J. KSIAM Vol.3, No.107-114, 1999

# PROPERTIES OF RANDOM SIGNALS IN WAVELET DOMAIN

Young Seock Lee, Sung Hwan Kim*

## Abstract

In many applications(e,g., identification of non-destructive testing signal and biomedical signal and multiscale analysis of image), it is of interest to analyze and identify phenomena occurring at the different scales. The recently introduced wavelet transforms provide a time-scale decomposition of signals that offers the possibility of such signals. However, there is no corresponding statistical properties to development of multiscale statistical signal processing. In this paper, we derive such properties of random signals in wavelet domain.

## I. Introduction

The wavelet transform have been used mainly in the fields of signal processing, image coding and compression, and in certain areas of mathematics, as in solution of partial differential equations or numerical analysis[1][2][3][4]. Recently an enormous interest has emerged on the use of wavelet transforms in several areas. One of these areas is to understand the statistical behavior of random signals in wavelet domain. Basseville et al.[5] studied random processes defined on a multiscale grid of wavelet decomposition coefficients but its relationship to conventional notions of stationarity for random processes is unclear. Wornell[6] used wavelet transform to synthesize 1/f processes. His work assumes a very simple correlation structure for the wavelet coefficients are all independent. And also Marsry[7] studied stationary increment processes on the wavelet domain and applied to fractional Brownian motion(fBm) but his work did not consider mismatching between wavelets in $L^2(R)$ and ensembles of stationary increment processes in $L^1(R)$. Dijkerman[8] used wavelet transform to analyze time domain AR processes but his research was confined that characteristic of time domain AR process representations in wavelet domain and has not tried to an AR modeling in wavelet domain.

While any deterministic signals in $L^2(R)$ is completely characterized by the coefficients of their wavelet transform, the same is not necessarily true of a random signals. In general, the sample paths of random signals may not be a subset of $L^2(R)$ signal space, in which case the inner products between wavelets and random signals are not well defined and dangerous to divergence. But if we confined random signals as wide sense stationary signals and choose

Keywords : wavelet transform, random signal, decomposition

a proper wavelet, it was possible for random signals to converge in $L^2(R)$ with probability one. In this paper we derive properties of wavelet-transformed wide sense stationary signal and apply real ultrasonic signal.

## II. Preliminary

One of the most powerful concepts in the class of random processes is that of stationarity and the extension and use of this concept to wavelet domain represents one of the goal of our works. For a stationary process $X(t)$, we focus an wide-sense stationary(WSS) process which is the mean value is independent of the time and that the autocorrelation depends only on the time difference as

$$E[X(t)] = C, \text{ where } C \text{ is a constant}, \tag{1}$$

$$E[X(t)(X(t-\tau)] = \gamma_{XX}(\tau). \tag{2}$$

The mean value $E[X(t)]$ will usually not enter our discussion because it is normally assumed to be zero. Given an WSS process $X(t)$, its probability distribution function $F_X(x)$ is Gaussian if the probability distribution function is

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^{x} \exp\left[-\frac{1}{2\sigma_x^2}(x-\mu_x)^2\right] dx. \tag{3}$$

In other word, the WSS signals with Gaussian distribution are characterized by (1),(2),(3) and two WSS processes $X(t)$ and $Y(t)$ are said to be jointly WSS if the process satisfies the following relation

$$\gamma_{XY}(\tau) = E[X(t)Y(t-\tau)] \tag{4}$$

and the autocorrelation functions (2) and (4) have finite values under assuming an WSS.

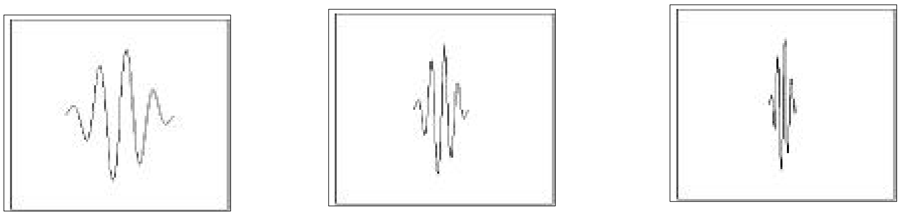
The basic fact about wavelet transforms is that they are localized in time, contrary to what happen to Fourier transform. This makes wavelet transforms allow us to analyze a series into both time and scale. The wavelet transforms are generated by a function $\Psi(t)$ called mother wavelet with

