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A NOTE ON SOME HIGHER ORDER CUMULANTS IN k PARAMETER NATURAL EXPONENTIAL FAMILY

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ABSTRACT. We show the cumulants of a minimal sufficient statistics in k parameter natural exponential family by parameter function and partial parameter function. We find the cumulants have some merits of central moments and general cumulants both. The first three cumulants are the central moments themselves and the fourth cumulant has the form related with kurtosis.

1. INTRODUCTION

In this paper, we found some interesting results about the higher order cumulants in k parameter natural exponential family. We will follow the notation of Bar-Lev[1].

Let $\mathbf{T} = (T_1, \cdots, T_k; k \geq 2)$be a minimal sufficient statistic for an exponential model that constitutes a k parameter natural exponential family. Consider a partition of $\mathbf{T}$ into $(\mathbf{T}_1, \mathbf{T}_2)$ where $\mathbf{T}_1 = (T_1, \cdots, T_r)$, and $\mathbf{T}_2 = (T_{r+1}, \cdots, T_k; 1 \leq r \leq k-1)$. We present some higher order cumulants of $\mathbf{T}$, and conditional cumulants of $\mathbf{T}_1$, given $\mathbf{T}_2 = \mathbf{t}_2$.

Assume that the model from which the sample observations $X_1, \cdots, X_n$ are taken has the form of k parameter exponential family. Then the joint pdf of $\mathbf{X} = (X_1, \cdots, X_n)$ may be represented as follows

$$f_{\mathbf{X}}(\mathbf{x}, \boldsymbol{\theta}) = \Big\{\prod_{i=1}^{n} h(x_i) I_S(x_i)\Big\} \exp\Big\{\sum_{i=1}^{k} \theta_i \sum_{j=1}^{n} u_i(x_j) - nl(\boldsymbol{\theta})\Big\} \tag{1.1}$$

where S is the common support of the X_i's, $I_s(\cdot)$ is the indicator function of the set S, and $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_k)$ is the vector of natural parameters $(\in \Theta)$. Define $T_i = \sum_{j=1}^{n} u_i(x_j), i = 1, \cdots, k$, and let $(\mathbf{T}_1, \mathbf{T}_2)$ be a minimal sufficient statistic for $\boldsymbol{\theta}$. In addition, consider a partition of $\boldsymbol{\theta}$ into $(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$, where $\boldsymbol{\theta}_1 = (\theta_1, \cdots, \theta_r)$, and $\boldsymbol{\theta}_2 = (\theta_{r+1}, \cdots, \theta_k)$.

The derivation of moments or conditional moments from the pdf (1.1) is cumbersome and difficult to carry out. Most authors make the pdf into natural exponential family through reparametrization (see, Bickel and Docksum [2, p.70]; Bar-Lev [1]; Lehmann [4, p.57]). The pdf of k parameter natural exponential family has the form

$$f_{\mathbf{T}}(\mathbf{t} : \boldsymbol{\theta}) = g(\mathbf{t}) \exp\Big\{\boldsymbol{\theta} \cdot \mathbf{t} - nl(\boldsymbol{\theta})\Big\} I_{S_{\mathbf{T}}}(\mathbf{t}) \tag{1.2}$$

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for some measurable function g. In the case of (1.2) we get the moment generating function of $\mathbf{T}$ easily.

For $\boldsymbol{\theta} \in \Theta$, the moment generating function of $\mathbf{T}$ is

$$E\Big[\exp\Big\{\sum_{i=1}^{k} s_i T_i\Big\}\Big] = \exp\Big\{nl(\theta_1 + s_1, \cdots, \theta_k + s_k) - nl(\theta_1, \cdots, \theta_k)\Big\} \tag{1.3}$$

from which moments of $\mathbf{T}$ can be obtained. Cumulants of $\mathbf{T}$ can be derived by taking logarithms in eq.(1.3) that follows, differentiating with respect to the s_i's and substituting $s_i = 0, i = 1, \cdots, k$. But we can calculate the cumulants easily by differentiating $l(\boldsymbol{\theta})$ also. We refer to $l(\boldsymbol{\theta})$ as parameter function.

Bar-Lev[1] defines other parameter function from which conditional cumulants of $\mathbf{T}_1$ given $\mathbf{T}_2 = \mathbf{t}_2$ can be obtained like moment generating function. We refer to it as partial parameter function.

$$b(\boldsymbol{\theta}_1 : \mathbf{t}_2) \equiv f_{\mathbf{T}_2}(\mathbf{t}_2 : \boldsymbol{\theta}) \exp\Big\{nl(\boldsymbol{\theta}) - \boldsymbol{\theta}_2 \cdot \mathbf{t}_2\Big\} \tag{1.4}$$

And conditional cumulants are calculated by differentiating $\log b(\boldsymbol{\theta}_1, \mathbf{t}_2)$.

He shows that the first two cumulants equal to that of moments. The results are same with the cumulants in Kendall and Stuart[3, p.73]. We show the higher order cumulants derived from the parameter function are same with the cumulants also, and the usefulness of them.

2. Results

We can get $l(\boldsymbol{\theta})$ by integrating eq.(1.2) since $f_{\mathbf{T}}(\mathbf{t} : \boldsymbol{\theta})$ is a pdf. The parameter function is

$$l(\boldsymbol{\theta}) = \frac{1}{n} \log \int g(\mathbf{t}) \exp(\boldsymbol{\theta} \cdot \mathbf{t}) d\mathbf{t} \tag{2.1}$$

Now by differentiating (2.1) with respect to $\theta_i, i = 1, \cdots, k$, we get following results.

