J. KSIAM **Vol.3**, No.2, 81-97, 1999

TIME OPTIMAL CONTROL PROBLEM OF RETARDED SEMILINEAR SYSTEMS WITH UNBOUNDED OPERATORS IN HILBERT SPACES

Jong-Yeoul Park, Jin-Mun Jeong, Yong-Han Kang

Abstract. This paper deals with the time optimal control problem for the retarded semilinear system by using the construction of fundamental solution in case where the principal operators are unbounded operators.

1. Introduction

Let H and V be complex Hilbert spaces such that the embedding $V \subset H$ is continuous. In this paper we deal with the time optimal control problem governed by semilinear parabolic type equation in Hilbert space H as follows.

$$\text{(RSE)} \qquad \begin{cases} \dfrac{d}{dt}x(t) = A_0 x(t) + A_1 x(t-h) \\ \qquad\qquad + \displaystyle\int_{-h}^{0} a(s) A_2 x(t+s)ds + f(t, x(t)) + k(t), \\ x(0) = \phi^0, \quad x(s) = \phi^1(s) \quad -h \le s < 0. \end{cases}$$

Let A_0 be the operator associated with a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding inequality. Then A_0 generates an analytic semigroup $S(t)$ in both H and V^* and so the equation (RSE) may be considered as an equation in both H and V^*.

Let $(\phi^0, \phi^1) \in H \times L^2(0, T; V)$ and $x(T; \phi, f, u)$ be a solution of the system (RSE) associated with nonlinear term f and control u at time T.

We now define the fundamental solution $W(t)$ of (RSE) by

$$W(t) = \begin{cases} x(t; (\phi^0, 0), 0, 0), & t \ge 0 \\ 0 & t < 0. \end{cases}$$

1991 *Mathematics Subject Classification.* Primary 35B37; Secondary 93C20.
Key words and phrases. semilinear evolution equation, regularity, optimal conrol, compact imbedding.

According to the above definition $W(t)$ is a unique solotion of

$$W(t) = S(t) + \int_0^t S(t-s)\{A_1 W(s-h) + \int_{-h}^0 a(\tau)A_2 W(s+\tau)d\tau\}ds$$

for $t \geq 0$ (cf. Nakagiri [5]). Under the conditions $a(\cdot) \in L^2(-h, 0; \mathcal{R})$ and $A_i (i = 1, 2)$ are bounded linear operators on H into itself, S. Nakariri in [5] proved the standard optimal control proplems and the time optimal control problem for linear retarded system (RSE) in case $f \equiv 0$ in Banch space. If $A_i (i = 0, 1, 2) : D(A_0) \subset H \to H$ are unbounded operators, G. Di Blasio, K. Kunish and E. sinestrari in [2] obtained global existence and uniqueness of the strict solution for linear retarded system in Hilbert spaces. With the more general Lipschitz continuity of nonlinear operator f from $\mathcal{R} \times V$ to H, in [4] they eatablished the problem for existencs and uniqueness of solution of the given system. But we can not immediately obtain the time optimal control problem as in [5; section 8] without the condition for boundedness of the fundamental solution $W(t)$. Since the integral of $A_0 S(t-s)$ has a sigularity at $t = s$ we can not solve directly the integral equation of $W(t)$. In [6], H. Tanabe was investigated the fundamental solution $W(t)$ by constructing the resolvent operators for integrodifferential equations of Volterra type(see (3.14), (3.21) of [6]) with the condition that $a(\cdot)$ is real valued and Hölder continuous on $[-h, 0]$.

This paper deals with the time optimal control problem by using the construction of fundamental solution, which is the same results of [5], in case where the principal operators $A_i (i = 0, 1, 2)$ are unbounded operators.

2. Retarded semilinear equations

The inner product and norm in H are denoted by $(\cdot, \cdot)$ and $|\cdot|$. The notations $||\cdot||$ and $||\cdot||_*$ denote the norms of V and V^* as usual, respectively. Hence we may regard that

$$||u||_* \leq |u| \leq ||u||, \quad u \in V. \tag{2.1}$$

Let $a(\cdot, \cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\text{Re } a(u, u) \geq c_0 ||u||^2 - c_1 |u|^2, \quad c_0 > 0, \quad c_1 \geq 0. \tag{2.2}$$

Let A_0 be the operator associated with the sesquilinear form $-a(\cdot, \cdot)$:

$$(A_0 u, v) = -a(u, v), \quad u, \ v \in V.$$

It follows from (2.2) that for every $u \in V$

$$\text{Re}\,((c_1 - A_0)u, u) \geq c_0 ||u||^2.$$

Then A_0 is a bounded linear operator from V to V^*, and its realization in H which is the restriction of A_0 to

$$D(A_0) = \{u \in V; A_0 u \in H\}$$

is also denoted by A_0. Then A_0 generates an analytic semigroup in both H and V^*. Hence we may assume that there exists a constant C_0 such that

$$||u|| \leq C_0 ||u||_{D(A_0)}^{1/2} |u|^{1/2}, \tag{2.3}$$

for every $u \in D(A_0)$, where

$$||u||_{D(A_0)} = (|A_0 u|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A_0)$.

First, we introduce the following linear retarded functional differential equation:

$$\text{(RE)} \qquad \begin{cases} \dfrac{d}{dt}x(t) = A_0 x(t) + A_1 x(t-h) \\ \qquad\qquad + \displaystyle\int_{-h}^{0} a(s) A_2 x(t+s) ds + k(t), \\ x(0) = \phi^0, \quad x(s) = \phi^1(s) \quad -h \leq s < 0. \end{cases}$$

Here, the operators A_1 and A_2 are bounded linear from V to V^* such that their restrictions to $D(A_0)$ are bounded linear operators from $D(A_0)$ to H. The function $a(\cdot)$ is assumed to be a real valued and Hölder continous in the interval $[-h, 0]$.