$$\Psi_{a,b}(t) = |a|^{-\frac{1}{2}} \Psi\left(\frac{t-b}{a}\right) \text{ for } a \in R - \{0\},\ b \in R, \tag{5}$$

the wavelet transform is defined as

$$(Wf)(a,b) = \int_{-\infty}^{\infty} f(t)\Psi^*_{a,b}(t)\,dt = \langle f, \Psi_{a,b}\rangle,\ f(t) \in L^2(R) \tag{6}$$

$*$ is denoted by complex conjugate. $(Wf)(a,b)$ is therefore the correlation of $f(t)$ with $\Psi$ shifted by $b$ and scaled by $a$. Another interpretation is that $(Wf)(a,b)\,\Psi_{a,b}(t)$ is simply the projection of onto $\Psi_{a,b}$.



a=2 a=1 a=0.5

Fig.1. wavelets for several scale values of $a$

Figure 1 displays wavelets along with scale 2, 1, 0.5. Since the analyzing functions for the transform are all shifted and scaled versions of the mother wavelet, the time localization afforded by the transform increases with decreasing scale at the expense of frequency localization.
The tradeoff between time and frequency localization with scale is a distinguishing feature of
wavelet transform. It makes the transform particularly well suited to the detection and analysis of signals.

## III. Properties of WSS signals in wavelet domain

In general, the sample paths of WSS processes may not be a subset of $L^2(R)$ but that of $L^1(R)$, in which case the inner products between wavelets and WSS signals are not well defined and dangerous to divergence. In this point, the statistical properties of transformed
signals are proven to be bounded.

Let $X(t),\ t \in R$ be an WSS signal with zero mean and $\gamma(u) = E[X(t)X(t+u)]$

be the autocorrelation function of $X(t)$. We assume wavelet $\Psi(t)$ is bounded and compactly

supported,
there is $A, B \in R$ such that $|\Psi(t)| \le A$ and $|supportness\ of\ \Psi(t)| \le B$ . We know that $\Psi_{a,b}(t) = 2^{-a/2}\Psi(2^{-a}t - b)$,
we have $|\Psi_{a,b}(\cdot)| = 2^{-a}A$ and $|supportness\ of\ \Psi_{a,b}(t)| \le 2^{a}B$ ,
and denote by $K$ , the variation of $\Psi(t)$. the wavelet transform of WSS signal is

$$(WX)(a,b) = \int_{-\infty}^{\infty} X(t)\Psi^{*}_{a,b}(t)\,dt \tag{7}$$

and its autocorrelation function is

$$\zeta_{a,b} = \int_{-\infty}^{\infty}(u)\,du \int_{0}^{\infty} dt\ \gamma(u)\ \Psi_{a,b}(t)\Psi_{a,b}(t+|u|). \tag{8}$$

From (8) we define an partial autocorrelation function

$$\xi_{a,b} = \int_{-T}^{T} du \int_{0}^{T} dt\ \gamma(u)\, \Psi_{a,b}(t)\, \Psi_{a,b}(t+|u|) \tag{9}$$

and the relation between (8) and (9) is $\zeta_{a,b} = \lim_{T\to\infty} \xi_{a,b}$ .
(9) is expanded following as

$$\xi_{a,b} = \int_{-T}^{T} du\ \gamma(u) \int_{0}^{T} dt\ \Psi_{a,b}(t)\, \Psi_{a,b}(t) + \varepsilon(t) \tag{10}$$

where $\int_{0}^{T} dt\ \Psi_{a,b}(t)\, \Psi_{a,b}(t)$ is expanded following as

$$\int_{0}^{T} dt\ \Psi_{a,b}(t)\, \Psi_{a,b}(t) \le (2^{-a/2}A)^2\ |support\Psi(t)| \le A^2 |support\Psi(t)| \tag{11}$$

and also

$$\varepsilon(t) = \gamma(u) \left\{ \int_{0}^{T} dt\ \Psi_{a,b}(t+u)\Psi_{a,b}(t) - \int_{0}^{T} dt \Psi_{a,b}(t)\, \Psi_{a,b}(t) \right\} \tag{12}$$

