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Ostrowski's Inequality for Monotonous Mappings and Applications

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Abstract

An inequality of Ostrowski's type for monotonous nondecreasing mappings is given. Applications for quadrature formulas are pointed out.

1 Introduction

The following theorem contains the integral inequality which is known in the literature as Ostrowski's inequality [4, p. 469].

THEOREM 1.1. *Let $f : [a,b] \to R$ be a differentiable mapping on (a,b) whose derivative is bounded on (a,b) and denote $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then for all $x \in [a,b]$ we have the inequality*

$$| f(x) - \frac{1}{b-a}\int_a^b f(t)dt | \le [\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}](b-a)\|f'\|_\infty.$$

The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

In the recent paper [1], S.S. Dragomir has proved the following Ostrowski's type inequality for mappings with bounded variation:

THEOREM 1.2. *Let $u : [a,b] \to R$ be mapping with bounded variation on $[a,b]$. Then for all $x \in [a,b]$, we have the inequality*

$$| \int_a^b u(t)dt - u(x)(b-a) | \le [\frac{1}{2}(b-a) + |x - \frac{a+b}{2}|]V_a^b(u). \quad (2.1)$$

where $V_a^b(u)$ denotes the total variation of u.

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The constant $\frac{1}{2}$ is the best possible one.

A corollary of this results is the following inequality for monotonous mappings

COROLLARY 1.3. *Let $u : [a,b] \to R$ be a monotonous mapping on $[a,b]$. Then we have the inequlity*

$$\mid \int_a^b u(t)dt - u(x)(b-a) \mid \leq [\frac{1}{2}(b-a) + |x - \frac{a+b}{2}|] \mid f(b) - f(a) \mid .$$

In this paper we prove an Ostrowski's type inequality for monotonous nondecreasing mappings which improves the above result and apply it in obtaining a Riemann's type quadrature formula for this class of mappings.

For some similar results for differentiable mappings see the recent papers [2-3] by Dragomir and Wang.

2 An Inequality for Monotonous Mappings

The following results of Ostrowski's type holds

THEOREM 2.1. *Let $u : [a,b] \to R$ be a monotonous nondecreasing mapping on $[a,b]$. Then for all $x \in [a,b]$, we have the inequality*

$$\mid u(x) - \tfrac{1}{b-a} \int_a^b u(t)dt \mid \tag{2.1}$$

$$\leq \frac{1}{b-a}\{[2x - (a+b)]u(x) + \int_a^b sgn(t-x)u(t)dt\}$$

$$\leq \frac{1}{b-a}[(x-a)(u(x)-u(a)) + (b-x)(u(b)-u(x))]$$

$$\leq [\frac{1}{2} + \frac{\mid x - \frac{a+b}{2} \mid}{b-a}](u(b)-u(a)).$$

All the inequalities in (2.1) are sharp and the constant $\frac{1}{2}$ is the best possible one.

Proof. Using the integration by parts formula for Riemann-Stieltjes integral, we have the identity

$$u(x) - \frac{1}{b-a}\int_a^b u(t)dt = \frac{1}{b-a}\int_a^b p(x,t)du(t) \tag{2.2}$$

where

$$p(x,t) := \begin{cases} t-a \text{ if } t \in [a,x] \\ t-b \text{ if } t \in (x,b] \end{cases}.$$

Indeed, we have

$$\int_a^x (t-a)du(t) = u(x)(x-a) - \int_a^x u(t)dt$$

and

$$\int_x^b (t-b)du(t) = u(x)(b-x) - \int_x^b u(t)dt.$$

If we add the above two equalities, we get

$$u(x)(b-a) - \int_a^b u(t)dt = \int_a^b p(x,t)du(t)$$

and the identity (2.2) is proved.