Let $W(\cdot)$ be the fundamental solution of the linear equaton associated with (RE) which is the operator valued function satisfying

$$\begin{aligned} W(t) &= S(t) + \int_0^t S(t-s)\{A_1 W(s-h) \\ &\quad + \int_{-h}^{0} a(\tau) A_2 W(s+\tau) d\tau\} ds, \quad t > 0, \\ W(0) &= I, \qquad W(s) = 0, \quad -h \leq s < 0, \end{aligned} \tag{2.4}$$

where $S(\cdot)$ is the semigroup generated by A_0. Then

$$\begin{aligned} x(t) &= W(t)\phi^0 + \int_{-h}^{0} U_t(s)\phi^1(s) ds + \int_0^t W(t-s) k(s) ds, \\ U_t(s) &= W(t-s-h) A_1 + \int_{-h}^{s} W(t-s+\sigma) a(\sigma) A_2 d\sigma. \end{aligned} \tag{2.5}$$

Recalling the formulation of mild solutions, we know that the mild solution of (RE) is also represented by

$$x(t) = \begin{cases} S(t)\phi^0 + \displaystyle\int_0^t S(t-s)\{A_1 x(s-h) \\ + \displaystyle\int_{-h}^{0} a(\tau) A_2 x(s+\tau) d\tau + k(s)\} ds, \ (t > 0), \\ \phi(s), \quad -h \leq s < 0. \end{cases}$$

¿From Theorem 1 in [6] it follows the following results.

Proposition 2.1. *The fundamental solution $W(t)$ to (RE) exists uniquely. The functions $A_0W(t)$ and $dW(t)/dt$ are strongly continuous except at $t = nh,\ h = 0, 1, 2,\ ...,$ and the following inequalities hold:*
for $i = 0, 1, 2$ and $n = 0, 1, 2,\ ...$

$$|A_iW(t)| \leq C_n/(t - nh), \tag{2.6}$$

$$|dW(t)/dt| \leq C_n/(t - nh), \tag{2.7}$$

$$|A_iW(t)A_0^{-1}| \leq C_n \tag{2.8}$$

in $(nh, (n + 1)h)$,

$$|\int_t^{t'} A_iW(\tau)d\tau| \leq C_n \tag{2.9}$$

for $nh \leq t < t' \leq (n + 1)h$. Let ρ be the order of Hölder continuity of $a(\cdot)$. Then for $nh \leq t < t' \leq (n + 1)h$ and $0 < \kappa < \rho$

$$|W(t') - W(t)| \leq C_{n,\kappa}(t' - t)^\kappa(t - nh)^{-\kappa}, \tag{2.10}$$

$$|A_i(W(t') - W(t))| \leq C_{n,\kappa}(t' - t)^\kappa(t - nh)^{-\kappa-1}, \tag{2.11}$$

$$|A_i(W(t') - W(t))A_0^{-1}| \leq C_{n,\kappa}(t' - t)^\kappa(t - nh)^{-\kappa}, \tag{2.12}$$

where C_n and $C_{n,\kappa}$ are constants dependent on n and n, κ, respectively, but not on t and t'.

Considering as an equation in V^* we also obtain the same norm eatimates of (2.6)-(2.12) in the space V^*. By virtue of Theorem 3.3 of [2] we have the following result on the linear equation (RE).

Proposition 2.2. *1) Let $F = (D(A_0), H)_{\frac{1}{2},2}$ where $(D(A_0), H)_{1/2,2}$ denote the real interpolation space between $D(A_0)$ and H. For $(\phi^0, \phi^1) \in F \times L^2(-h, 0; D(A_0))$ and $k \in L^2(0, T; H)$, $T > 0$, there exists a unique solution x of (RE) belonging to*

$$L^2(-h, T; D(A_0)) \cap W^{1,2}(0, T; H) \subset C([0, T]; F)$$

and satisfying

$$\begin{aligned} ||x||_{L^2(-h,T;D(A_0))\cap W^{1,2}(0,T;H)} &\leq C_1'(||\phi^0||_F \\ &+ ||\phi^1||_{L^2(-h,0;D(A_0))} + ||k||_{L^2(0,T;H)}), \end{aligned} \tag{2.13}$$

where C_1' is a constant depending on T.
2) Let $(\phi^0, \phi^1) \in H \times L^2(-h, 0; V)$ and $k \in L^2(0, T; V^*)$, $T > 0$. Then there exists a unique solution x of (RE) belonging to

$$L^2(-h, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H)$$

and satisfying

$$\begin{aligned} ||x||_{L^2(-h,T;V)\cap W^{1,2}(0,T;V^*)} &\leq C_1'(|\phi^0| \\ &+ ||\phi^1||_{L^2(-h,0;V)} + ||k||_{L^2(0,T;V^*)}). \end{aligned} \tag{2.14}$$

In what follows we assume that

$$||W(t)|| \leq M, \quad t > 0$$

for the sake of simplicity.

Proposition 2.3. *Let $k \in L^2(0,T;H)$ and $x(t) = \int_0^t W(t-s)k(s)ds$. Then there exists a constant C_1' such that for $T > 0$*

$$||x||_{L^2(0,T;D(A_0))} \le C_1'||k||_{L^2(0,T;H)}, \tag{2.15}$$

$$||x||_{L^2(0,T;H)} \le MT||k||_{L^2(0,T;H)}, \tag{2.16}$$

and

$$||x||_{L^2(0,T;V)} \le (C_1'MT)^{\frac{1}{2}}||k||_{L^2(0,T;H)}. \tag{2.17}$$

Proof. The assertion (2.15) is immediately obtained from Proposition 2.2 for the equation (RE) with $(\phi^0, \phi^1) = (0,0)$. Since

$$\begin{aligned} ||x||^2_{L^2(0,T;H)} &= \int_0^T |\int_0^t W(t-s)k(s)ds|^2 dt \\ &\le M^2 \int_0^T (\int_0^t |k(s)|ds)^2 dt \\ &\le M^2 \int_0^T t \int_0^t |k(s)|^2 ds dt \\ &\le M^2 \frac{T^2}{2} \int_0^T |k(s)|^2 ds \end{aligned}$$

it follows that

$$||x||_{L^2(0,T;H)} \le MT||k||_{L^2(0,T;H)}.$$

¿From (2.3), (2.15), and (2.16) it holds that

$$||x||_{L^2(0,T;V)} \le (C_1'MT)^{\frac{1}{2}}||k||_{L^2(0,T;H)}. \quad \square$$

Let f be a nonlinear mapping from $\mathcal{R} \times V$ into H. We assume that for any x_1, $x_2 \in V$ there exists a constant $L > 0$ such that

$$|f(t,x_1) - f(t,x_2)| \le L||x_1 - x_2||, \tag{F1}$$

$$f(t,0) = 0. \tag{F2}$$

The following result on (RSE) is obtained from theorem 2.1 in [4].