From (12), $\left\{ \int_{0}^{T} dt\ \Psi_{a,b}(t+u)\Psi_{a,b(t)} - \int_{0}^{T} dt \Psi_{a,b}(t)\, \Psi_{a,b}(t) \right\}$ is expanded following as

$$\begin{aligned} &\left| \int_{0}^{T} dt\ \Psi_{a,b}(t+u)\Psi_{a,b}(t) - \int_{0}^{T} dt \Psi_{a,b}(t)\, \Psi_{a,b}(t) \right| \\ &\qquad \le \int_{0}^{T} dt\ |\Psi_{a,b}(t)|\,|\Psi_{a,b}(t+u) - \Psi_{a,b}(t)| \end{aligned} \tag{13}$$

and the right side of (13) is expanded to

$$\begin{aligned} &\int_{0}^{T} dt\ |\Psi_{a,b}(t)|\,|\Psi_{a,b}(t+u) - \Psi_{a,b}(t)| \\ &\qquad \le 2^{-a/2}A \int_{0}^{T} dt\ |\Psi_{a,b}(t+u) - \Psi_{a,b}(t+u-1) + \Psi_{a,b}(t+u-2) \\ &\qquad + \Psi_{a,b}(T+u-2) - \cdots\cdots + \Psi_{a,b}(t+1) - \Psi_{a,b}(t)| \end{aligned} \tag{14}$$

and also the right side of (14) holds (15) by Cauchy-Schwartz inequality

$$\begin{aligned} &2^{-a/2}A \int_{0}^{T} dt\ |\Psi_{a,b}(t+u) - \Psi_{a,b}(t+u-1) + \Psi_{a,b}(t+u-2) \\ &\qquad + \Psi_{a,b}(T+u-2) - \cdots\cdots + \Psi_{a,b}(t+1) - \Psi_{a,b}(t)| \\ &\quad \le 2^{-a/2}A \int_{0}^{T} dt\ |\Psi_{a,b}(t+u) - \Psi_{a,b}(t+u-1)| + |\Psi_{a,b}(t+u-2) \\ &\qquad + \Psi_{a,b}(T+u-2)| - \cdots\cdots + |\Psi_{a,b}(t+1) - \Psi_{a,b}(t)| \end{aligned} \tag{15}$$

$$\leq 2^{-a/2} A \; 2^{-a/2} K \; |u| \; = \; 2^{-a} A K \; |u| \; < \; \infty .$$

Consequently $|\varepsilon(t)|$ is following as

$$|\varepsilon(t)| \leq A K \, 2^{-a} \int_{-T}^{T} du \; |\gamma(u)| u \, | . \tag{16}$$

By (11) and (16), $\xi_{a,b}$ is

$$\xi_{a,b} \;\leq\; \int_{-T}^{T} du \{ \gamma(u) A^{2} \; |support \;\; \Psi(t)| \; + \; 2^{-a} A K \;\; \gamma(u) |u| \} \tag{17}$$

so,

$$\zeta_{a,b} = \lim_{T \to \infty} \xi_{a,b} < \infty . \tag{18}$$

Hence, by (17) and (18), we have that $\xi_{a,b}$ is bounded and also $\zeta_{a,b}$ is bounded by relation $\zeta_{a,b} = \lim_{T \to \infty} \xi_{a,b}$ . So the derivated results imply that the WSS signals in the transform domain are bounded and at least not diverge if wavelet is bounded and compactly supported and it holds crosscorrelation between another scale wavelet representations of WSS signal.

In discrete time domain, wavelet transform is computed by a filtering operation followed by decimation by a fact of two. This filtering operation called perfect reconstruction filter bank(PRQMF) is depicted by Figure 2. in which $H_0(z)$ is a high pass filter, $G_0(z)$ is a low pass filter and the two filters construct analysis stage of PRQMF and also synthesis stage is constructed by $H_1(z)$ and $G_1(z)$ has a power complementary relationship to analysis stage.

By the multiresolution analysis property of wavelet transform, the downsampled resulting sequence of $G_0(z)$ are then generated by performing the same filtering operation of analysis stage and so on recursively.

