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THE BOUNDARY ELEMENT METHOD FOR POTENTIAL PROBLEMS WITH SINGULARITIES

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ABSTRACT. A new procedure of the boundary element method(BEM),say, singular BEM for the potential problems with singularities is presented. To obtain the numerical solution of which asymptotic behavior near the singularities is close to that of the analytic solution, we use particular elements on the boundary segments containing singularities. The Motz problem and the crack problem are taken as the typical examples, and numerical results of these cases show the efficiency of the present method.

1. Introduction. The general potential boundary value problem in the plane can be written by

$$\begin{aligned} \Delta u &= 0\,, \qquad \text{in} \quad \Omega \\ u &= \overline{u}\,, \qquad \text{on} \quad \Gamma_u \\ q &= \overline{q}\,, \qquad \text{on} \quad \Gamma_q\,, \end{aligned} \tag{1.1}$$

where $\Gamma = \Gamma_u + \Gamma_q$ is the piecewisely smooth boundary of the domain Ω and $q = \frac{\partial u}{\partial n}$ is the normal derivative of the potential u with the outward unit normal vector n.

There are many numerical methods for solving boundary value problems such as finite element method(FEM), finite difference method(FDM), Ritz-Galerkin method and boundary element method(BEM). Each of these methods has its own advantages and disadvantages. When the solutions have singularities, which are due to the complicated boundary conditions or the geometries of the boundaries, the traditional numerical schemes are not useful. Therefore some special manipulation is needed to overcome this difficulty [1 – 4].

The BEM is prevailing recently in many engineering disciplines because of the reduction in the dimensionality of the problem which results in a much smaller system of algebraic equations to be solved numerically. The present work is concerned with the simple and effective numerical implementation of the traditional BEM [5,6], for solving the potential problems with singularities.

It is well known that, for interior points P in Ω, the solution of the problem (1.1) satisfies

$$u(P) \;+\; \int_\Gamma q^*(P,Q)\, u(Q)\, d\Gamma(Q) \;=\; \int_\Gamma u^*(P,Q)\, q(Q)\, d\Gamma(Q)\,, \tag{1.2}$$

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in which $u^*(P,Q)$ and $q^*(P,Q)$ are fundamental solutions of the Laplace equation such as

$$u^*(P,Q) \;=\; \frac{1}{2\pi}\log\frac{1}{r} \tag{1.3}$$

and

$$q^*(P,Q) \;=\; \frac{\partial u^*}{\partial n_Q}(P,Q) \;=\; -\frac{1}{2\pi}\frac{1}{r^2}\left(r_1 n_1 \,+\, r_2 n_2\right). \tag{1.4}$$

In the formulae (1.3) and (1.4), for the points $P=(p_1,p_2)$ and $Q=(q_1,q_2)$,

$$\begin{aligned} r \;&=\; |Q-P| \;=\; \sqrt{r_1^2+r_2^2}\,, \\ r_1 \;&=\; q_1-p_1\,, \qquad r_2 \;=\; q_2-p_2\,, \\ n_Q \;&=\; (n_1,\, n_2)\,. \end{aligned} \tag{1.5}$$

Limiting process of the equation (1.2) to the boundary point induces the boundary integral equation

$$\frac{1}{2}u(P) \,+\, \int_\Gamma q^*(P,Q)\,u(Q)\,d\Gamma(Q) \;=\; \int_\Gamma u^*(P,Q)\,q(Q)\,d\Gamma(Q)\,, \quad P \;\in\; \Gamma\,. \tag{1.6}$$

First, the numerical scheme for the traditional BEM is given in the section 2. A simple modification of the boundary elements near the singularities is proposed in the section 3, and applications of the present method to the Motz problem and crack problem are studied in the last section.

2. Traditional BEM with Constant Elements and Linear Discretization.

In this section we review the complete algorithm of the traditional BEM, for simplicity, based on the constant element with the linear discretization of the boundary.