Proposition 2.4. *Suppose that the assumptions (F1), (F2) are satisfied. Then for any $(\phi^0, \phi^1) \in H \times L^2(-h,0;V)$ and $k \in L^2(0,T;V^*)$, $T > 0$, the solution x of (RE) exists and is unique in $L^2(-h,T;V) \cap W^{1,2}(0,T;V^*)$, and there exists a constant C_2' depending on T such that*

$$\begin{aligned} ||x||_{L^2(-h,T;V)\cap W^{1,2}(0,T;V^*)} \le C_2'(1 + |\phi^0| \\ + ||\phi^1||_{L^2(-h,0;V)} + ||k||_{L^2(0,T;V^*)}). \end{aligned} \tag{2.18}$$

3. Lemmas for fundamental solutions

For the sake of simplicity we assume that $S(t)$ is uniformly bounded. Then

$$|S(t)| \le M_0 (t \ge 0),\ |A_0 S(t)| \le M_0/t (t > 0),\ |A_0^2 S(t)| \le K/t^2 (t > 0) \tag{3.1}$$

for some constant M_0(e.g., [6]). we also assume that $a(\cdot)$ is Hölder continuous of oder ρ:

$$|a(\cdot)| \le H_0, \quad |a(s) - a(\tau)| \le H_1 (s - \tau)^\rho \tag{3.2}$$

for some constants H_0, H_1.

Lemma 3.1. *For* $0 < s < t$ *and* $0 < \alpha < 1$

$$|S(t) - S(s)| \le \frac{M_0}{\alpha} (\frac{t - s}{s})^\alpha, \tag{3.3}$$

$$|A_0 S(t) - A_0 S(s)| \le M_0 (t - s)^\alpha s^{-\alpha - 1}. \tag{3.4}$$

Proof. ¿From (3.1) for $0 < s < t$

$$|S(t) - S(s)| = |\int_s^t A_0 S(\tau) d\tau| \le M_0 \log \frac{t}{s}. \tag{3.5}$$

It is easily seen that for any $t > 0$ and $0 < \alpha < 1$

$$\log(1 + t) \le t^\alpha / \alpha. \tag{3.6}$$

Combining (3.6) with (3.5) we get (3.3). For $0 < s < t$

$$|A_0 S(t) - A_0 S(s)| = |\int_s^t A_0^2 S(\tau) d\tau| \le M_0 (t - s)/ts. \tag{3.7}$$

Noting that $(t - s)/s \le ((t - s)/s)^\alpha$ for $0 < \alpha < 1$, we obtain (3.4) from (3.7). □

According to Tanabe [6] we set

$$V(t) = \begin{cases} A_0(W(t) - S(t)), & t \in (0, h] \\ A_0(W(t) - \int_{nh}^t S(t - s) A_1 W(s - h) ds), & \end{cases} \tag{3.8}$$

where $t \in (nh, (n+1)h] (n = 1, 2, \ \dots\ )$ in the second line of the right term of (3.8). For $0 < t \le h$

$$W(t) = S(t) + A_0^{-1} V(t)$$

and from (2.4) we have

$$W(t) = S(t) + \int_0^t \int_\tau^t S(t - s) a(\tau - s) ds A_2 W(\tau) d\tau.$$

Hence,

$$V(t)=V_0(t)+\int_0^t A_0\int_\tau^t S(t-s)a(\tau-s)dsA_2A_0^{-1}V(\tau)d\tau$$

where

$$V_0(t)=\int_0^t A_0\int_\tau^t S(t-s)a(\tau-s)dsA_2S(\tau)d\tau.$$

For $nh\le t\le(n+1)h(n=0,1,2,\ \dots\ )$ the fundamental solution $W(t)$ is represented by

$$\begin{aligned}W(t)=&S(t)+\int_{nh}^t S(t-s)A_1W(s-h)ds\\&+\int_0^{t-h}\int_\tau^{\tau+h}S(t-s)a(\tau-s)dsA_2W(\tau)d\tau\\&+\int_{t-h}^{nh}\int_\tau^t S(t-s)a(\tau-s)dsA_2W(\tau)d\tau\\&+\int_{nh}^t\int_\tau^t S(t-s)a(\tau-s)dsA_2W(\tau)d\tau.\end{aligned}$$

The integral equation to be satisfied by (3.8) is

$$V(t)=V_0(t)+\int_{nh}^t A_0\int_\tau^t S(t-s)a(\tau-s)dsA_2A_0^{-1}V(\tau)d\tau$$

where

$$\begin{aligned}V_0(t)=&A_0S(t)+A_0\int_h^{nh}S(t-s)A_1W(s-h)ds\\&+\int_0^{t-h}A_0\int_\tau^{\tau+h}S(t-s)a(\tau-s)dsA_2W(\tau)d\tau\\&+\int_{t-h}^{nh}A_0\int_0^t S(t-s)a(\tau-s)dsA_2W(\tau)d\tau\\&+\int_{nh}^t A_0\int_\tau^t S(t-s)a(\tau-s)dsA_2\int_{nh}^\tau S(\tau-\sigma)A_1W(\sigma-h)d\sigma d\tau.\end{aligned}$$

Thus, the integral equation (3.8) can be solved by succesive approximation and $V(t)$ is uniformly bounded in $[nh,(n+1)h]$(e.g. (3.16) and the preceding part of (3.40) in [6]).It is not difficult to show that for $n>1$

$$V(nh+0)\neq V(nh-0),\quad \text{and}\quad W(nh+0)=W(nh-0).$$

Moreover, we obtain the following result.