Results 1 : Cumulants of minimal sufficient statistic

$$\frac{n\partial l(\boldsymbol{\theta})}{\partial \theta_i} = E(T_i) = \mu_i$$

$$\frac{n\partial^2 l(\boldsymbol{\theta})}{\partial \theta_i^2} = E((T_i - \mu_i)^2) = \sigma_i^2$$

$$\frac{n\partial^3 l(\boldsymbol{\theta})}{\partial \theta_i^3} = E((T_i - \mu_i)^3)$$

$$\frac{n\partial^4 l(\boldsymbol{\theta})}{\partial \theta_i^4} = E((T_i - \mu_i)^4) - 3\sigma_i^4$$

The first cumulant is the first moment μ_i, and it is same with the Kendall and Stuart's first cumulant. And we can find the kurtosis of $\mathbf{T}_i, i = 1, \cdots, k$ from the results above.

$$k_i = \frac{n\partial^4 l(\boldsymbol{\theta})}{\partial \theta_i^4} \Big/ \Big(\frac{n\partial^2 l(\boldsymbol{\theta})}{\partial \theta_i^2}\Big)^2 = \frac{E((T_i - \mu_i)^4)}{\sigma_i^4} - 3 \tag{2.2}$$

It is interesting also. We usually define basic kurtosis as subtracting 3 from general kurtosis, because that of normal distribution is 3. If we calculate kurtosis from $l(\boldsymbol{\theta})$, it is lessened by 3 from general kurtosis, and resulted basic kurtosis as 0.

And now, we get conditional cumulants of $\mathbf{T}_1$, given $\mathbf{T}_2 = \mathbf{t}_2$, by differentiating $\log b(\boldsymbol{\theta}_1 : \mathbf{t}_2)$ with respect to $\theta_i, i = 1, \cdots, r$.

Results 2 : Conditional cumulants of minimal sufficient statistics

$$\frac{\partial \log b(\boldsymbol{\theta}_1 : \mathbf{t}_2)}{\partial \theta_i} = E(T_i|\mathbf{T}_2 = \mathbf{t}_2) = \mu_i^*$$

$$\frac{\partial^2 \log b(\boldsymbol{\theta}_1 : \mathbf{t}_2)}{\partial \theta_i^2} = E((T_i - \mu_i^*)^2|\mathbf{T}_2 = \mathbf{t}_2) = {\sigma_i}^{*2}$$

$$\frac{\partial^3 \log b(\boldsymbol{\theta}_1 : \mathbf{t}_2)}{\partial \theta_i^3} = E((T_i - \mu_i^*)^3|\mathbf{T}_2 = \mathbf{t}_2)$$

$$\frac{\partial^4 \log b(\boldsymbol{\theta}_1 : \mathbf{t}_2)}{\partial \theta_i^4} = E((T_i - \mu_i^*)^4|\mathbf{T}_2 = \mathbf{t}_2) - 3{\sigma_i}^{*4}$$

They present consistent results with Results 1 above.

3. Concluding Remarks

In this section, we show some types of moments with our results derived from parameter function $l(\boldsymbol{\theta})$ and partial parameter function $b(\boldsymbol{\theta}_1 : \mathbf{t}_2)$. We can find our results are more convenient than others in Table 1.

In general, it is some statistics(mean, variance, skewness, kurtosis etc.) to know when we calculate moments. In this view, no type of moment is always superior to other usual types of moment. But our results show that the cumulants from parameter function are always superior to other types of moments especially in fourth order moment for calculating kurtosis. And it is very easy to calculate the cumulants.

References

1. S. K. Bar-Lev, A Derivation of Conditional Cumulants in Exponential Models, The American Statistician, 48(2), 1994, 126-129.
2. P. J. Bickel and K. A. Doksum, Mathematical Statistics, Holden-Day Inc., 1977.
3. M. Kendall and A. Stuart, The Advanced Theory of Statistics Vol.1 Distribution Theory, Charles Griffin & Co., 1977.
4. E. L. Lehmann, Testing Statistical Hypothesis second ed., Wiley, 1986.

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Table 1. Comparison with other types of moments

types and order	first	second	third	fourth
$\mu'_k = E(T_i)^k$	μ_i	$\mu_i^2 + \sigma_i^2$	$E(T_i^3)$	$E(T_i^4)$
$\mu_k = E((T_i - \mu_i)^k)$	0	σ_i^2	$E((T_i - \mu_i)^3)$	$E((T_i - \mu_i)^4)$
$\eta_k = E(T_i(T_i - 1) \cdots (T_i - k - 1))$	μ_i	$\sigma_i^2 + \mu_i^2 - \mu_i$	$E(T_i(T_i - 1)(T_i - 2))$	$E(T_i(T_i - 1)(T_i - 2)(T_i - 3))$
$\frac{n\partial^k l(\theta)}{\partial \theta_i^k}$	μ_i	σ_i^2	$E((T_i - \mu_i)^3)$	$E((T_i - \mu_i)^4) - 3\sigma_i^4$
$\frac{\partial^k \log b(\theta_1 : \mathbf{t}_2)}{\partial \theta_i^k}$	μ_i^*	${\sigma_i}^{2*}$	$E((T_i - \mu_i)^3 \vert \mathbf{T}_2 = \mathbf{t}_2)$	$E((T_i - \mu_i)^4 \vert \mathbf{T}_2 = \mathbf{t}_2) - 3{\sigma_i}^{4*}$