Fig. 2. Filter structure corresponding to discrete time wavelet transform analysis and synthesis stage

In Figure 2, given an WSS signal $X(n)$ as an input of PRQMF, the statistical behavior of output is unknown. But the decimated version $X_D(n)$ of WSS signal $X(n)$ is WSS because

$$X_D(n) = X(2^i n) \tag{19}$$

and

$$\begin{aligned} E[X_D(n)X_D^*(n+\tau)] &= E[X(2^i n)X^*(2^i(n+\tau))] \\ &= E[X(m)X(m+\tau')^*] \\ &= \gamma(\tau'), \quad \text{where } 2^i n = m \text{ and } \tau' = 2^i\tau. \end{aligned} \tag{20}$$

Since the transfer functions $H_0(z)$ and $G_0(z)$ are operators of linear transformation, so their outputs are WSS by

$$\begin{aligned} &E[u_0(n)u_0^*(n+\tau)] \\ &= E[X(n)X^*(n+\tau)] \otimes u_0(\tau) \otimes u_0(-\tau) \\ &= \gamma(\tau) \otimes u_0(\tau) \otimes u_0(-\tau), \end{aligned} \tag{21}$$

where $u_i \in \{g_0, h_0\}$ and $\otimes$ is a convolution operator.

Let $v_i(n)$ be the transfer function of the i-th scale, the i-th output of PRQMF, $X_i(n)$ is

$$X_i(n) = \sum_k v_i(k)X(2^i n - k), \tag{22}$$

its autocorrelation function is

$$\begin{aligned} E[X_i(n)X_i^*(m)] &= E[\sum_k\sum_{k'} v_i(k)X(2^i n - k)v_i^*(k')X^*((2^i m - k')] \\ &= \sum_k v_i(k)v_i^*(k')\gamma(2^i(n-m) - k', -k') \end{aligned} \tag{23}$$

and also the crosscorrelation of $X_i$ and $X_j$ corresponding to two distinct scales, $i \neq j$ is

$$\begin{aligned} E[X_i(n)X_j^*(m)] &= E[\sum_k v_i(k)X(2^i n - k)v_j^*(k')X_j^*((2^j m - k')] \\ &= \sum_k v_i(k)v_j^*(k')\gamma(\tau' - k', -k') \end{aligned} \tag{24}$$

by Fubini's theorem and substituting $\tau' = 2^i n - 2^j m - k + k'$.

From observation of (23) and (24), it implies that the wavelet coefficients of WSS signal are WSS at same scale and jointly WSS at two distinct scales and that an autocorrelation function, (23) of wavelet coefficients has much stronger autocorrelated-property than that of the time series.

For example, assume the autocorrelation function $\gamma(n-m) = \lambda e^{-\alpha|n-m|}$, $R^+ \supset \{\lambda, \alpha\}$, which is exponentially decreasing to the time difference $|n-m|$, it is frequently considered for the Gaussian process, the corresponding autocorrelation function $\gamma_i(n-m)$ of wavelet coefficients at the i-th scale is $\lambda e^{-2^i\alpha|n-m|}$ which is exponentially decreasing to the translation difference $2^i|n-m|$ as figure 3. It imply that an autocorrelation function in wavelet domain has much whitening comparing with that in time domain as good properties.

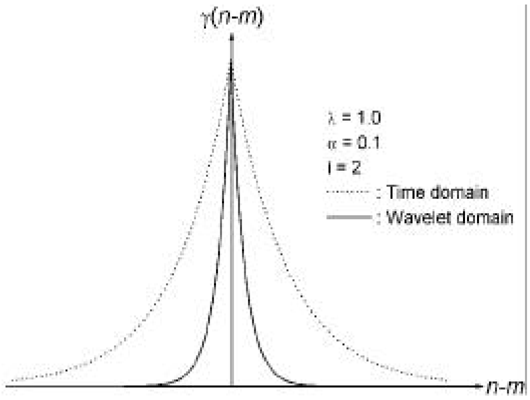



Fig. 3. Autocorrelation functions in time domain and wavelet domain

For real signal applications, we generate ultrasonic signal as fig. 4, it is collected from type 303 stainless steel by 6MHz unfocused ultrasonic sensor(A109, Panametrics Ltd.).