Lemma 3.2. *There exists a constant $C'_n > 0$ such that*

$$|\int_{nh}^{t} a(\tau - s)A_i W(\tau)d\tau| \leq C'_n, \quad i = 1, 2, \tag{3.9}$$

for $n = 0, 1, 2, \ ..., \ t \in [nh, (n+1)h]$ and $t \leq s \leq t + h$.

Proof. For $t \in [0, h]$(i.e.,$n = 0$), from (3.8) it follows

$$\begin{aligned}\int_0^t a(\tau - s)A_i W(\tau)d\tau &= \int_0^t a(\tau - s)ds A_i A_0^{-1}(A_0 S(\tau) + V(\tau))d\tau \\ &= \int_0^t (a(\tau - s) - a(s))A_i A_0^{-1} A_0 S(\tau)d\tau + a(s)A_i A_0^{-1}(S(t) - I) \\ &+ \int_0^t a(\tau - s)A_i A_0^{-1} V(\tau)d\tau.\end{aligned}$$

Noting that

$$|\int_0^t (a(\tau - s) - a(s))A_i A_0^{-1} A_0 S(\tau)d\tau| \leq M_0 H_1 |A_i A_0^{-1}| \int_0^t \tau^{\rho - 1} d\tau,$$

we have

$$\begin{aligned}|\int_0^t a(\tau - s)A_i W(\tau)d\tau| \leq &|A_i A_0^{-1}|\{h^\rho M_0 H_1 + H_0(M + 1) \\ &+ hH_0(\sup_{0 \leq t \leq h} |V(t)|)\}.\end{aligned}$$

Thus the assertion (3.9) holds in $[0, h]$. For $t \in [nh, (n+1)h], \ n \geq 1$,

$$\begin{aligned}\int_{nh}^t a(\tau - s)A_i W(\tau)d\tau &= \int_{nh}^t a(\tau - s)A_i A_0^{-1} V(\tau)d\tau \\ &+ \int_{nh}^t a(\tau - s)A_i \int_{nh}^{\tau} S(\tau - \xi)A_1 W(\xi - h)d\xi d\tau.\end{aligned} \tag{3.10}$$

The first term of the right of (3.10) is estimated as

$$|\int_{nh}^t a(\tau - s)A_i A_0^{-1} V(\tau)d\tau| \leq hH_0 |A_i A_0^{-1}|(\sup_{nh \leq t \leq (n+1)h} |V(t)|)\}.$$

Let $\sigma = (\tau + nh)/2$ for $nh < \tau < (n+1)h$. Then

(3.11)
$$
\begin{aligned}
&|A_0 \int_{nh}^{\tau} S(\tau-\xi)A_1 W(\xi-h)d\xi| \\
&\quad \le |\int_{\sigma}^{\tau} A_0 S(\tau-\xi)(A_1 W(\xi-h) - A_1(W(\tau-h))d\xi \\
&\quad + (S((\tau-nh)/2) - I)A_1 W(\tau-h) \\
&\quad + \int_{nh}^{\sigma} (A_0 S(\tau-\xi) - A_0 S(\tau-nh))A_1 W(\xi-h)d\xi \\
&\quad + A_0 S(\tau-nh)\int_{nh}^{\sigma} A_1 W(\xi-h)d\xi| \\
&\quad \le \int_{\sigma}^{\tau} \frac{M_0}{\tau-\sigma} C_{n-1,\kappa}(\tau-\xi)^{\kappa}(\xi-nh)^{-\kappa-1}d\xi + (M_0+1)\frac{C_{n-1}}{\tau-nh} \\
&\quad + \int_{nh}^{\sigma} \frac{M_0(\xi-nh)}{(\tau-\xi)(\tau-nh)}\frac{C_{n-1}}{\xi-nh}d\xi + \frac{M_0 C_{n-1}}{\tau-nh} \\
&\le M_0 C_{n-1,\kappa}\int_{nh}^{\tau}(\tau-\xi)^{\kappa-1}(\xi-nh)^{-\kappa}d\xi \frac{2}{\tau-nh} \\
&+ \frac{(2M_0+1)C_{n-1}}{\tau-nh} + \frac{M_0 C_{n-1}}{\tau-nh}\log 2 \\
&= \{2M_0 C_{n-1,\kappa}B(\kappa, 1-\kappa) + (2M_0 + 1 + M_0\log 2)C_{n-1}\}/(\tau-nh) \\
&\equiv C'_{n,\kappa}/(\tau-nh)
\end{aligned}
$$

where $B(\cdot,\cdot)$ is the Beta function. Noting that

$$
\frac{d}{d\tau}\int_{nh}^{\tau} S(\tau-\xi)A_1 W(\xi-h)d\xi = A_1 W(\tau-h) + A_0 \int_{nh}^{\tau} S(\tau-\xi)A_1 W(\xi-h)d\xi,
$$

and integrating this equality on $[nh, t]$

$$
\begin{aligned}
&\int_{nh}^{t} A_0 \int_{nh}^{\tau} S(\tau-\xi)A_1 W(\xi-h)d\xi d\tau \\
&\qquad = \int_{nh}^{t} S(t-\xi)A_1 W(\xi-h)d\xi - \int_{nh}^{t} A_1 W(\tau-h)d\tau.
\end{aligned} \tag{3.12}
$$

