c) is a convergence space. But there is no topology $\Im$ on X such that $(X, \omega(\Im)) = (X, c)$. Hence ω is not dense.

Proposition 2.8

The functor τ is faithful and dense.

Proof. Faithfulness is immediate from the definition of τ. Let $(X, \Im)$ be any topological spaces. Then $(X, \omega(\Im)) \in$ Conv and $\tau(X, \omega(\Im)) = (X, \tau(\omega(\Im))) = (X, \Im)$. Hence τ is dense.

Remark. The function τ is neither full nor embedding.

Let (X, c) be the convergence space given by the previous remarks and (X, c') be the convergence space gives by $c'(x) = F(X)$, where $F(X)$ is the set of all filters on X.

Then $1_X : (X, \tau(c')) \to (X, \tau(c))$ is continuous, because $\tau(c)$ is the indiscrete space. But $1_X : (X, c') \to (X, c)$ is not continuous, so τ is not full. Also $\tau(X, c') = \tau(X, c)$ but $(X, c') \neq (X, c)$, so τ is not embedding.

Let Seq be the category of all sequential convergence spaces and continuous maps between them (Park, 1996).

For $X \in$ Seq and $A \subseteq X$, let $\overline{A} = \{x \in X \mid$ there is a sequence $\langle x_n \rangle$ in A converging to $x\}$.

Proposition 2.9

For any sequential convergence space X and $A, B \subseteq X$, the followings hold :

1) $A \subseteq \overline{A}$
2) $\overline{\emptyset} = \emptyset$
3) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Proof. 1) and 2) are trivial. For 3), suppose that $x \in \overline{A \cup B}$. Then there is a sequence $\langle x_n \rangle$ in $A \cup B$ converging to x. If A contains infinitely many terms of x_n, then $x \in \overline{A}$ or if B contains infinitely many terms of x_n, then $x \in \overline{B}$.

Hence $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. The converse is trivial.

For a sequential convergence space X and $A \subseteq X$, A is said to be closed if $\overline{A} = A$. Hence by the above proposition, the family $\{A^c \subseteq X \mid A$ is closed in $X\}$ is a topology on X, called the topology induced by the sequential convergence space X.

Proposition 2.10

For a sequential convergence space X and $A \subseteq X$, A is open in the induced topology if and only if for any sequence $\langle x_n \rangle$ converging to a point p in A there exists an

$n_0 \in \mathbb{N}$ such that $x_n \in A$ for all $n \geq n_0$.

Proof. Suppose that there is a sequence $< x_n >$ converging to $p \in A$ such that for all natural number N, there exists an $n \geq N$ with $x_n \notin A$. Then we can construct a subsequence $< x_{s(n)} >$ of $< x_n >$ with $x_{s(n)} \notin A$, i.e, $x_{s(n)} \in A^c$. But since $< x_{s(n)} >$ converges to p, $p \in \overline{A^c}$. This is a contradiction, since $\overline{A^c} = A^c$. Conversely, let $p \in \overline{A^c}$. Then there is a sequence $< x_n >$ in A^c converging to p.

If $p \in A$, by assumption, there exists an $n_0 \in \mathbb{N}$ such that $x_n \in A$ for all $n \geq n_0$.

But this is a contradiction, since $x_n \in A^c$ for all $n \in \mathbb{N}$. So $p \in A^c$, i.e. $\overline{A^c} \subseteq A^c$. Therefore A^c is closed, and hence A is open in the induced topology.

Theorem 2.11

Let $f : X \to Y$ be a sequentially continuous map and $\Im$ and $\Im'$ be induced topologies on X and Y respectively. Then

1) for $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$

2) $f : (X, \Im) \to (Y, \Im')$ is continuous

Proof. 1) is trivial, For 2), let U be open in $(Y, \Im')$ and $< x_n >$ be a sequence in X which converges to $p \in f^{-1}(U)$. Then $f(x_n)$ converges to $f(p) \in U$. Since U is open, there exists an $n_0 \in \mathbb{N}$ such that $f(x_n) \in U$ for all $n \geq n_0$.

Hence $x_n \in f^{-1}(U)$. So $f^{-1}(U)$ is open in $(X, \Im)$. Hence f is continuous.

Thus, we can define a functor τ_s : Seq $\to$ Top by $\tau_s(X, \Xi) = (X, \tau_s(\Xi))$, where $\tau_s(\Xi)$ is the topology induced by X and $\tau_s(f) = f$. Conversely, we can also define a functor ω_s : Top $\to$ Seq by $\omega_s(X, \Im) = (X, \omega_s(\Im))$ where $\omega_s(\Im) = \{(< u_n >, x) \mid u_n \to x,\ x \in X\}$.

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