2.1. *Discretization*

Let the boundary Γ is discretized by the line segments Γ_j $(j \;=\; 1,2,\cdots,n)$, of which end points are (x_j,y_j) and (x_{j+1},y_{j+1}). Then every point $Q=(x,y) \;\in\; \Gamma_j$ can be written by

$$\begin{aligned} x \;&=\; \xi_j(t) \;=\; \frac{1}{2}\left[(x_{j+1}-x_j)\,t \,+\, (x_{j+1}+x_j)\right], \\ y \;&=\; \eta_j(t) \;=\; \frac{1}{2}\left[(y_{j+1}-y_j)\,t \,+\, (y_{j+1}+y_j)\right], \qquad -1\le t\le 1\,, \end{aligned} \tag{2.1}$$

with

$$\begin{aligned} d\Gamma_j(Q) \;&=\; \sqrt{\xi_j'(t)^2+\eta_j'(t)^2}\,dt \\ &=\; \frac{1}{2}\sqrt{(x_{j+1}-x_j)^2+(y_{j+1}-y_j)^2}\,dt \;\equiv\; \frac{1}{2}L_j\,dt\,, \end{aligned} \tag{2.2}$$

and the outward unit normal vector is

$$n_Q \,=\, (n_1, n_2) \,=\, (y_{j+1} - y_j,\, x_j - x_{j+1})/L_j\,. \tag{2.3}$$

On the other hand we take the node point on Γ_i as

$$P_i \,=\, (\hat{x}_i,\, \hat{y}_i)\,, \qquad \hat{x}_i \,=\, (x_i + x_{i+1})/2\,, \quad \hat{y}_i \,=\, (y_i + y_{i+1})/2\,. \tag{2.4}$$

Then the distance between P_i and $Q \,\in\, \Gamma_j$ is

$$r(t) \,=\, |Q - P_i| \,=\, \sqrt{r_1(t)^2 + r_2(t)^2}\,, \tag{2.5}$$

with

$$r_1(t) \,=\, \xi_j(t) - \hat{x}_i \qquad \text{and} \quad r_2(t) \,=\, \eta_j(t) - \hat{y}_i\,. \tag{2.6}$$

On each boundary segment Γ_j, we take the constant values for the potential and flux, say,

$$u(Q) \,=\, u(P_j) \,=\, u^j\,, \qquad q(Q) \,=\, q(P_j) \,=\, q^j\,, \qquad \text{for all } Q \,\in\, \Gamma_j\,. \tag{2.7}$$

Then, for every node point $P = P_i \,\in\, \Gamma_i$, the boundary integral equation (1.6) results in

$$\frac{1}{2}u^i \,+\, \sum_{j=1}^{n} \left[\int_{\Gamma_j} q^*(P_i, Q)\, d\Gamma_j(Q)\right] u^j \,=\, \sum_{j=1}^{n} \left[\int_{\Gamma_j} u^*(P_i, Q)\, d\Gamma_j(Q)\right] q^j\,, \tag{2.8}$$

$i = 1, 2, \cdots, n\,.$

The equation (2.8) is rewritten by

$$\sum_{j=1}^{n} H^{ij} u^j \,=\, \sum_{j=1}^{n} G^{ij} q^j\,, \qquad i = 1, 2, \cdots, n\,, \tag{2.9}$$

in which the integrals G^{ij} and H^{ij} can be evaluated numerically by the Gauss quadrature rule. That is, referring to the formulae (2.1) – (2.6),

$$\begin{aligned} G^{ij} \,&=\, \int_{\Gamma_j} u^*(P_i, Q)\, d\Gamma_j(Q) \\ &=\, -\frac{1}{2\pi} \int_{-1}^{1} \log r(t) \left[\frac{1}{2} L_j\right] dt \\ &\approx\, \left(-\frac{L_j}{4\pi}\right) \sum_{m=1}^{M} \omega_m \log r(t_m)\,, \end{aligned} \tag{2.10}$$

where ω_m and t_m are weights and nodes of Gauss quadrature rule in the interval $-1 \le t \le 1$. In particular, when $i = j$,

$$\begin{aligned} G^{ii} \,&=\, \left(-\frac{L_i}{4\pi}\right) \int_{-1}^{1} \log \left|\frac{L_i}{2} t\right| dt \\ &=\, \left(-\frac{L_i}{2\pi}\right) \left\{\log\left(\frac{L_i}{2}\right) \,-\, 1\right\}\,. \end{aligned} \tag{2.11}$$