By Lemma 3.1 and the induction hypothesis, the first term of the right of (3.12) is estimated as

$$
\begin{aligned}
&|\int_{nh}^{\tau} S(\tau-\xi)A_1 W(\xi-h)d\xi| \\
&\qquad = |\int_{nh}^{\tau} (S(\tau-\xi) - S(\tau-nh))A_1 W(\xi-h)d\xi \\
&\qquad + S(\tau-nh)\int_{nh}^{\tau} A_1 W(\xi-h)d\xi| \\
&\qquad \le \int_{nh}^{\tau} M_0 \log\frac{\tau-nh}{\tau-\xi}\frac{C_{n-1}}{\xi-nh}d\xi + M_0 C_{n-1} \\
&\qquad \le M_0 C_{n-1} c_0 + M_0 C_{n-1}
\end{aligned} \tag{3.13}
$$

where

$$c_0 = \int_0^1 \log \frac{1}{1-\sigma} \frac{d\sigma}{\sigma}.$$

Thus, combining the above inequarity with (2.9) we get

$$|\int_{nh}^t A_0 \int_{nh}^\tau S(\tau - s) A_i W(s-h) ds d\tau| \leq (M_0 c_0 + M_0 + 1) C_{n-1}. \tag{3.14}$$

Therefore, from (3.11), (3.14) the second term of the right of (3.10) is estimated as

$$\begin{aligned}
&|\int_{nh}^t a(\tau - s) A_i \int_{nh}^\tau S(\tau-\xi) A_1 W(\xi - h) d\xi d\tau| \\
&\quad = |\int_{nh}^t (a(\tau - s) - a(s-nh)) A_i \int_{nh}^\tau S(\tau-\xi) A_1 W(\xi - h) d\xi d\tau \\
&\quad + a(s-nh) \int_{nh}^t A_i \int_{nh}^\tau S(\tau-\xi) A_1 W(\xi - h) d\xi d\tau| \\
&\quad \leq \int_{nh}^t H_1 (\tau - nh)^\rho |A_i A_0^{-1}| C'_{n.\kappa} (\tau - nh)^{-1} d\tau \\
&\quad + |a(s-nh)||A_i A_0^{-1}|(M_0 c_0 + M_0 + 1) C_{n-1} \\
&\quad \leq H_1 C'_{n.\kappa} |A_i A_0^{-1}| (t-nh)^\rho + H_0 |A_i A_0^{-1}| (M c_0 + M + 1) C_{n-1}.
\end{aligned}$$

Hence, we get the assertion (3.9). □

We define the operator $K_1(t', t) : H \to H($ or $V^* \to V^*)$ by

$$K_1(t', t) = \int_t^{t'} S(t' - s) A_1 W(s-h) ds, \tag{3.15}$$

for $nh \leq t < t' < (n+1)h$. In terms of (3.13) $K_1(t', t)$ is uniformly bounded in $(nh, (n+1)h]$. And we remark that $K_1(t', t)$ converges to 0 as $t' \to t$ at any element of $D(A_0)$ in view of (2.8). We introduce another operator $K_2(t', t) : H \to H($ or $V^* \to V^*)$ by

$$K_2(t', t) = \int_t^{t'} S(t' - s) \int_{-h}^0 a(\tau) A_2 W(s+\tau) d\tau ds, \tag{3.16}$$

for $nh \leq t < t' < (n+1)h$.

Lemma 3.3. *Let $nh \leq t < t' < (n+1)h$. Then there exists a constant C'_n such that and*

$$|K_2(t', t)| \leq 3 M_0 C'_n (t' - t). \tag{3.17}$$

Proof. In $[0,h]$, we transform $K_2(t',t)$ by suitable change of variables and Fubini's theorem as

$$\begin{aligned}
K_2(t',t) &= \int_t^{t'} S(t'-s)\int_0^s a(\tau-s)A_2W(\tau)d\tau ds \\
&= \int_0^t \int_t^{t'} S(t'-s)a(\tau-s)A_2W(\tau)dsd\tau \\
&\quad + \int_t^{t'} \int_\tau^{t'} S(t'-s)a(\tau-s)A_2W(\tau)dsd\tau \\
&= \int_t^{t'} S(t'-s)\int_0^t a(\tau-s)A_2W(\tau)d\tau ds \\
&\quad + \int_t^{t'} S(t'-s)\int_t^s a(\tau-s)A_2W(\tau)d\tau ds.
\end{aligned}$$

Thus from Lemma 3.2 we have

$$|K_2(t',t)| \le 2M_0C_n'(t'-t).$$

In $[nh,(n+1)h)$, by the similar way mentioned above we get

$$\begin{aligned}
K_2(t',t) &= \int_t^{t'} S(t'-s)\int_{-h}^0 a(\tau)A_2W(\tau+s)d\tau ds \\
&= \int_t^{t'} S(t'-s)\int_{s-h}^s a(\tau-s)A_2W(\tau)d\tau ds \\
&= \int_{t-h}^{t'-h} \int_t^{\tau+h} S(t'-s)a(\tau-s)A_2W(\tau)dsd\tau \\
&\quad + \int_{t'-h}^{t} \int_t^{t'} S(t'-s)a(\tau-s)A_2W(\tau)dsd\tau \\
&\quad + \int_t^{t'} \int_\tau^{t'} S(t'-s)a(\tau-s)A_2W(\tau)dsd\tau \\
&= \int_t^{t'} S(t'-s)\int_{s-h}^{t'-h} a(\tau-s)A_2W(\tau)d\tau ds \\
&\quad + \int_t^{t'} S(t'-s)\int_{t'-h}^{t} a(\tau-s)A_2W(\tau)d\tau ds \\
&\quad + \int_t^{t'} S(t'-s)\int_t^s a(\tau-s)A_2W(\tau)d\tau ds.
\end{aligned}$$

Therefore, by Lemma 3.2 it holds (3.17) $\square$

4. Time optimal control

Let Y be a real Banach space. In what follows the admissible set U_{ad} be weakly compact subset in $L^2(0,T;Y)$. Consider the following hereditary controlled system:

$$
\text{(RSC)} \qquad \begin{cases} \dfrac{d}{dt}x(t) = A_0x(t) + A_1x(t-h) \\ \qquad\qquad + \displaystyle\int_{-h}^{0} a(s)A_2x(t+s)ds + f(t,x(t)) + Bu(t), \\ x(0) = \phi^0, \quad x(s) = \phi^1(s) \quad -h \le s < 0, \\ u \in U_{ad}. \end{cases}
$$

