J. KSIAM **Vol.3**, No.2, 1-4, 1999

Optimal ρ acceleration parameter for the ADI iteration for the real three dimensional Helmholtz equation with nonnegative ω

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Abstract

The Helmholtz equation is very important in physics and engineering. However, solution of the Helmholtz equation is in general known as a very difficult phenomenon. For if the ω is negative, the FDM discretized linear system becomes indefinite, whose solution by iterative method requires a very clever preconditioner. In this paper we assume that ω is nonnegative, and determine the optimal ρ parameter for the three dimensional ADI iteration for the Helmholtz equation. The ADI(Alternating Direction Implicit) method is also getting new attentions due to the fact that it is very suitable to the vector/parallel computers, for example, as a preconditioner to the Krylov subspace methods. However, classical ADI was developed for two dimensions, and for three dimensions it is known that its convergence behaviour is quite different from that in two dimensions. So far, in three dimensions the so-called Douglas-Rachford form of ADI was developed. It is known to converge for a relatively wide range of ρ values but its convergence is very slow. In this paper we determine the necessary conditions of the ρ parameter for the convergence and optimal ρ for the three dimensional ADI iteration of the Peaceman-Rachford form for the real Helmholtz equation with nonnegative ω. Also, we conducted some experiments which is in close agreement with our theory. This straightforward extension of Peaceman-rachford ADI into three dimensions will be useful as an iterative solver itself or as a preconditioner to the the Krylov subspace methods, such as CG(Conjugate Gradient) method or GMRES(m).

1 Three Dimensional Extension into the Helmholtz equation

For three dimensional Poisson problems Douglas[1] proposed a variant of the classical ADI, which has more smooth convergence behavior.

ALGORITHM **1.1 DO3-ADI(Douglas ADI)**

$$
\begin{aligned}
(H+\rho_i I)u_{i+1/3} &= -(H+2V+2W-\rho_i I)u_i + 2b \\
(V+\rho_i I)u_{i+2/3} &= -(H+V+2W-\rho_i I)u_i - Hu_{i+1/3} + 2b \\
(W+\rho_i I)u_{i+1} &= -(H+V+W-\rho_i I)u_i - Hu_{i+1/3} - Vu_{i+2/3} + 2b
\end{aligned} \tag{1}
$$

This work was supported by project for supprting leading trial schools in IT from the Ministry of Information and Communication

Douglas[1] has proven that in the case where the matrices H, V, and W all commute, the above iteration is convergent for **fixed** $\rho > 0$. However, it is demonstrated by experiments that DO3-ADI converges for a wider range of values of ρ, but the convergence rate is very slow. Due to the slow convergence rate it has rarely been used as an iterative method.

So writing

$$\begin{aligned} A &= (\tilde{H} + \tfrac{\omega}{3}I \ + \ \rho_i I) + (A - \tilde{H} - \tfrac{\omega}{3}I \ - \ \rho_i I) \\ &= (\tilde{V} + \tfrac{\omega}{3}I \ + \ \rho_i I) + (A - \tilde{V} - \tfrac{\omega}{3}I \ - \ \rho_i I) \\ &= (\tilde{W} + \tfrac{\omega}{3}I \ + \ \rho_i I) + (A - \tilde{W} - \tfrac{\omega}{3}I \ - \ \rho_i I) \end{aligned}$$

ALGORITHM **1.2 Peaceman-Rachford ADI in three dimensions(PR3-ADI)**

$$\begin{aligned} (H + \rho_i I)u_{i+1/3} &= -(V + w - \rho_i I)u_i + b \\ (V + \rho_i I)u_{i+2/3} &= -(H + w - \rho_i I)u_{i+1/3} + b \\ (W + \rho_i I)u_{i+1} &= -(H + V - \rho_i I)u_{i+2/3} + b \end{aligned} \tag{2}$$

where $H = \tilde{H} + \frac{\omega}{3}I$, $V = \tilde{V} + \frac{\omega}{3}I$, $W = \tilde{W} + \frac{\omega}{3}I$. The convergence behavior of this algorithm is quite different from that of PR2-ADI in two dimension. Assume that H, V, and W are pairwise commutative, and that

$$a \le \sigma(H), \sigma(V), \sigma(W) \le b,$$

where $\sigma(M)$ denote the spectrum of the matrix M.

Then, a and b are known to be

$$a = 4\sin^2(\frac{\pi}{2(n+1)}) + \frac{\omega}{3}, \ b = 4\sin^2(\frac{n\pi}{2(n+1)}) + \frac{\omega}{3}$$

where $N = n^3$.