In the formula (2.9) $H^{ij} = \frac{1}{2}\delta_{ij} + \hat{H}^{ij}$, and $\hat{H}^{ij}$ is approximated by

$$\begin{aligned} \hat{H}^{ij} &= \int_{\Gamma_j} q^*(P_i, Q)\, d\Gamma_j(Q) \\ &= \left(-\frac{L_j}{4\pi}\right) \int_{-1}^{1} \frac{1}{r(t)^2} \left[r_1(t)n_1 + r_2(t)n_2\right] dt \\ &\approx \left(-\frac{L_j}{4\pi}\right) \sum_{m=1}^{M} \frac{\omega_m}{r(t_m)^2} \left[r_1(t_m)n_1 + r_2(t_m)n_2\right] . \end{aligned} \tag{2.12}$$

When $i = j$, $[r_1(t)n_1 + r_2(t)n_2] = 0$ so that

$$\hat{H}^{ii} = 0 . \tag{2.13}$$

2.2. *Solving the system of boundary integral equations*
If we define the following matrices

$$H = \left[H^{ij}\right]_{n\times n} , \qquad G = \left[G^{ij}\right]_{n\times n}$$

and vectors

$$\mathbf{u} = \{u^j\}_{n\times 1} , \qquad \mathbf{q} = \{q^j\}_{n\times 1} ,$$

the equation (2.9) can be written by

$$H\mathbf{u} = G\mathbf{q} . \tag{2.14}$$

Assume that the input data is given as

$$\begin{aligned} \mathrm{EP} &= \{x_j,\, y_j\}_{n\times 2} , \\ \mathbf{f} &= \{f^j\}_{n\times 1} , \\ T &= \{T[j]\}_{n\times 1} \qquad (\, T[j] = 0,\ \text{or } 1\,) , \end{aligned} \tag{2.15}$$

in which EP is a set of the extreme points of the boundary segments, and $\mathbf{f}$ is a set of boundary conditions at the node points. T indicates the type of boundary conditions at the nodes ; $T[j] = 0$ means that the value of the potential is known at the node j, that is, $u^j = f^j$ while the flux q^j is unknown. $T[j] = 1$ means that $q^j = f^j$ with the potential u^j unknown.

To find the unknown values of u and q by substituting the boundary conditions into (2.14) one has to rearrange the system by moving columns of H and G from one side to the other. If all the unknowns are passed to the left hand side, then the system (2.14) is translated into

$$\begin{aligned} A\mathbf{x} &= \mathbf{y} , \\ A &= [a_{ij}]_{n\times n} , \\ \mathbf{x} &= \{x^j\}_{n\times 1} , \qquad \mathbf{y} = \{y^j\}_{n\times 1} , \end{aligned} \tag{2.16}$$

where $\mathbf{x}$ is a vector of unknowns for u's and q's. $\mathbf{y}$ is found by multiplying the corresponding columns of the translated matrix of G by known values for u's and q's.

Based on the statement given above, we introduce an algorithm to translate the system (2.14) into (2.16) as following :

DO $j = 1, 2, \cdots, n$
  DO $i = 1, 2, \cdots, n$

$$\begin{aligned} a_{ij} &= (T[j]-1)G^{ij} + T[j]H^{ij} \\ B^{ij} &= T[j]G^{ij} + (T[j]-1)H^{ij} \end{aligned} \tag{2.17}$$

CONTINUE
DO $i = 1, 2, \cdots, n$

$$y^i = \sum_{k=1}^{n} B^{ik} f^k \tag{2.18}$$

CONTINUE

Once the vector of unknowns, $\mathbf{x} = \{x^j\}_{n\times 1}$ is obtained by solving the equation (2.16), values of the potential and the flux at each node are given by

$$\begin{aligned} u^j &= T[j]x^j + (1-T[j])f^j \\ q^j &= (1-T[j])x^j + T[j]f^j\,, \qquad j = 1, 2, \cdots n\,. \end{aligned} \tag{2.19}$$

2.3. *Evaluation at the internal points*
After the unknown coefficients of $\{u^j\}$ and $\{q^j\}$ are obtained, the values of potential and flux at the interior point can be evaluated such as

$$u(P) = \sum_{j=1}^{n} G^j(P)q^j - \sum_{j=1}^{n} \hat{H}^j(P)u^j\,, \quad P = (x, y) \in \Omega\,. \tag{2.20}$$

In this formula $G^j(P)$ and $\hat{H}^j(P)$ are same as G^{ij} and $\hat{H}^{ij}$ given in (2.10) and (2.12), respectively if we replace $r_i(t)$ $(i = 1, 2)$ in (2.6) by

$$r_1(t) = \xi_j(t) - x \qquad \text{and} \quad r_2(t) = \eta_j(t) - y\,. \tag{2.21}$$