Now, assume that $\Delta_n : a = x_0^{(n)} < x_1^{(n)} < ... < x_{n-1}^{(n)} < x_n^{(n)} = b$ is a sequence of divisions with $\nu(\Delta_n) \to 0$ as $n \to \infty$, where $\nu(\Delta_n) := \max_{i\in\{0,\dots,n-1\}}(x_{i+1}^{(n)} - x_i^{(n)})$ and $\xi_i^{(n)} \in [x_i^{(n)}, x_{i+1}^{(n)}]$. If $p : [a,b] \to R$ is continuous on $[a,b]$ and $v : [a,b] \to R$ is monotonous nondecreasing on $[a,b]$, then

$$\left| \int_a^b p(x)dv(x) \right| = \left| \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} p(\xi_i^{(n)})[v(x_{i+1}^{(n)}) - v(x_i^{(n)})] \right|$$

$$\leq \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} | p(\xi_i^{(n)}) || v(x_{i+1}^{(n)}) - v(x_i^{(n)})|$$

$$\leq \lim_{\nu(\Delta_n)\to 0} \sum_{i=0}^{n-1} | p(\xi_i^{(n)}) | (v(x_{i+1}^{(n)}) - v(x_i^{(n)}))$$

$$= \int_a^b | p(x) | dv(x).$$

As u is monotonous nondecreasing on $[a,b]$, and $p(x,.)$ is continuous on the portions, then using the above inequality we can state that

$$|\int_a^b p(x,t)du(t)| \le \int_a^b |p(x,t)|du(t). \tag{2.3}$$

Now, let us observe that

$$\int_a^b |p(x,t)|du(t) = \int_a^x |t-a|du(t) + \int_x^b |t-b|du(t)$$

$$= \int_a^x (t-a)du(t) + \int_x^b (b-t)du(t)$$

$$= (t-a)u(t)]_a^x - \int_a^x u(t)dt - (b-t)u(t)]_x^b + \int_x^b u(t)dt$$

$$= [2x-(a+b)]u(x) - \int_a^x u(t)dt + \int_x^b u(t)dt$$

$$= [2x-(a+b)]u(x) + \int_a^b sgn(t-x)u(t)dt.$$

Using the inequality (2.3) and the identity (2.2) we get the first part of (2.1). Now let us observe that

$$\int_a^b sgn(t-x)u(t)dt = -\int_a^x u(t)dt + \int_x^b u(t)dt.$$

As u is monotonous nondecreasing on $[a,b]$, we can state that

$$\int_a^x u(t)dt \ge (x-a)u(a)$$

and

$$\int_x^b u(t)dt \le (b-x)u(b)$$

and then

$$\int_a^b sgn(t-x)u(t)dt \leq (b-x)u(b) - (x-a)u(a).$$

Consequently

$$[2x-(a+b)]u(x) + \int_a^b sgn(t-x)u(t)dt$$

$$\leq [2x-(a+b)]u(x) + (b-x)u(b) - (x-a)u(a)$$

$$= (b-x)(u(b)-u(x)) + (x-a)(u(x)-u(a))$$

and the second part of (2.1) is proved.
Finally, let us observe that

$$(b-x)(u(b)-u(x)) + (x-a)(u(x)-u(a))$$

$$\leq \max\{b-x, x-a\}[u(b)-u(x)+u(x)-u(a)]$$

$$= [\frac{b-a}{2} + \mid x - \frac{a+b}{2} \mid](u(b)-u(a))$$

and the inequality (2.1) is thus proved.
Assume that (2.1) holds with a constant C instead of $\frac{1}{2}$, i.e.,

(2.1')
$$\mid u(x) - \tfrac{1}{b-a}\int_a^b u(t)dt \mid$$

$$\leq \frac{1}{b-a}\{[2x-(a+b)]u(x) + \int_a^b sgn(t-x)u(t)dt\}$$

$$\leq \frac{1}{b-a}[(x-a)(u(x)-u(a)) + (b-x)(u(b)-u(x))]$$

$$\leq [C + \frac{\mid x - \frac{a+b}{2} \mid}{b-a}](u(b)-u(a)).$$

Consider the mapping $u_0 : [a,b] \to R$ given by

$$u_0(x) := \begin{cases} -1 \text{ if } x = a \\ 0 \text{ if } x \in (a,b] \end{cases}.$$