Let T_ρ be the operator associated with PR3-ADI. Then,

$$T_\rho = (W + \rho I)^{-1}(H + V - \rho I)(V + \rho I)^{-1}(H + W - \rho I)(H + \rho I)^{-1}(V + W - \rho I) \tag{3}$$

Since the given equation is separable, $HV = VH$, $HW = WH$, and $VW = WV$ and H, V, and W share common set of eigenvectors. Let v be any such vector, and

$$Hv = \mu v, \ , Vv = \nu v, Wv = \xi v.$$

Then,

$$T_\rho v = \frac{(\mu + \nu - \rho)(\nu + \xi - \rho)(\mu + \xi - \rho)}{(\mu + \rho)(\nu + \rho) \ (\xi + \rho)} v \tag{4}$$

Then, the spectral radius of T_ρ is given by

$$S_p(T_\rho) = \max_{a \le \mu, \nu, \xi \le b} \left| \frac{(\mu + \nu - \rho)(\nu + \xi - \rho)(\mu + \xi - \rho)}{(\mu + \rho)(\nu + \rho)(\xi + \rho)} \right| \tag{5}$$

Now, we are looking for ρ such that (5) becomes smaller than 1. Now, we introduce several functions. Let

$$\phi_1(\rho) = \max_{a\le\mu,\nu,\xi\le b}\left|\frac{\nu+\xi-\rho}{\mu+\rho}\right| \tag{6}$$

$$\phi_2(\rho) = \max_{a\le\mu,\nu,\xi\le b}\left|\frac{\mu+\xi-\rho}{\nu+\rho}\right| \tag{7}$$

$$\phi_3(\rho) = \max_{a\le\mu,\nu,\xi\le b}\left|\frac{\mu+\nu-\rho}{\xi+\rho}\right| \tag{8}$$

and

$$\psi_1(\rho) = \max_{a\le\mu\le b}\left|\frac{2\mu-\rho}{\mu+\rho}\right| \tag{9}$$

$$\psi_2(\rho) = \max_{a\le\nu\le b}\left|\frac{2\nu-\rho}{\nu+\rho}\right| \tag{10}$$

$$\psi_3(\rho) = \max_{a\le\xi\le b}\left|\frac{2\xi-\rho}{\xi+\rho}\right| \tag{11}$$

Theorem 1.1 *Assume that $\rho > b/2$. Then,*

$$\phi_1(\rho) \equiv \psi_1(\rho)$$

Corollary 1.1 *With the same hypotheses as in theorem 1.1 the necessary and sufficient condition that the PR3-ADI iteration is convergent is that $\rho > b/2$.*

Proof. $S_p(T_\rho) = \phi_1(\rho)^3$, *hence if $\rho > b/2$ then $S_p(T_\rho) < 1$.* □

Theorem 1.2 *ρ minimizing $S_p(T_\rho)$ is given by $\rho = \rho^*$, where*

$$\rho^* = \frac{a+b+\sqrt{(a+b)^2+32ab}}{4}.$$

Proof.

$$\frac{\partial\phi_1}{\partial\rho} = -\frac{3b}{(b+\rho)^2} < 0, \quad \rho < \rho^*$$

and

$$\frac{\partial\phi_1}{\partial\rho} = \frac{3a}{(a+\rho)^2} > 0, \quad \rho > \rho^*.$$

So, the minimum is obtained when $\rho = \rho^*$. □

2 Experiments

The above tables shows that Douglas-Rachford ADI is indeed always convergent for any positive ρ, while the convergence speed is very slow. Also, the minimum number of iteration for PR3-ADI seems to happen around $\rho = 2.0$, which is close to our theoretical ρ^*.

	ρ												
	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
PR3	SL	SL	SL	SL	SL	SL	SL	87	58	62	63	80	61
DO3	145	140	133	134	134	128	122	154	170	174	176	183	195

Table 1: Poission Problem with N=48x48x48, iteration of CG-ADI with constant ρ until $\frac{||r_k||}{||r_0||} \leq 10^{-6}$

3 Conclusion

In three dimensions for the real Helmholtz equation with nonnegative ω the optimal ρ parameter for the stationary ADI iteration was determined. We believe that for the specific Helmholtz equation straightforward extension of Peaceman-Rachford ADI with properly predetermined ρ might converge faster our result might turn out to be useful. than the Douglas-rachford ADI for three dimensions. We believe that as a preconditioner to the Krylov subspace methods, such as CG(Conjugate Gradient) or GMRES(m),

References

[1] J. Douglas, "Alternating direction methods for three space variables", *Numerische Mathematik*, Vol. 4, pp. 41-63, 1962

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