In order to evaluate the derivatives of the potential at the internal points, we consider the following equations resulting from (1.2).

$$\begin{aligned} \frac{\partial u}{\partial x}(P) &= \int_\Gamma \frac{\partial}{\partial x}u^*(P,Q)\,q(Q)\,d\Gamma(Q) - \int_\Gamma \frac{\partial}{\partial x}q^*(P,Q)\,u(Q)\,d\Gamma(Q)\,, \\ \frac{\partial u}{\partial y}(P) &= \int_\Gamma \frac{\partial}{\partial y}u^*(P,Q)\,q(Q)\,d\Gamma(Q) - \int_\Gamma \frac{\partial}{\partial y}q^*(P,Q)\,u(Q)\,d\Gamma(Q)\,, \end{aligned} \tag{2.22}$$

for $P = (x, y) \in \Omega$. In these formulae the kernels are

$$\begin{aligned}
\frac{\partial}{\partial x}u^*(P,Q) &= \frac{1}{2\pi}\frac{r_1}{r^2}, \qquad \frac{\partial}{\partial y}u^*(P,Q) = \frac{1}{2\pi}\frac{r_2}{r^2}, \\
\frac{\partial}{\partial x}q^*(P,Q) &= -\frac{1}{2\pi}\frac{1}{r^2}\left\{\frac{2}{r^2}r_1(r_1n_1 + r_2n_2) - n_1\right\}, \\
\frac{\partial}{\partial y}q^*(P,Q) &= -\frac{1}{2\pi}\frac{1}{r^2}\left\{\frac{2}{r^2}r_2(r_1n_1 + r_2n_2) - n_2\right\}.
\end{aligned} \tag{2.23}$$

Discretization of the equation (2.22) gives

$$\begin{aligned}
\frac{\partial u}{\partial x}(P) &= \sum_{j=1}^{n}\left\{G_x^j(P)q^j - H_x^j(P)u^j\right\}, \\
\frac{\partial u}{\partial y}(P) &= \sum_{j=1}^{n}\left\{G_y^j(P)q^j - H_y^j(P)u^j\right\},
\end{aligned} \tag{2.24}$$

where

$$\begin{aligned}
G_x^j(P) &= \int_{\Gamma_j}\frac{\partial u^*}{\partial x}\,d\Gamma_j, \qquad G_y^j(P) = \int_{\Gamma_j}\frac{\partial u^*}{\partial y}\,d\Gamma_j, \\
H_x^j(P) &= \int_{\Gamma_j}\frac{\partial q^*}{\partial x}\,d\Gamma_j, \qquad H_y^j(P) = \int_{\Gamma_j}\frac{\partial q^*}{\partial y}\,d\Gamma_j.
\end{aligned} \tag{2.25}$$

Using the formulae (2.23)–(2.25) one can obtain the derivatives of the potential at every internal points.

3. Boundary Elements near the Singular Points.

Assume that Γ is a boundary of a bounded region and that behavior of the solutions for the potential and the flux at the internal point P near $P_\circ$ is like as

$$u(P) = O(|P_\circ - P|^\alpha), \qquad q(P) = O(|P_\circ - P|^{\alpha-1}), \tag{3.1}$$

$0 < \alpha < 1$. That is $P_\circ$ is the singular point for the flux. For some integer k, one may take the sequential boundary segments Γ_k and Γ_{k+1} of which common extreme point is $P_\circ$. In this case, instead of the constant elements, we present particular boundary elements on these segments Γ_k and Γ_{k+1} such as

$$\begin{aligned}
u_k(t) &= u^k(1-t)^\alpha, \qquad q_k(t) = g^k(1-t)^{\alpha-1} \qquad \text{on } \Gamma_k, \\
u_{k+1}(t) &= u^{k+1}(1+t)^\alpha, \quad q_{k+1}(t) = g^{k+1}(1+t)^{\alpha-1} \qquad \text{on } \Gamma_{k+1},
\end{aligned} \tag{3.2}$$

$-1 \le t \le 1$. It should be noted that $u_k(t)$, $u_{k+1}(t)$, $q_k(t)$ and $q_{k+1}(t)$ satisfy the conditions in (3.1) near the singular point $P_\circ$.