Here the controller B is a bounded linear operator from Y to H. We denote the solution $x(t)$ in (RSC) by $x_u(t)$ to express the dependence on $u \in U_{ad}$. That is, x_u is trajectory corresponding to the controll u. Suppose the target set W is weakly compact in H and define

$$U_0 = \{u \in U_{ad} : x_u(t) \in W \text{ for some } \ t \in [0,T]\}$$

for $T > 0$ and suppose that $U_0 \neq \emptyset$. The optimal time is defined by low limit t_0 of t such that $x_u(t) \in W$ for some admissible control u. For each $u \in U_0$ we can define the first time $\tilde{t}(u)$ such that $x_u(\tilde{t}) \in W$. The our problem is to find a control $\bar{u} \in U_0$ such that

$$\tilde{t}(\bar{u}) \le \tilde{t}(u) \quad \text{for all } u \in U_0$$

subject to the constraint (RSC).
Since $x_u \in C([0,T];H)$, the transition time $\tilde{t}(u)$ is well defined for each $u \in U_{ad}$.

Theorem 4.1. *1) Let $F = (D(A_0),H)_{1/2,2})$. If $(\phi^0,\phi^1) \in F \times L^2(-h,0;\, D(A_0))$ and $k \in L^2(0,T;H)$, then the solution x of the equation (RSE) belonging to $L^2(-h,T;D(A_0))$ $\cap W^{1,2}(0,T;H)$, and the mapping $F \times L^2(-h,0;D(A_0\ )) \times L^2(0,T;H) \ni (\phi^0,\phi^1,k) \mapsto x \in L^2(-h,T;D(A_0)) \cap\, W^{1,2}(0,T;H)$ is continuous.*
2) If $(\phi^0,\phi^1) \in H \times L^2(-h,0;V)$ and $k \in L^2(0,T;V^*)$, then the solution x of the equation (RSE) belonging to $L^2(-h,T;V)) \cap W^{1,2}(0,T;V^*)$, and the mapping $H \times L^2(-h,0;V\ \ )) \times L^2(0,T;V^*) \ni (\phi^0,\phi^1,k) \mapsto x \in L^2(-h,T;V) \cap W^{1,2}(0,T;V^*)$ is continuous.

Proof. 1) We know that x belongs to $L^2(0,T;D(A_0)) \cap\, W^{1,2}(0,T;H)$ from Proposition 2.2. Let $(\phi_i^0,\phi_i^1,k_i) \in F \times L^2(-h,0;D(A_0)) \times L^2(0,T;H)$, and x_i be the solution of (RSE) with (ϕ_i^0,ϕ_i^1,k_i) in place of (ϕ^0,ϕ^1,k) for $i = 1,\ 2$. Then in view of Proposition 2.2 we have

(4.1)
$$
\begin{aligned}
||x_1 - x_2||_{L^2(-h,T;D(A_0)) \cap W^{1,2}(0,T;H)} &\le C_1'\{||\phi_1^0 - \phi_2^0||_F \\
&\quad + ||\phi_1^1 - \phi_2^1||_{L^2(-h,0:D(A_0))} + ||f(\cdot,x_1) - f(\cdot,x_2)||_{L^2(0,T;H)} \\
&\quad + ||k_1 - k_2||_{L^2(0,T;H)}\} \\
&\le C_1'\{||\phi_1^0 - \phi_2^0||_F + ||\phi_1^1 - \phi_2^1||_{L^2(-h,0:D(A_0))} + ||k_1 - k_2||_{L^2(0,T;H)} \\
&\quad + L||x_1 - x_2||_{L^2(0,T:V)}\}.
\end{aligned}
$$

Since

$$x_1(t) - x_2(t) = \phi_1^0 - \phi_2^0 + \int_0^t (\dot{x}_1(s) - \dot{x}_2(s))ds,$$

we get

$$||x_1 - x_2||_{L^2(0,T;H)} \leq \sqrt{T}|\phi_0^1 - \phi_2^0| + \frac{T}{\sqrt{2}}||x_1 - x_2||_{W^{1,2}(0,T;H)}.$$

Hence arguing as in (2.3) we get

$$\begin{aligned} (4.2) \qquad ||x_1 - x_2||_{L^2(0,T;V)} &\leq C_0||x_1 - x_2||^{1/2}_{L^2(0,T;D(A_0))}||x_1 - x_2||^{1/2}_{L^2(0,T;H)} \\ &\leq C_0||x_1 - x_2||^{1/2}_{L^2(0,T;D(A_0))} \\ &\quad \times \{T^{1/4}|\phi_1^0 - \phi_2^0|^{1/2} + (\frac{T}{\sqrt{2}})^{1/2}||x_1 - x_2||^{1/2}_{W^{1,2}(0,T;H)}\} \\ &\leq C_0 T^{1/4}|\phi_1^0 - \phi_2^0|^{1/2}||x_1 - x_2||^{1/2}_{L^2(0,T;D(A_0))} \\ &\quad + C_0(\frac{T}{\sqrt{2}})^{1/2}||x_1 - x_2||_{L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)} \\ &\leq 2^{-7/4}C_0|\phi_1^0 - \phi_2^0| \\ &\quad + 2C_0(\frac{T}{\sqrt{2}})^{1/2}||x_1 - x_2||_{L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)}. \end{aligned}$$

Combining (4.1) and (4.2) we obtain

$$\begin{aligned} (4.3) \qquad ||x_1 - x_2||_{L^2(-h,T;D(A_0))\cap W^{1,2}(0,T;H)} &\leq C_1'\{||\phi_1^0 - \phi_2^0||_F \\ &\quad + ||\phi_1^1 - \phi_2^1||_{L^2(-h,0:D(A_0))} + ||k_1 - k_2||_{L^2(0,T;H)} \\ &\quad + 2^{-7/4}C_0 L|\phi_1^0 - \phi_2^0| \\ &\quad + 2C_0(\frac{T}{\sqrt{2}})^{1/2}L||x_1 - x_2||_{L^2(0,T;D(A_0))\cap W^{1,2}(0,T;H)}\}. \end{aligned}$$