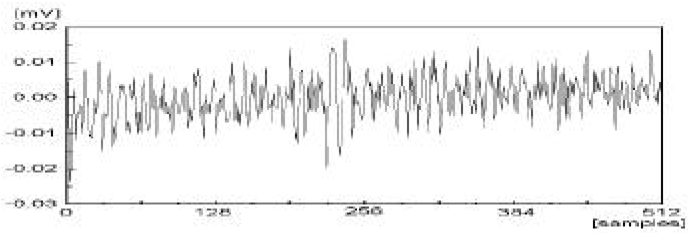



Fig. 4. Collected ultrasonic signal

Their autocorrelation in time and wavelet domain are fig. 5, fig. 6, respectively.

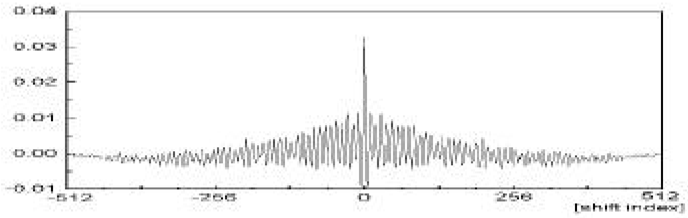



Fig. 5. Autocorrelation in time domain

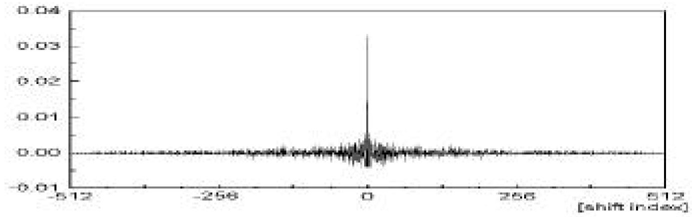



Fig. 6. Autocorrelation in wavelet domain

Figure 5 and figure 6 imply that the wavelet coefficients has much stronger autocorrelated-property than that of the time series for real signals.

## Ⅳ. Conclusions

In this paper, we analyze WSS signals and drives their properties in wavelet domain. In wavelet domain autocorrelation function of wavelet coefficients is bounded and exponentially depending on translation difference $(n-m)$. its result implies that an autocorrelation function in wavelet domain has much whitening comparing with that in time domain as good properties. These results will be applied to wavelet domain statistical modeling, system identification and various fields of statistical signal processing.

## REFERENCES

[1] I. Doroslovachi, H. Fan, "Wavelet Based Linear System Modeling and adaptive Filtering," Trans. on SP, Vol. 44, No. 5, pp. 1156-1167, May 1996.
[2] S. Narayan, A. M. Peterson and M. J. Narasima, " Transform Domain LMS Algorithm," IEEE Trans. on ASSP, Vol. 31, pp. 609-615, June 1993.
[3] O. Rioul, M. Vetterli, "Wavelet and Signal Processing," IEEE Signal Processing Magazine, Vol. 8, No. 4, pp. 14-38, October 1991.
[4] P. Vaidynathan, Z. Doganata, "The Role of Lossless Systems in Modern Digital Signal Processing," IEEE Trans. on Education, Special Issue on Circuits and Systems, Vol. 32, No. 3, pp. 181-197, August 1989.
[5] M. Basseville et al., "Modeling and Estimation of Multiresolution Stochastic Processes," IEEE Trans. on Inform. Theory, Vol. 38, pp. 766-784, March 1992.
[6] G.W. Wornell, "A Karhunnen-Loeve-like expansion for 1/f processes via wavelets," IEEE Trans. on SP, Vol. 40, pp.611-623, March 1992.
[7] Stamatis Cambanis, Elias Masry, "Wavelet Approximation of Deterministic and Random Signals:Convergence Properties and Rates," IEEE Trans. on Information Theory, Vol. 40, No. 4, July 1994.
[8] W. Dijkerman, R. Mazumdar, "Wavelet Representations of Stochastic Processes and Multiresolution Stochastic Models," Trans. on SP, Vol. 42, pp. 1640-1652, July 1994.

*Department of Electronics Eng., Chung-Woon Univ.
yslee@cwunet.ac.kr
Department of Electrical Eng., Univ. of Seoul