Then, for the singular boundary segments Γ_k and Γ_{k+1}, the integrals in (2.10) and (2.12) are replaced by

$$\begin{aligned} G^{ik} &\approx \left(-\frac{L_k}{4\pi}\right) \sum_{m=1}^{M} \omega_m (1-t_m)^{\alpha-1} \log r(t_m)\,, \\ \hat{H}^{ik} &\approx \left(-\frac{L_k}{4\pi}\right) \sum_{m=1}^{M} (1-t_m)^{\alpha} \frac{\omega_m}{r(t_m)^2} \left[r_1(t_m)n_1 \,+\, r_2(t_m)n_2\right]\,, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} G^{i(k+1)} &\approx \left(-\frac{L_{k+1}}{4\pi}\right) \sum_{m=1}^{M} \omega_m (1+t_m)^{\alpha-1} \log r(t_m)\,, \\ \hat{H}^{i(k+1)} &\approx \left(-\frac{L_{k+1}}{4\pi}\right) \sum_{m=1}^{M} (1+t_m)^{\alpha} \frac{\omega_m}{r(t_m)^2} \left[r_1(t_m)n_1 \,+\, r_2(t_m)n_2\right]\,. \end{aligned} \tag{3.4}$$

When Γ is an open arc, say, a crack and thus the singularities occur at the two crack tips, one may take Γ_1 and Γ_n so that they contain the singular points in the left and right hand sides, respectively. Therefore, in this case, the formulae (3.3) and (3.4) hold, by replacing k and $k+1$ by n and 1, respectively.

4. Applications of the Singular BEM.

We introduce two typical examples, a Motz problem and a crack problem, in sequence, to show the efficiency of the singular BEM introduced in this article. The number of the nodes of Gauss quadrature rule, in the formulae (2.10) and (2.12), is taken as $M = 4$.

4.1. The Motz problem.

As a typical singularity problem we consider the Motz problem, as shown in Figure 1, in a rectangular domain $\Omega = \{(x,y) \mid -1 < x < 1, \quad 0 < y < 1\}$ with the boundary conditions :

$$\begin{aligned} u|_{y=0,x<0} \,&=\, 0, \qquad u|_{x=1} \,=\, 500, \\ q|_{y=1} \,&=\, q|_{y=0,x>0} \,=\, q|_{x=-1} \,=\, 0\,. \end{aligned} \tag{4.1}$$

Fig 1 is located hear.

It is known that the solution of (4.1) has a singularity at the origin. In fact the exact solution can be expressed in a series as [3]

$$u(r,\theta) = \sum_{j=0}^{\infty} b_j r^{j+\frac{1}{2}} \cos(j+\frac{1}{2})\theta\,, \tag{4.2}$$

where (r,θ) are polar coordinates.

Applying the singular BEM to this problem, we have a good approximation to the exact solution near the singular point. Referring to the fact that the derivatives of the exact solution has $O(r^{-\frac{1}{2}})$ singularity in the vicinity of the origin, one has to take the singular boundary elements in (3.2) such as

$$\begin{aligned} u_k(t) &= u^k(1-t)^{\frac{1}{2}}\,, & q_k(t) &= g^k(1-t)^{-\frac{1}{2}} \\ u_{k+1}(t) &= u^{k+1}(1+t)^{\frac{1}{2}}\,, & q_{k+1}(t) &= g^{k+1}(1+t)^{-\frac{1}{2}}. \end{aligned} \tag{4.3}$$

Figure 2 shows the behavior of the traditional BEM solution, u_n^T and the singular BEM solution, u_n^S near the singular point, respectively . The subscript n indicates the number of the boundary elements. Comparison of the relative errors of these two results are given in Figure 3. We have taken u as an exact solution which is evaluated from the truncated series with sufficiently large number of terms in (4.2).

Fig 2 is located hear.

Fig 3 is located hear.

In Table 1 relative errors of the traditional and the singular BEM solutions with $n = 30$ for the potential and normal derivatives are given. The selected points are such as

$$P_1 = (-\frac{1}{2}, \frac{1}{4}), \qquad P_2 = (\frac{1}{2}, \frac{1}{2}), \qquad P_3 = (\frac{1}{2}, \frac{3}{4}).$$

The relative errors are defined as

$$E^T = \left|\frac{u - u_n^T}{u}\right|, \quad E_1^T = \left|\frac{\partial}{\partial x}\left(u - u_n^T\right) \Big/ \frac{\partial u}{\partial x}\right|, \quad E_2^T = \left|\frac{\partial}{\partial y}\left(u - u_n^T\right) \Big/ \frac{\partial u}{\partial y}\right|,$$

and E^S, E_1^S, E_2^S are similarly defined. Table 1 shows that the singular BEM solution is satisfactory on the whole interior points as well as near the singular points.