Suppose that $(\phi_n^0, \phi_n^1, k_n) \to (\phi^0, \phi^1, k)$ in $F \times L^2(-h, 0; D(A_0)) \times L^2(0, T; H)$, and let x_n and x be the solutions (RSE) with $(\phi_n^0, \phi_n^1, k_n)$ and (ϕ^0, ϕ^1, k) respectively. Let $0 < T_1 \leq T$ be such that

$$2C_0C_1'(T_1/\sqrt{2})^{1/2}L < 1.$$

Then by virtue of (4.3) with T replaced by T_1

we see that $x_n \to x$ in $L^2(-h, T_1; D(A_0)) \cap W^{1,2}(0, T_1; H)$. This implies that $(x_n(T_1), (x_n)_{T_1})$ $\mapsto (x(T_1), x_{T_1})$ in $F \times L^2(-h, 0; D(A_0))$. Hence the same argument shows that $x_n \to x$ in

$$L^2(T_1, \min\{2T_1, T\}; D(A_0)) \cap W^{1,2}(T_1, \min\{2T_1, T\}; H).$$

Repeating this process we conclude that $x_n \to x$ in $L^2(-h, T; D(A_0)) \cap W^{1,2}(0, T; H)$. ∎

2) From proposition 2.2 or 2.4 we have

$$\begin{aligned}||x_1 - x_2||_{L^2(-h,T;V)\cap W^{1,2}(0,T;V^*)} &\leq C_1'\{|\phi_1^0 - \phi_2^0| \\ &+ ||\phi_1^1 - \phi_2^1||_{L^2(-h,0:V)} + ||f(\cdot,x_1) - f(\cdot,x_2)||_{L^2(0,T;V^*)} \\ &+ ||k_1 - k_2||_{L^2(0,T;V^*)}\} \\ &\leq C_1'\{|\phi_1^0 - \phi_2^0| + ||\phi_1^1 - \phi_2^1||_{L^2(-h,0:V)} + ||k_1 - k_2||_{L^2(0,T;V^*)} \\ &+ L||x_1 - x_2||_{L^2(0,T:V)}\}.\end{aligned}$$

Hence, in vrtue of (4.2) and since the embedding $L^2(-h,T;D(A_0))\cap W^{1,2}(0,T;H) \subset L^2(-h,T;V)\cap W^{1,2}(0,T;V^*)$ is continuous, by the similar way of 1) we can obtain the result of 2) □

Theorem 4.2. *Assume that $U_0 \neq \emptyset$. Then there exists a time optimal control.*

Proof. Let $t_n \to t_0 + 0$, u_n be an admissible control and suppose that the trajectory x_n corresponding to u_n belongs to W. Let $\mathcal{F}$ and $\mathcal{B}$ be the Nemitsky operators corresponding to the maps f and B,which are defined by

$$(\mathcal{F}u)(\cdot) = f(\cdot, x_u), \quad \text{and} \quad (\mathcal{B}u)(\cdot) = Bu(\cdot),$$

respectively. Then

$$\begin{aligned}x_n(t_n) =& x(t_n;\phi,0) + \int_0^{t_0} W(t_n - s)((\mathcal{F}+\mathcal{B})u_n)(s)ds, \\ &+ \int_{t_0}^{t_n} W(t_n - s)((\mathcal{F}+\mathcal{B})u)(s)ds\end{aligned} \tag{4.4}$$

where

$$x(t_n;\phi,0) = W(t)\phi^0 + \int_{-h}^0 U_t(s)\phi^1(s)ds.$$

¿From Proposition 2.4 it follows that

$$x(t_n,\phi,0) \to x(t_0;\phi,0) \quad \text{strongly in } H. \tag{4.5}$$

The third term in (4.4) tends to zero as $t_n \to t_0 + 0$ from the fact that

(4.6)

$$\begin{aligned}&|\int_{t_0}^{t_n} W(t_n - s)((\mathcal{F}+\mathcal{B})u)(s)ds| \\ &\leq (\sup_{t\in[0,T]} ||W(t)||)\{LC_2'(|\phi^0| + ||\phi^1||_{L^2(0,T;V)} + ||u||_{L^2(0,T;Y)}) + |f(0)| \\ &+ ||B||||u||_{L^2(0,T;Y)}\}(t_n - t_0)^{1/2}.\end{aligned}$$

By the definition of fundamental solution $W(t)$ it holds

$$\begin{aligned} W(t+\epsilon)-S(\epsilon)W(t) &= S(t+\epsilon)+\int_0^{t+\epsilon} S(t+\epsilon-s)\{A_1W(s-h) \\ &\quad+\int_{-h}^0 a(\tau)A_2W(s+\tau)d\tau\}ds \\ &\quad-S(\epsilon)\{S(t)+\int_0^t S(t-s)\{A_1W(s-h) \\ &\quad+\int_{-h}^0 a(\tau)A_2W(s+\tau)d\tau\}ds \\ &=\int_t^{t+\epsilon} S(t+\epsilon-s)\{A_1W(s-h) \\ &\quad+\int_{-h}^0 a(\tau)A_2W(s+\tau)d\tau\}ds \\ &=K_1(t+\epsilon,t)+K_2(t+\epsilon,t). \end{aligned}$$

Hence, since

$$W(t_n-s)=S(t_n-t_0)W(t_0-s)+K_1(t_n-s,t_0-s)+K_2(t_n-s,t_0-s)$$

the second term of (4.4) is represented as

$$\begin{aligned} &\int_0^{t_0} S(t_n-t_0)W(t_0-s)((\mathcal{F}+\mathcal{B})u_n)(s)ds \\ &+\int_0^{t_0}(K_1(t_n-s,t_0-s)+K_2(t_n-s,t_0-s))((\mathcal{F}+\mathcal{B})u_n)(s)ds. \end{aligned} \tag{4.7}$$

The second term of the (4.7) tends to zero as $\epsilon\to 0$ in terms of Lemma 3.3.

We denote $x_n(t_n)$ by w_n. Since W and U_{ad} are weakly compact, there exist an $u_0\in U_0$, $w_0\in W$ such that we may assume that $w-\lim u_n=u$ in U_{ad} and $w-\lim w_n=w_0$ in $L^2\cap W^{1,2}$.