Puting in (2.1') $u = u_0$ and $x = a$, we get

$$| u(x) - \tfrac{1}{b-a} \int_a^b u(t)dt |$$

$$= \frac{1}{b-a}\{[2x-(a+b)]u(x) + \int_a^b sgn(t-x)u(t)dt\}$$

$$= \frac{1}{b-a}[(x-a)(u(x)-u(a)) + (b-x)(u(b)-u(x))] = 1$$

$$\leq [C + \tfrac{|x-\frac{a+b}{2}|}{b-a}](u(b)-u(a)) = (C+\tfrac{1}{2})$$

which prove the sharpness of the first two inequalities and the fact that C should not be less than $\frac{1}{2}$. ■

The following corollaries are interesting:

COROLLARY 2.2. *Let u be as above. Then we have the midpoint inequality:*

(2.4)
$$| u(\tfrac{a+b}{2}) - \tfrac{1}{b-a} \int_a^b u(t)dt |$$
$$\leq \frac{1}{b-a}\int_a^b sgn(t - \frac{a+b}{2})u(t)dt \leq \frac{1}{2}[u(b)-u(a)].$$

Also, we have the following *"trapezoid inequality"* for monotonous nondecreasing mappings.

COROLLARY 2.3. *Under the above assumptions, we have*

(2.5)
$$| \tfrac{u(a)+u(b)}{2} - \tfrac{1}{b-a} \int_a^b u(t)dt | \leq \tfrac{1}{2}[u(b)-u(a)].$$

Proof. Let us choose in Theorem 2.1, $x = a$ and $x = b$ to obtain

$$| u(a) - \frac{1}{b-a}\int_a^b u(t)dt | \leq \frac{1}{b-a}[-(b-a)u(a) + \int_a^b u(t)dt]$$

and

$$| u(b) - \frac{1}{b-a}\int_a^b u(t)dt |\leq \frac{1}{b-a}[(b-a)u(b) - \int_a^b u(t)dt].$$

Summing the above inequalities, using the triangle inequality and deviding by 2,we get the desired inequality (2.5). ■

3 A Qadrature Formula

Let $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be a division of the interval $[a, b]$ and $\xi_i \in [x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ a sequence of intermediate points for I_n. Construct the Riemann sums

$$R_n(f, I_n, \boldsymbol{\xi}) = \sum_{i=0}^{n-1} f(\xi_i)h_i$$

where $h_i := x_{i+1} - x_i$.

We have the following quadrature formula

THEOREM 3.1. *Let $f : [a, b] \to R$ be a monotonous nondecreasing mapping on $[a, b]$ and I_n, ξ_i $(i = 0, ..., n-1)$ be as above. Then we have the Riemann quadrature formula*

$$\int_a^b f(x)dx = R_n(f, I_n, \boldsymbol{\xi}) + W_n(f, I_n, \boldsymbol{\xi}) \qquad (3.1)$$

where the remainder satisfies the estimation

$$| W_n(f, I_n, \boldsymbol{\xi}) |\leq 2\sum_{i=0}^{n-1}(\xi_i - \frac{x_i + x_{i+1}}{2})f(\xi_i) + \int_a^b S(t, I_n, \boldsymbol{\xi})f(t)dt$$

$$\leq \sum_{i=0}^{n-1}[(\xi_i - x_i)(f(\xi_i) - f(x_i)) + (x_{i+1} - \xi_i)(f(x_{i+1}) - f(\xi_i))]$$

$$\leq \sup_{i=0,...,n} [\frac{1}{2}h_i + | \xi_i - \frac{x_i + x_{i+1}}{2} |](f(b) - f(a))$$

$$\leq [\frac{1}{2}\nu(h) + \sup_{i=0,\dots,n} \mid \xi_i - \frac{x_i + x_{i+1}}{2} \mid](f(b) - f(a)) \leq \nu(h)(f(b) - f(a)) \quad (3.2)$$

for all ξ_i $(i = 0, ..., n-1)$ *as above, where* $\nu(h) := \max_{i=0,\dots,n}\{h_i\}$ *and*

$$S(t, I_n, \boldsymbol{\xi}) = sgn(t - \zeta_i) if\ t \in [x_i, x_{i+1})(i = 0, ..., n-1).$$