Table 1 is located hear.

4.2. The crack problem.

In this case we consider the Dirichlet problem on a crack, Γ such as

$$\begin{aligned} &\Delta u(P) = 0, \qquad P \in R^2 \setminus \Gamma, \\ &u(P) = f(P), \qquad P \in \Gamma, \\ &\sup_{P \in R^2} |u(P)| < \infty. \end{aligned} \tag{4.4}$$

Recently several numerical methods for this type of problems are studied by the indirect BEM [7,8,9]. Even though they have given complete approximation schemes and convergence analysis, numerical results for the singular fields near the crack tips are not presented.

Above all it should be noted that, for crack problems as given above, the double layer potential $\int_\Gamma q^*(P,Q)\,u(Q)\,d\Gamma(Q)$ in (1.2) is canceled by the opposite signs of the fundamental solution q^* on the upper and lower crack faces. Thus the integral equation (1.2) should be replaced by

$$u(P) \;=\; \int_\Gamma u^*(P,Q)\,q(Q)\,d\Gamma(Q) \;+\; \alpha\,, \tag{4.5}$$

where α is an unknown constant. As mentioned in the literature [8], the addition of the unknown constant α is due to the following constraint

$$\int_\Gamma q(Q)\,d\Gamma(Q) \;=\; 0\,, \tag{4.6}$$

which results from the boundedness condition in the problem (4.4). In fact the constant α means the limit value of the potential $u(P)$ as $|P| \to \infty$.

It is known that the flux has also $O(r^{-\frac{1}{2}})$ singularity near the crack tips. Taking the first boundary segment, Γ_1 and the last one, Γ_n so that they contain the left and right hand side crack tips, respectively, we denote the singular boundary elements such as

$$q_1(t) \;=\; q^1(1+t)^{-\frac{1}{2}} \quad \text{on } \Gamma_1\,, \qquad q_n(t) \;=\; g^n(1-t)^{-\frac{1}{2}} \quad \text{on } \Gamma_n\,. \tag{4.7}$$

Then the discretization scheme given in the section 2 and the condition (4.6) imply that

$$\sum_{j=1}^{n} \frac{L_j}{2} \int_{-1}^{1} q_j(t)\,dt \;=\; \sqrt{2}\left\{L_1 q^1 \;+\; L_n q^n\right\} \;+\; \sum_{j=2}^{n-1} L_j q^j \;=\; 0\,. \tag{4.8}$$

If one chooses the boundary segments so that $L_1 = L_2 = \cdots = L_n$, equations (4.5) and (4.8) result in the system

$$\begin{bmatrix} G^{11} & G^{12} & & \cdots\cdots & & G^{1n} & 1 \\ G^{21} & G^{22} & & \cdots\cdots & & G^{2n} & 1 \\ \vdots & \vdots & & & & \vdots & \vdots \\ G^{n1} & G^{n2} & & \cdots\cdots & & G^{nn} & 1 \\ \sqrt{2} & 1 & 1 & \cdots\cdots & 1 & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} q^1 \\ q^2 \\ \vdots \\ q^n \\ \alpha \end{bmatrix} \;=\; \begin{bmatrix} u^1 \\ u^2 \\ \vdots \\ u^n \\ 0 \end{bmatrix} \tag{4.9}$$

As an example we take Γ as the line segments on the x-axis, that is, $\Gamma = [-1,1]$, and suppose the boundary condition is given by

$$f(x) \;=\; e^{-x}\cos\sqrt{1-x^2}\,, \qquad x \in \Gamma\,. \tag{4.10}$$

The exact solution of (4.4) is known as [7]

$$u(x_1,x_2) \;=\; \mathrm{Re}\left[e^{\sqrt{z^2-1}-z}\right]\,, \qquad z = x_1 + ix_2\,. \tag{4.11}$$

Figure 4 shows the behavior of the indirect BEM solution in the ref. [8], u_n^I and the singular BEM solution, u_n^S near the right crack tip, respectively. Figure 5 gives comparison of the relative errors of these solutions with respective to the exact solution (4.11). Figure 4 and Figure 5 prove that the present method is very effective near the singularities.

Fig 4 is located hear.

Fig 5 is located hear.

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