Let $p\in H$. Then $S^*(t_n-t_0)p\to p$ strongly in H and by (F1) and Theorem 4.1,

$$W(t_0-\cdot)((\mathcal{F}+\mathcal{B})u_n)(\cdot)\to W(t_0-\cdot)((\mathcal{F}+\mathcal{B})u_0)(\cdot) \tag{4.8}$$

weakly $L^2(0,T;V)$. Hence from (4.5)-(4.8) it follows that

$$(w_0,p)=(x(t_0;\phi,0),p)+\int_0^{t_0}(W(t_0-s)((\mathcal{F}+\mathcal{B})u_0)(s),p)ds$$

by tending $n\to\infty$. Since p is arbitrary, we have

$$w_0=x(t_0;\phi,0)+\int_0^{t_0} W(t_0-s)((\mathcal{F}+\mathcal{B})u_0)(s)ds\in W$$

and hence w_0 is the trajectory correspondiny to u_0, i.e., $u_0 \in U_0$. $\square$

Now we consider the case where the target set W is singleton.

Consider that $W = w_0$ such that $\phi^0 \neq w_0$ and $\phi^1(s) \neq w_0$ for some $s \in [-h, 0)$. Then we can choose a decreasing sequence $\{W_n\}$ of weakly compact sets with nonempty interior such that

$$w_0 \in \bigcap_{n=1}^{\infty} W_n, \text{ and } \operatorname{dist}(w_0, W) = \sup_{x \in W_n} |x - w_0| \to 0 (n \to \infty). \tag{4.9}$$

Define

$$U_0^n = \{u \in U_{ad} : x_u(t) \in W_n \text{ for some } t \in [0, T]\}.$$

Then, we may assume that u_n is the time optimal control with the optimal time t_n to the target set W_n, $n = 1, 2, \ldots$.

Theorem 4.3. *Let $\{W_n\}$ be a sequence of closed convex in X satisfying the condition (4.9) and $U_0^n \neq \emptyset$. Then there exists a time optimal control u_0 with the optimal time $t_0 = \sup_{n \geq 1}\{t_n\}$ to the point target set $\{w_0\}$ which is given by the weak limit of some subsequence of $\{u_n\}$ in $L^2(0, t_0; Y)$.*

Proof. Since (4.9) is satisfied and U_{ad} is weakly compact, there exists $w_n = x_n(t_n) \in W_n \to w_0$ strongly in H. Since U_{ad} is weakly compact, there exists $u_0 \in U_{ad}$ such that $u_n \to u_0$ weakly in $L^2(0, t_0; Y)$. Thus, from the similar argument used in the proof of Theorem 4.2 we can easily prove that u_0 is the time optimal control and t_0 is the optimal time to the target $\{w_0\}$. $\square$

Remark 1. Let x_u be the solution of (RSC) corresponding to u. Then the mapping $u \mapsto x_u$ is compact from $L^2(0, T; Y)$ to $L^2(0, T; H)$. We define the soluton mapping S from $L^2(0, T; Y)$ to $L^2(0, T; H)$ by

$$(Su)(t) = x_u(t), \quad u \in L^2(0, T; Y).$$

In virtue of Proposition 2.4

$$||Su||_{L^2(0,T;V) \cap W^{1,2}(0,T;V^*)} = ||x_u|| \leq C_2'\{|x_0| + || + Bu||_{L^2(0,T;H)}.$$

Hence if u is bounded in $L^2(0, T; Y)$, then so is x_u in $L^2(0, T; V) \cap W^{1,2}(0, T; V^*)$. Since V is compactly imbedded in H by assumption, the imbedding $L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset L^2(0, T; H))$ is also compact in view of Theorem 2 of J. P. Aubin [1]. Hence, the mapping $u \mapsto Su = x_u$ is compact from $L^2(0, T; Y)$ to $L^2(0, T; H)$.

Since $\{x_n\}$ is bounded in $L^2 \cap W^{1,2}$ and $L^2 \cap W^{1,2} \subset L^2(0, T; H)$ compacvtively it holds $x_n \to x$ strongly in $L^2(0, T; H)$. Since $x_n \to x$ weakly in $L^2 \cap W^{1,2}$ we have $x_n \to x$ strongly in $L^2(0, T; H)$. ¿From (f1) and Lemma 3.1 we see that $\mathcal{F}$ is a compact operator from $L^2(0, T; Y)$ to $L^2(0, T; H)$ and hence, it holds $\mathcal{F}u_n \to \mathcal{F}u$ strongly in $L^2(0, T; V^*)$. Therefore $(\mathcal{F}u_n, x^*) = (\mathcal{F}u_0, x^*)$.

References

1. J. P. Aubin, *Un thèoréme de composité*, C. R. Acad. Sci. **256** (1963), 5042–5044.
2. G. Di Blasio, K. Kunisch and E. Sinestrari, L^2*−regularity for parabolic partial integrodifferential equations with delay in the highest-order derivatives*, J. Math. Anal. Appl. **102** (1984), 38–57.
3. J. M. Jeong, *Retarded functional differential equations with* L^1*-valued controller*, Funkcialaj Ekvacioj **36** (1993), 71–93.
4. J. Y. Park, J. M. Jeong and Y. C. Kwun, *Regularity and controllability for semilinear control system*, Indian J. pure appl. Math. **29(3)** (1998), 239-252.
5. S. Nakagiri, *Optimal control of linea retarded systems in Banach spaces*, J. Math. Anal. Appl. **120(1)** (1986), 169-210.
6. H. Tanabe, *Fundamental solutions for linear retarded functional differential equations in Banach space*, Funkcialaj Ekvacioj **35(1)** (1992), 149–177.

Department of Mathematics,
Pusan National University,
Pusan 609-739, Korea

Division of Mathematical Sciences,
Pukyong National University,
Pusan 608-737, Korea

Department of Mathematics,
Pusan National University,
Pusan 609-739, Korea