Proof. Apply Theorem 2.1 on the interval $[x_i, x_{i+1}]$ to get

$$\mid \int_{x_i}^{x_{i+1}} f(x)dx - f(\xi_i)h_i \mid \leq 2(\xi_i - \frac{x_i + x_{i+1}}{2})f(\xi_i) + \int_{x_i}^{x_{i+1}} S(t, I_n, \boldsymbol{\xi})f(t)dt$$

$$\leq (\xi_i - x_i)(f(\xi_i) - f(x_i)) + (x_{i+1} - \xi_i)(f(x_{i+1}) - f(\xi_i))$$

$$\leq [\frac{1}{2}h_i + \mid \xi_i - \frac{x_i + x_{i+1}}{2} \mid](f(x_{i+1}) - f(x_i)). \quad (3.3)$$

Summing over i from 0 to $n-1$ and using the generalized triangle inequality we get

$$\mid W_n(f, I_n, \boldsymbol{\xi}) \mid \leq \sum_{i=0}^{n-1} \mid \int_{x_i}^{x_{i+1}} f(x)dx - f(\xi_i)h_i \mid$$

$$\leq 2\sum_{i=0}^{n-1}[(\xi_i - \frac{x_i + x_{i+1}}{2})f(\xi_i) + \int_{x_i}^{x_{i+1}} S(t, I_n, \boldsymbol{\xi})f(t)dt]$$

$$\leq \sum_{i=0}^{n-1}[(\xi_i - x_i)(f(\xi_i) - f(x_i)) + (x_{i+1} - \xi_i)(f(x_{i+1}) - f(\xi_i))]$$

$$\leq \sum_{i=0}^{n-1}[\frac{1}{2}h_i + \mid \xi_i - \frac{x_i + x_{i+1}}{2} \mid](f(x_{i+1}) - f(x_i))$$

$$\leq \sup_{i=0,\dots,n}[\frac{1}{2}h_i + \mid \xi_i - \frac{x_i + x_{i+1}}{2} \mid] \sum_{i=0}^{n-1}(f(x_{i+1}) - f(x_i))$$

$$= \sup_{i=0,\dots,n}[\frac{1}{2}h_i + \mid \xi_i - \frac{x_i + x_{i+1}}{2} \mid](f(b) - f(a)).$$

The fourth inequality follows by the properties of sup(.).
Now, as

$$| \xi_i - \frac{x_i + x_{i+1}}{2} | \leq \frac{1}{2} h_i$$

for all $\xi_i \in [x_i, x_{i+1}] (i = 0, ..., n-1)$ the last part of (3.2) is also proved. ■

COROLLARY 3.2. *Let f, I_n be as in Theorem 3.1. Then we have the midpoint rule*

$$\int_a^b f(x)dx = M_n(f, I_n) + S_n(f, I_n)$$

where

$$M_n(f, I_n) = \sum_{i=0}^{n-1} f(\frac{x_i + x_{i+1}}{2}) h_i$$

and the remainder $S_n(f, I_n)$ satisfies the estimation

$$| S_n(f, I_n) | \leq \int_a^b \mu(I_n) f(t) dt \leq \frac{1}{2} \nu(h)(f(b) - f(a))$$

where

$$\mu(I_n) = sgn(t - \frac{x_i + x_{i+1}}{2}) if \ t \in [x_i, x_{i+1})(i = 0, ..., n-1).$$

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